A New Markov Chain Based Acceptance Sampling Policy via the Minimum Angle Method

M. S. Fallah Nezhad[*]

We develop an optimization model based on Markovian approach to determine the optimum value of thresholds in a proposed acceptance sampling design. Consider an acceptance sampling plan where items are inspected and when the number of conforming items between successive defective items falls below a lower control threshold value, then the batch is rejected, and if it falls above a control threshold value, then the batch is accepted and if it falls within the thresholds, the process of inspecting the items continues. A decision is made to accept or reject the batch. We begin with developing a Markov model for determining performance measures of sampling designs, resulting in an acceptance sampling plan optimized based on the minimum angle method. Then, the performance measures of the acceptance sampling plan are determined and the optimum values of thresholds are selected in order to optimize the objective functions. In order to demonstrate the application of the proposed methodology, numerical examples are illustrated.

1. Introduction

Acceptance sampling plans are statistical tools for rectifying the quality assurance. The sampling plans provide the vendor and buyer with decision rules for product acceptance to meet the present product quality requirements. Several types of decision rules have been proposed for the acceptance sampling problem but work on using the number of successive conforming items to control quality of the received lot based on the minimum angle method is scare. The idea of using the run-lengths of successive conforming items as an indicator of process performance has been around for a long time (Bourke [2]). Calvin [6] proposed a control chart based on the run-lengths of successive conforming items. Goh [11] proposed a charting procedure to control the low-nonconformity production. Bourke [4] proposed monitoring statistics based on the sums and CUSUMs of such conforming run-lengths for the case of 100% inspection. Bourke [2] noted that based on the conventional measures of performance of sampling plans such as the average outgoing quality, average fraction inspected, and the proportion passed under sampling inspection, the idea of using the run-lengths of successive conforming items as an indicator of process performance turns to a better performance. Also, Bourke [3] proposed switching rules based on a cumulative sum of the observed run-lengths of conforming items between successive defective items.

In recent decades, applying acceptance sampling methods have raised a number of questions in quality control. The main target being production specification and reduction of manufacturing tolerances, in many cases because of human and manufacturing system errors, acceptance sampling is a desired method (Arshadi Khamseh et al. [1]). Varderman (20) and Schilling (16) worked on how much accuracy of the acceptance sampling designs is used in practical environments. Hamilton and Lesperance (12) developed a method for single and multi variable acceptance sampling assuming that the process quality can be determined from the number of defects in the lot while the variance and mean are known.

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[*] Department of Industrial Engineering, Yazd University, Yazd, Iran. Email: Fallahnezhad@Yazduni.ac.ir

Tagares (18) proposed an economical model for the single variable acceptance sampling plan based on the Taguchi loss functions in the absence of inspection errors. Klassen (13) proposed a credit-based acceptance sampling system. The credit of the producer was defined as the total number of items accepted since the last rejection. In this system, the sample size from a lot is given by a simple function on the lot size, the credit and the chosen guaranteed upper limit on the outgoing quality. Niaki and Fallahnezhad (15) applied Bayesian inferences concept to design an acceptance sampling design. They used a stochastic dynamic programming model to minimize the ratio of the system cost to the system correct choice probability. Fallahnezhad and Hosseininasab (7) proposed a single stage acceptance sampling plan based on the control threshold policy. The objective of their model is to design an economic acceptance sampling model. Fallahnezhad and Niaki (17) proposed a Markov model for single sampling plan based on the control threshold policy considering the run-lengths of successive conforming items as an indicator of the process performance. Fallahnezhad et al. (9) extended this approach to the sum of run-lengths of successive conforming items. Fallahnezhad et al. (10) proposed a decision tree approach for designing the economic models of sampling plans.

Some authors applied the Markovian models to machine maintenance policy to derive the optimal process control. Tagaras (19) studied the joint process control and machine maintenance problem of a Markovian deteriorating machine. Kuo (14) developed an optimal adaptive control policy for the joint machine maintenance and product quality control. Bowling et al. (5) proposed a Markovian approach to determine the optimal process target levels for a multi-stage serial production system.

Here, a general model for acceptance sampling plans is developed incorporating the number of conforming items between successive defective items in its design. It is assumed that when the number of conforming items between successive defective items is more than a control threshold value, then the batch is accepted and when it is less than a control threshold value, then the batch is rejected. This paper provides an optimized acceptance sampling plan for given values of the acceptable quality level and limiting quality level using the minimum angle technique.

The rest of the paper is organized as follows. We present the preliminaries in Section 2 and give the model development in Section 3. The proposed methodology is described in Section 4. Section 5 provides a summary of results for the proposed method.

2. Preliminaries

Our notations are summarized below:

p: the proportion of the defective items in the batch

AQL: the maximum acceptable level of the batch quality (Accepted Quality Level)

LQL: the minimum rejectable level of the batch quality (Limiting Quality Level)

P: transition probability matrix

Q: a square matrix containing transition probabilities of going from any non-absorbing state to any other non-absorbing state

R: a matrix containing all probabilities of going from any non-absorbing state to an absorbing state (i.e., accepted or rejected batch)

A: an identity matrix representing the probability of staying in a state

O: a matrix representing the probabilities of escaping an absorbing state (always zero)

M: fundamental matrix containing the expected number of transitions from any non-absorbing state to any other non-absorbing state before an absorption

F: the absorption probability matrix containing the long run probabilities of the transition from any non-absorbing state to any absorbing state

p_{ij} : probability of going from state *i* to state *j* in a single step

$m_{ij}(p)$: expected number of transitions from any non-absorbing state *i* to any other non-absorbing state *(j)* before absorption occurs when proportion of the defective items is p

$f_{ij}(p)$: long run probability of going from any non-absorbing state *i* to any absorbing state *j* when proportion of the defective items is p .

3. Model Development

The model is to develop a Markovian approach for determining an optimal value of threshold for the accepting or rejecting the lot. Assume that in an acceptance sampling plan, Y_i is defined as the number of conforming items between the successive *(i-1)*th and *i*th defective items. The decision rule is defined as follows:

If $Y_i \geq U$, then the lot is in a good state and accepted. If $Y_i \leq L$, then the lot is in a bad state and rejected. If $U > Y_i > L$, then inspection of the items continues; where, U is an upper control threshold and L is a lower control threshold value.

The states of the problem is defined as follows:

State 1 ($U > Y_i > L$) : the value of Y_i is between the control thresholds L and U and thus inspection of the items continues.

State 2 ($Y_i \geq U$) : the batch is in a good state and accepted.

State 3 ($Y_i \leq L$) : the batch is in a bad state and rejected.

Thus, it is concluded that:

probability of inspecting more items $= p_{11} = P\{U > Y_i > L\}$,

probability of accepting the batch $= p_{12} = P\{Y_i \geq U\}$,   (1)

probability of rejecting the batch$= p_{13} = P\{Y_i \leq L\}$,

where $P(Y_i = r) = (1-p)^r p$, with p denoting the proportion of the defective items in the lot.

The transition probability matrix among the states of the lot is determined as follows:

$$P = \begin{matrix} & \begin{matrix} 1 & \ 2 & \ 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix}. \tag{2}$$

The matrix *P* is an absorbing Markov chain with states 2 and 3 being absorbing and state 1 being transient. To analyze this matrix, first the transition probability matrix is rearranged into the following form:

$$\begin{bmatrix} A & O \\ R & Q \end{bmatrix}. \tag{3}$$

Rearranging the *P* Matrix results in the following matrix:

$$\begin{array}{c} \\ 2 \\ 3 \\ 1 \end{array} \begin{matrix} 2 & 3 & 1 \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ p_{12} & p_{13} & p_{11} \end{bmatrix} \end{matrix}. \tag{4}$$

The fundamental matrix M can be determined as follows (Bowling et al., 2004):

$$M = m_{11}(p) = (I - Q)^{-1} = \frac{1}{1 - p_{11}} = \frac{1}{1 - P\{U > Y_i > L\}}, \tag{5}$$

where I is the identity matrix. The value $m_{11}(p)$ denotes the expected number of times in the long run that the transient state 1 is occupied before absorption occurs (i.e., the batch accepted or rejected), given that the initial state is 1. The long-run absorption probability matrix F is calculated as follows (Bowling et al., 2004):

$$F = M \times R == \begin{bmatrix} f_{12}(p) & f_{13}(p) \end{bmatrix} = \begin{bmatrix} \dfrac{p_{12}}{1 - p_{11}} & \dfrac{p_{13}}{1 - p_{11}} \end{bmatrix} \tag{6}$$

The elements of matrix F, $f_{12}(p), f_{13}(p)$ denote the probabilities of the batch being accepted and rejected, respectively.

The practical performance of any sampling plan is determined through its operating characteristic curve. When producer and consumer are negotiating for designing sampling plans, it is important specially to minimize the consumer's risk. In order to minimize the consumer's risk, the ideal OC curve could be made to pass as closely through [AQL, $1 - \alpha$] and [AQL, β]. One approach to minimize the consumer's risks for ideal condition is minimization of angle ϕ between the lines joining the points [AQL, $1-\alpha$], [AQL, β] and [AQL, $1-\alpha$], [LQL, β]. In this case, the value of the performance criterion in the minimum angle method is:

$$Cot(\phi) = \left(\frac{P_a(LQL) - P_a(AQL)}{LQL - AQL} \right), \tag{7}$$

where $P_a(LQL), P_a(AQL)$ are the probabilities of accepting the batch when the proportion of defective items in the batch are respectively LQL, AQL. Assume A is the point [AQL, $1-\alpha$], B is the point [AQL, β] and C is the point [LQL, β]. Thus the smaller the value of $Cot(\phi)$, the closer the angle ϕ approaching zero, the ideal condition for the chord AC approaching AB. For example, in a design, we have $P_a(LQL = 0.2) = 0.1, P_a(AQL = 0.05) = 0.9$. The simple plot of the OC curve is shown in Figure 1.

It is seen that as ϕ approaches zero, the chord AC approaches the, reaching the ideal condition.

The values of $P_a(LQL), P_a(AQL)$ are determined as follows:

$$p = AQL \rightarrow P_a(AQL) = f_{12}(AQL) = \frac{P\{U \leq Y_i\}}{1 - P\{U > Y_i > L\}}$$

$$p = LQL \rightarrow 1 - P_a(LQL) = 1 - f_{12}(LQL) = 1 - \frac{P\{U \leq Y_i\}}{1 - P\{U > Y_i > L\}} \tag{8}$$

Since the values of LQL, AQL are constant and $LQL > AQL$, therefore the objective function is determined as to be,

$$V = \underset{L,U}{Min}\left\{P_a\left(LQL\right) - P_a\left(AQL\right)\right\}. \qquad (9)$$
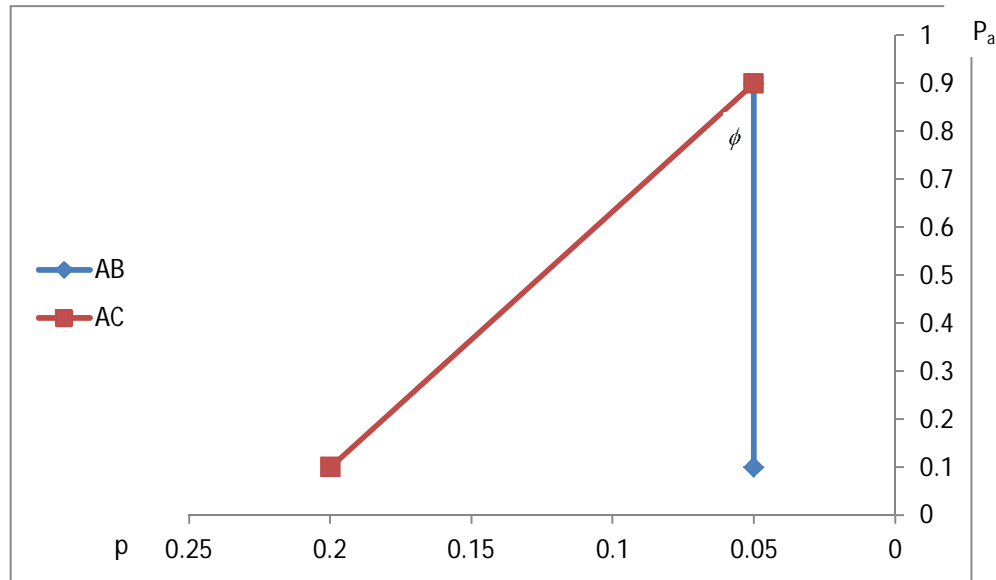

Figure 1. The simple plot of the OC curve

Another performance measure for the acceptance sampling plan is the expected number of inspected items. Since sampling and inspecting usually incur cost, therefore designs minimizing this measure while satisfying the first and the second error inequalities are considered to be optimal sampling plans. Since the proportion of defective items is not known in the start of the process, in order to consider this property in designing the acceptance sampling plans, we try to minimize the expected number of inspected items for acceptable and rejectable lots simultaneously. Therefore, the optimal acceptance sampling plan should have certain properties: it should have a minimized value for the objective function of the minimum angle method that is resulted from the ideal OC curve. It should also minimize the expected number of inspected items either in the decision of rejecting or accepting the lot. Therefore, a second objective function is defined as the expected number of items inspected. The value of this objective function is determined based on the value of $m_{11}(p)$, where $m_{11}(p)$ is the expected number of times in the long run that the transient state 1 is occupied before absorption occurs. Since for visits to a transient state, the average number of inspections is $\dfrac{1}{p}$, the expected number of items inspected is given by $\dfrac{1}{p}m_{11}(p)$. Now, the objective functions, W and Z are defined as the expected number of items inspected respectively in the acceptable condition $(p = AQL)$ and rejectable condition $(p = LQL)$:

$$W = \underset{L,U}{Min}\left\{\frac{1}{AQL}m_{11}\left(AQL\right)\right\},$$

$$Z = \underset{L,U}{Min}\left\{\frac{1}{LQL}m_{11}\left(LQL\right)\right\}. \qquad (10)$$

One approach to optimize the objective functions simultaneously is to define control limits for the objective functions Z,W and then try to minimize the value of the objective function V . For

example, if parameters Z_1, W_1 are defined as the upper control limits for Z, W, respectively, then the optimization problem can be defined as follows:

$$\begin{array}{l} \underset{L,U}{Max\, V} \\ s.t. \\ Z < Z_1, \\ W < W_1. \end{array} \tag{11}$$

Optimal values of L, U can be determined by solving the above nonlinear optimization problem using search procedures or other optimization tools.

Therefore, there are three objective functions and the optimal design is based on the value of these objective functions. The first objective function is defined as

$$V = \underset{L,U}{Min}\left\{ P_a\left(LQL\right) - P_a\left(AQL\right)\right\},$$

where,

$$P_a\left(AQL\right) = \frac{P\left\{U \le Y_i\right\}}{1 - P\left\{U > Y_i > L\right\}}.$$

The probabilities $P\left\{U \le Y_i\right\}$ and $P\left\{U > Y_i > L\right\}$ are determined using the fact that Y_i follows a geometric distribution with the success probability being equal to $p = AQL$. Also, $P_a\left(LQL\right)$ is determined by a similar reasoning. The second objective function is defined as $W = \underset{L,U}{Min}\left\{\frac{1}{AQL}m_{11}\left(AQL\right)\right\}$, where $m_{11}\left(AQL\right) = \frac{1}{1 - P\left\{U > Y_i > L\right\}}$. Again, the probability $P\left\{U > Y_i > L\right\}$ is determined using the fact that Y_i follows a geometric distribution with the success probability being equal to $p = AQL$. The third objective function is defined by a similar argument.

4. A Numerical Example

To demonstrate the application of the proposed methodology in an acceptance sampling design, a numerical example is solved. Consider a sampling problem, where $AQL = 0.05, LQL = 0.2$. The values of the objective functions for different alternative values of L and U among the existing alternatives are obtained using (9) and (10). Table 1 shows 16 different alternative combinations of L and U together with their objective functions values.

The optimization model is used to determine the optimal solution. Assume that the values of control limits are $Z_1 = 6, W_1 = 21$. The feasible values of L and U among the existing alternatives are obtained using the constraints of the optimization model (11). These feasible values are shown in Table 2. Based on the results, the best combination value is $L = 3$ and $U = 5$ with the maximum value of $V = -0.45$.

As mentioned, the optimal acceptance sampling design is determined based on the values of the three objective functions. In other words, the design with the optimized value for each objective function is selected as optimal. Since the optimum of the objective functions is not expected to occur at a single point, we need some compromised methods in optimizing the objective functions.

Table 1. Objective functions values

Z	W	V	L	U
7.02	22.04	-0.23	0.00	3.00
8.20	23.13	-0.27	0.00	4.00
9.48	24.28	-0.32	0.00	5.00
10.82	25.47	-0.37	0.00	6.00
5.73	20.95	-0.31	1.00	3.00
6.50	21.93	-0.36	1.00	4.00
7.27	22.95	-0.41	1.00	5.00
8.04	24.02	-0.46	1.00	6.00
5.00	20.00	-0.35	2.00	3.00
5.57	20.90	-0.39	2.00	4.00
6.13	21.82	-0.44	2.00	5.00
6.67	22.79	-0.49	2.00	6.00
5.00	20.00	-0.40	3.00	4.00
5.45	20.85	-0.45	3.00	5.00
5.86	21.73	-0.49	3.00	6.00
6.25	22.63	-0.53	3.00	7.00

Table 2. Feasible values of L and U among the existing alternatives

Z	W	V	L	U
5.73	20.95	-0.31	1.00	3.00
5.00	20.00	-0.35	2.00	3.00
5.57	20.90	-0.39	2.00	4.00
5.00	20.00	-0.40	3.00	4.00
5.45	20.85	-0.45	3.00	5.00

Methods used to optimize the objective functions can be interactive to consider priorities of the objectives. In general, since the OC curve is the most important performance measure of sampling plans, the method which maximizes the objective function V considering the two inequalities of expected number of items inspected is suggested to be applied in practice.

5. Conclusion

Here, a new approach for acceptance sampling plan was introduced. The number of conforming items between successive defective items was selected as the decision making criterion. Three objective functions were developed to optimize the performance measures of the acceptance sampling plan. Then, a procedure was proposed to optimize the objective functions simultaneously. In the case that group inspection was impossible and the items were to be inspected consecutively, the proposed approach is a suitable alternative. Since the proposed acceptance sampling plan considers different performance measures of acceptance sampling plan, it is beneficial for the practitioners to use it in the quality control environments. For further research, it is recommended to use the cumulative sum of the observed run-lengths of conforming items between successive defective items as the indicator of the process performance. Also, economic design of acceptance sampling plan using the proposed approach is another subject for future research.

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