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Close interval approximation of piecewise quadratic fuzzy numbers for fuzzy fractional program

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The fuzzy approach has undergone a profound structural transformation in the past few decades. Numerous studies have been undertaken to explain fuzzy approach for linear and nonlinear programs. While, the findings in earlier studies have been conflicting, recent studies of competitive situations indicate that fractional programming problem has a positive impact on comparative scenario. We propose one of the best interval approximations, close interval approximation of piecewise quadratic fuzzy numbers for solving fuzzy number fractional programming problem without converting it to a crisp problem. A new form of simplex method is introduced here for solving fuzzy number fractional programming problem using fuzzy arithmetic. The fuzzy analogue of some important theorems of fuzzy fractional programming problem proved. A fuzzy fractional programming problem is worked out as an example to illustrate the proposed method.

Key words: *Piecewise quadratic fuzzy number, Fuzzy fractional programming problem Fuzzy linear equations, Close interval approximation.*

1. Introduction

Decision-making problem occurring in real world are generally programming problems, and criteria involved in programming problems are often conflicting non commensurable and fuzzy in nature. Fuzzy mathematical programming is a family of optimization problems in which the optimization model parameters are not well defined. This means that the objective function and constraint coefficients are not exactly known and some of the inequalities involved may also be subject to unsharp boundaries. Precise values are not known and approximations can be specified more comfortably than point values. Linear/fractional programming is one of the operations research techniques, which is widely used and have found many achievements in both applications and theories. Generally speaking, in fuzzy mathematical programming problems, the coefficients of decision variables are fuzzy numbers while decision variables are crisp ones. This means that in an uncertain environment, a crisp decision is made to meet some decision criteria.

Initially, Bellman and Zadeh [2] has introduced the concept of decision making in fuzzy environment. Lai and Hwang [7], Tong [14], Buckley [3], Negi [14], among others, considered the mathematical programming problem in which all parameters are fuzzy. Lai and Hwang [7] assumed that the parameters have a triangular possibility distribution and used an auxiliary model solved by multiobjective linear programming methods. The special L-R fuzzy numbers and triangular fuzzy numbers were studied by Appadoo [1]. Tong [19] defined a goal for the objective function. Buckley [3] and Negi [14] obtained an optimal solution by using possibility concepts. Maleki, Tata and Mashinchi [8] obtained an optimal solution by using fuzzy variables. Yager [2] developed a mathematical programming with fuzzy constraints and preference on the objectives. Zimmermann [24] developed fuzzy mathematical programming to solve the problem with several

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objective functions and discussed equivalent and nonequivalent objectives. Rommelfanger [10] investigated the fuzzy linear optimization problem with several objectives and suggested an approach for interactive decision-making in fuzzy environment. Jain and Lachhwani [6] discussed the solution of multiobjective fractional programming problem by converting it into fuzzy programming problem. Zangiabadi and Maleki [22] defined the concepts of optimality for linear programming problems with fuzzy parameters based on those for multiobjective linear programming problems. Maleki and Mashinchi [4] proposed a method for solving fuzzy number linear programming problems using the concept of expectation and variance as ranking functions. Nasseri and Ardil [13] proposed a new method for solving fuzzy variable linear programming problems directly using linear ranking function. Gani and Dinagar [4] used close interval approximation for solving LPP. Wang and Qiao [20] studied fuzzy programming with fuzzy random variable coefficients.

Numerous methods for optimization problems have been suggested in the literature. Each method appears to have some advantages as well as disadvantages. In the context of each application, some methods seem to be more appropriate than others. However, the issue of choosing a proper method in a given context is still a subject of active research. Traditional mathematical programming models are based on the assumption that the decision making has a single, quantifiable, objective such as maximization of profit or minimization of inefficiency or cost. In general, fuzzy programming problems are first converted into equivalent crisp linear or nonlinear programming problems, which are then solved by standard method. The final result of fuzzy programming problem deals with real numbers, which represent a compromise in terms of the fuzzy numbers involved.

A number of researchers have worked on fuzzy linear programming problems while very few have considered fuzzy fractional programming problems. Here, we propose a close interval approximation of piecewise quadratic fuzzy number for solving fuzzy fractional programming problems without converting them to equivalent crisp linear or nonlinear programming problems. Gani and Dinagar [14] results are generalized here by framing new form of simplex method for fuzzy number fractional programming problem using fuzzy arithmetic. The proposed interval approximation, called close interval approximation, is the best one with respect to certain measure of distance between fuzzy numbers. We prove fuzzy analogue of some important theorems of fractional programming problems. A numerical example involving piecewise quadratic fuzzy number is also given to illustrate our approach.

The remainder of the paper is organized as follows. Section contains 2 some definitions used in our work including piecewise quadratic fuzzy number and close interval approximation descriptions. Details of Fuzzy arithmetic operations, fuzzy equations and fuzzy ranking are given in Section 3. Fuzzy number fractional programming model and related propositions are given in Section 4. Section 5 represents the explanation of our proposed model in form of numerical example and finally conclusion is drawn in Section 6.

2. Preliminaries

Before we proceed with the analysis, let us introduce some concepts on fuzzy numbers.

Definition 1: Let X denote a universal set, then the characteristic function which assigns certain values or a membership grade to the elements of the universal set within a specified range [0, 1] is known as the *membership function,* and the set thus defined is called a *fuzzy set*. The membership grades correspond to the degree to which an element is compatible with the concept represented by the fuzzy set. If μ_A is the membership function defining a fuzzy set A, then $\mu_A : X \longrightarrow [0, 1]$.

Definition 2: An *α- cut* of a fuzzy set A is a crisp set A_α that contains all the elements of the universal set X that have a membership grade in A greater than or equal to the specified value of α. Thus, $A_\alpha = \{x \in X : \mu_A \geq \alpha\},\ 0 \leq \alpha \leq 1$.

Definition 3: A convex and normalized fuzzy set defined on real numbers whose membership function is piecewise continuous is called a fuzzy *number,*

$$\overline{A} = \{(x, \mu_{\overline{A}}(x)) \mid x \in X, \mu_{\overline{A}}(x) \in [0,1]\}.$$

A fuzzy set is called normal when at least one of its elements attains the maximum possible membership grade, that is $\forall x \in R,\ \underset{x}{Max}\, \mu_A(x) = 1$

A fuzzy set is convex if and only if

$$\mu_A(\lambda p + (1-\lambda) q) \geq \min[\mu_A(p), \mu_A(q)], \forall p, q \in R^n, \lambda \in [0,1].$$

Definition 4: A *Piecewise-Quadratic Fuzzy Number*, A PQFN, $\overline{a} = (a_1, a_2, a_3, a_4, a_5)$ is defined by its membership function as

$$\mu_{\overline{a}} = \begin{cases} \dfrac{1}{2(a_2 - a_1)^2}(x - a_1)^2 & for\ a_1 \leq x \leq a_2 \\ \dfrac{1}{2(a_3 - a_2)^2}(x - a_3)^2 + 1 & for\ a_2 \leq x \leq a_3 \\ \dfrac{1}{2(a_4 - a_3)^2}(x - a_3)^2 + 1 & for\ a_3 \leq x \leq a_4 \\ \dfrac{1}{2(a_5 - a_4)^2}(x - a_5)^2 & for\ a_4 \leq x \leq a_5 \\ 0 & otherwise \end{cases}$$

A PQFN can be characterized by defining the interval of confidence at level α. Thus, $\forall \alpha \in [0,1],\ \alpha = [a_1 + 2(a_2 - a_1)\alpha,\ a_5 - 2(a_5 - a_4)\alpha]$, an interval of confidence in R, represening a type of uncertainty.

The PQFN is a bell shaped function symmetric about the line $x = a_3$, and has a supporting interval $A = [a_1, a_5]$. Moreover, $a_3 = [a_1 + a_5]/2$ and $a_3 - a_2 = a_4 - a_3$ The α–cut at level α = ½ between the points (a_2, a_4) is called cross over points. A diagrammatic representation of PQFN is given in Figure 1.

Definition 5: If $\overline{a} = (a_1, a_2, a_3, a_4, a_5)$ is a Piecewise-Quadratic Fuzzy Number (PQFN), then its associated (crisp) number is given by $a = (a_1+a_2+a_3+a_4+a_5)/5$. We can find the same ordinary number by $a = (a_1+a_2+2a_3+a_4+a_5)/6$ also.

Definition 6: An interval approximation $[a_\alpha^L, a_\alpha^U]$ of a piecewise quadratic fuzzy number A is said to be close interval approximation if

$$a_\alpha^L = \inf\{x \in R \mid \mu_A \geq 0.5\}$$
$$a_\alpha^U = \sup\{x \in R \mid \mu_A \geq 0.5\}$$

which we denote by [A]. Mathematically, if A = (a_1, a_2, a_3, a_4, a_5), then the close interval approximation of PQFN is [A] = [a_2, a_4].

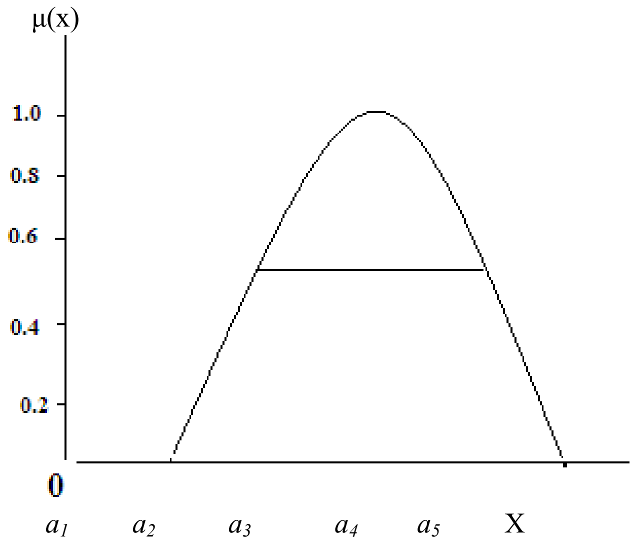

Figure 1: Diagrammatic representation of a PQFN

Definition7: If [A] = $\left[a_\alpha^L,\, a_\alpha^U\right]$ is the close interval approximation of PQFN, then associated real number of [A] is denoted by Re[A], Re[A] = $\left[\dfrac{a_\alpha^L + a_\alpha^U}{2}\right]$. If Re[A] > 0, then [A] is positive and if Re[A] < 0, then [A] is negative.

3. Fuzzy operations

Let [A] = $\left[a_\alpha^L,\, a_\alpha^U\right]$ and [B] = $\left[b_\alpha^L,\, b_\alpha^U\right]$ be the close interval approximations of PQFNs. Then, fuzzy arithmetic operations are defined as:

Fuzzy Addition: [A] + [B] = $\left[a_\alpha^L + b_\alpha^L,\, a_\alpha^U + b_\alpha^U\right]$.

Fuzzy Subtraction: [A] - [B] = $\left[a_\alpha^L - b_\alpha^U,\, a_\alpha^U - b_\alpha^L\right]$.

Fuzzy Scalar Multiplication: $\alpha[A] = \begin{cases} \left[\alpha a_\alpha^L, \alpha a_\alpha^U\right], & for\ \alpha > 0 \\ \left[\alpha a_\alpha^U, \alpha a_\alpha^L\right], & for\ \alpha < 0 \end{cases}$.

Fuzzy Multiplication: [A] . [B] = $\left[\frac{1}{2}\left(a_\alpha^U b_\alpha^L + a_\alpha^L b_\alpha^U\right), \frac{1}{2}\left(a_\alpha^L b_\alpha^L + a_\alpha^U b_\alpha^U\right)\right]$

Fuzzy Division:

$$\frac{[A]}{[B]} = \begin{cases} \left[2\left(\dfrac{a_\alpha^L}{b_\alpha^L + b_\alpha^U}\right), 2\left(\dfrac{a_\alpha^U}{b_\alpha^L + b_\alpha^U}\right)\right] & when\ [B] > 0\ and\ \ (b_\alpha^L + b_\alpha^U) \neq 0 \\ \left[2\left(\dfrac{a_\alpha^U}{b_\alpha^L + b_\alpha^U}\right), 2\left(\dfrac{a_\alpha^L}{b_\alpha^L + b_\alpha^U}\right)\right] & when\ [B] < 0\ and\ \ (b_\alpha^L + b_\alpha^U) \neq 0 \end{cases}$$

Fuzzy Equations

A linear equation system $\sum_{i=1}^{n}\overline{a_{ij}}x_j=\overline{b_j},\ j=1,2,...,m$, is said to be fuzzy linear equation if $\overline{a_{ij}},\overline{b_j}$ are PQFNs. Let A, B and X be fuzzy numbers Then. A+X =B and A.X = B are two simple fuzzy equations, where A and B are known and X is unknown. We are to determine fuzzy number X such that both equations hold for the given fuzzy numbers as represented by close interval approximations. Let [A] = $\left[a_\alpha^L,\ a_\alpha^U\right]$, [B] = $\left[b_\alpha^L,\ b_\alpha^U\right]$ and [X] = $\left[x_\alpha^L,\ x_\alpha^U\right]$, Then, A + X = B becomes X = B – A or $\left[x_\alpha^L,\ x_\alpha^U\right]$ = $\left[b_\alpha^L,\ b_\alpha^U\right]$ - $\left[a_\alpha^L,\ a_\alpha^U\right]$ = $\left[b_\alpha^L-a_\alpha^U,\ b_\alpha^U-a_\alpha^L\right]$. Equation *A.X=B* becomes X=B/A or, $\left[x_\alpha^L,\ x_\alpha^U\right]=\left[b_\alpha^L,\ b_\alpha^U\right]/\left[a_\alpha^L,\ a_\alpha^U\right]$

$$=\begin{cases}\left[2\left(\dfrac{b_\alpha^L}{a_\alpha^L+a_\alpha^U}\right),2\left(\dfrac{b_\alpha^U}{a_\alpha^L+a_\alpha^U}\right)\right] & when\ [A]>0\ and\ (a_\alpha^L+a_\alpha^U)\neq 0\\ \left[2\left(\dfrac{b_\alpha^U}{a_\alpha^L+a_\alpha^U}\right),2\left(\dfrac{b_\alpha^L}{a_\alpha^L+a_\alpha^U}\right)\right] & when\ [A]<0\ and\ (a_\alpha^L+a_\alpha^U)\neq 0\end{cases}$$

Fuzzy Ranking

A convenient method for comparing fuzzy numbers makes use of ranking functions. Let [A] and [B] be two close interval approximations of PQFNs. Then,

Fuzzy Equal: [A] and [B] are said to be equal if $a_\alpha^L=b_\alpha^L\quad and\quad a_\alpha^U=b_\alpha^U$.

Fuzzy Equivalent: [A] and [B] are said to be equivalent if $a_\alpha^L+a_\alpha^U=b_\alpha^L+b_\alpha^U$.

Fuzzy Less than: [A] is less than [B], it

$[A]\leq[B]$ if $a_\alpha^L\leq b_\alpha^L\quad and\quad a_\alpha^U\leq b_\alpha^U$ or $a_\alpha^L+a_\alpha^U\leq b_\alpha^L+b_\alpha^U$.

4. **Fuzzy fractional programming model**

The PQFN-FPP using close interval approximation is defined as:

$$Max/Min.\ Z=\frac{\sum_{j=1}^{p}[c_j]x_j+[\alpha\ ]}{\sum_{j=1}^{p}[d_j]x_j+[\beta\ ]}$$

$$\begin{aligned}&Subject\ to, && \sum_{j=i}^{p}[a_{ij}]x_j\leq[b_i], && i=1,2,...,m_0 && (1)\\ & && \sum_{j=i}^{p}[a_{ij}]x_j\geq[b_i], && i=m_0+1,2,...,m\\ &and && x_j\geq 0, && j=1,2,...,p.\end{aligned}$$

$$\begin{aligned}&where\quad [\alpha],[\beta],[a_{ij}]=\left(a_{ij\alpha}^L,\ a_{ij\alpha}^U\right),[b_{ij}]=\left(b_{ij\alpha}^L,\ b_{ij\alpha}^U\right),[c_j]=\left(c_{j\alpha}^L,\ c_{j\alpha}^U\right)\\ &and\quad [d_j]=\left(d_{ij\alpha}^L,\ d_{ij\alpha}^U\right)\in P(R),\end{aligned}$$

all a families are of closed intervals on the real line. Clearly P(R) ⊂ F(P), a space of PQFNs. System (1) is called close interval approximation of fuzzy number fractional programming problem.

Any x_j,j=1/0000/p, which satisfies the set of constraints (1) is called a feasible solution of (1). Let Q be the set of all feasible solutions of (1). We say that $x^0 \in$ Q is an optimal feasible solution of (1) if for all $x \in$ Q,

$$\frac{[c]x^0}{[d]x^0} \geq or \leq \frac{[c]x}{[d]x}.$$

For solving the fuzzy fractional program with piecewise quadratic fuzzy numbers, we first apply the close interval approximation and get a reduced fuzzy fractional programming problem. Then, the solution of reduced fractional programming problem is obtained by simplex method without converting the problem into a crisp one. The fuzzy arithmetic and following theorems are used to modify simplex method for the solution of fuzzy fractional programming problems.

Propositions

Proposition 1. *If the systems of fuzzy linear equations* $\sum_{i=1}^{n} a_{ij} x_j = b_j,\ j = 1,2,...,m$ *, has a crisp solution, and fuzzy numbers are replaced by close interval approximation of fuzzy numbers , then following conditions as hold:* $\sum_{i=1}^{n} \left(a_{ij\alpha}^{L} + a_{ij\alpha}^{U}\right) x_i = b_{j\alpha}^{L} + b_{j\alpha}^{U},\ j = 1,2,...,m$.

Proof. Let the x_i be a crisp solution of $\sum_{i=1}^{n} a_{ij} x_j = b_j,\ j = 1,2,...,m$. Replace the fuzzy numbers by close interval approximation of fuzzy numbers to get,

$$\sum_{i=1}^{n} [a_{ij}] x_i = [b_j],\ j = 1,2,...,m$$

$$\Rightarrow \sum_{i=1}^{n} \left[a_{ij\alpha}^{L}, a_{ij\alpha}^{U}\right] x_i = \left[b_{j\alpha}^{L}, b_{j\alpha}^{U}\right]$$

$$\Rightarrow \sum_{i=1}^{n} \left[a_{ij\alpha}^{L} x_i, a_{ij\alpha}^{U} x_i\right] = \left[b_{j\alpha}^{L}, b_{j\alpha}^{U}\right] \qquad (on\ fuzzy\ scalar\ multiplication)$$

$$\Rightarrow \sum_{i=1}^{n} \left(\frac{a_{ij\alpha}^{L} x_i + a_{ij\alpha}^{U} x_i}{2}\right) = \left(\frac{b_{j\alpha}^{L} + b_{j\alpha}^{U}}{2}\right) \qquad (on\ real\ association)$$

$$\Rightarrow \sum_{i=1}^{n} \left(a_{ij\alpha}^{L} + a_{ij\alpha}^{U}\right) x_i = \left(b_{j\alpha}^{L} + b_{j\alpha}^{U}\right),\ j = 1,2,...,m.$$

Proposition 2. *If the system of fuzzy linear equations* $\sum_{i=1}^{n} a_{ij} x_j = b_j,\ j = 1,2,...,m$ *, has a fuzzy solution, and fuzzy numbers are replaced by close interval approximation of fuzzy numbers, then the following conditions hold.* $\sum_{i=1}^{n} \left[\left(a_{ij\alpha}^{L} + a_{ij\alpha}^{U}\right) x_{i\alpha}^{L} + \left(a_{ij\alpha}^{L} + a_{ij\alpha}^{U}\right) x_{i\alpha}^{U}\right] = 2\left(b_{j\alpha}^{L} + b_{j\alpha}^{U}\right)$.

Proof: Replacing the fuzzy numbers by close interval approximation we get,

$$\sum_{i=1}^{n}\left[a_{ij}\right]x_j=\left[b_j\right],\ \ j=1,2,...,m_i Let\ [x_i]\, be\ the\ \ fuzzy\ solution\ then\ we\ have$$

$$\sum_{i=1}^{n}\left[a_{ij}\right]\left[x_i\right]=\left[b_j\right],\ \ j=1,2,...,m_i$$

$$\Rightarrow\ \ \sum_{i=1}^{n}\left[a_{ij\alpha}^{L},a_{ij\alpha}^{U}\right]\left[x_{i\alpha}^{L},x_{i\alpha}^{U}\right]=\left[b_{j\alpha}^{L},b_{j\alpha}^{U}\right]$$

On fuzzy multiplication, we have

$$\sum_{i=1}^{n}\left[\tfrac{1}{2}\left(a_{ij\alpha}^{U}x_{i\alpha}^{L}+a_{ij\alpha}^{L}x_{i\alpha}^{U}\right),\tfrac{1}{2}\left(a_{ij\alpha}^{L}x_{i\alpha}^{L}+a_{ij\alpha}^{U}x_{i\alpha}^{U}\right)\right]=\left[b_{j\alpha}^{L},b_{j\alpha}^{U}\right]$$

$$\Rightarrow\ \ \sum_{i=1}^{n}\left[\frac{\left(a_{ij\alpha}^{U}x_{i\alpha}^{L}+a_{ij\alpha}^{L}x_{i\alpha}^{U}\right)+\left(a_{ij\alpha}^{L}x_{i\alpha}^{L}+a_{ij\alpha}^{U}x_{i\alpha}^{U}\right)}{4}\right]=\left[\frac{b_{j\alpha}^{L}+b_{j\alpha}^{U}}{2}\right]\quad on\ real\ association$$

$$\Rightarrow\ \ \sum_{i=1}^{n}\left[\left(a_{ij\alpha}^{U}+a_{ij\alpha}^{L}\right)x_{i\alpha}^{L}+\left(a_{ij\alpha}^{L}+a_{ij\alpha}^{U}\right)x_{i\alpha}^{U}\right]=2\left[b_{j\alpha}^{L}+b_{j\alpha}^{U}\right]$$

Proposition 3. *The following fuzzy and classic forms of FPP are equivalent:*

$$Max\ Z_1=\frac{\sum_{j=1}^{p}[c_j]x_j}{\sum_{j=1}^{p}[d_j]x_j}\qquad\qquad Max\ Z_2=\frac{\sum_{j=1}^{p}c_j\,x_j}{\sum_{j=1}^{p}d_j\,x_j}$$

$$Subject\ to,\ \ \sum_{j=i}^{p}[a_{ij}]x_j\le[b_i],\qquad Subject\ to,\ \ \sum_{j=i}^{p}a_{ij}\,x_j\le b_i,$$

$$\sum_{j=i}^{p}[a_{ij}]x_j\ge[b_i],\qquad \sum_{j=i}^{p}a_{ij}\,x_j\ge b_i,$$

$$and\qquad x_j\ge 0,\qquad and\qquad x_j\ge 0,$$

Proof: Let Q_1 and Q_2 be the set of all feasible solutions of these FPP s respectively. Then, $x\in Q_1$ implies

$$\left.\begin{array}{l}\sum_{j=i}^{p}[a_{ij}]x_j \le [b_i],\\ \sum_{j=i}^{p}[a_{ij}]x_j \ge [b_i],\\ and \quad x_j \ge 0\end{array}\right\} \Rightarrow \left.\begin{array}{l}\sum_{j=i}^{p}[a_{ij\alpha}^{L}, a_{ij\alpha}^{U}]x_j \le [b_{i\alpha}^{L}, b_{i\alpha}^{U}],\\ \sum_{j=i}^{p}[a_{ij\alpha}^{L}, a_{ij\alpha}^{U}]x_j \ge [b_{i\alpha}^{L}, b_{i\alpha}^{U}],\\ and \quad x_j \ge 0\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} on\ \ scalar\ \ multilpication \\ \sum_{j=i}^{p}[x_j a_{ij\alpha}^{L}, x_j a_{ij\alpha}^{U}] \le [b_{i\alpha}^{L}, b_{i\alpha}^{U}],\\ \sum_{j=i}^{p}[x_j a_{ij\alpha}^{L}, x_j a_{ij\alpha}^{U}] \ge [b_{i\alpha}^{L}, b_{i\alpha}^{U}],\\ and \quad x_j \ge 0\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} on\ \ real\ \ association \\ \sum_{j=i}^{p}[\left(a_{ij\alpha}^{L} + a_{ij\alpha}^{U}\right)/2]x_j \le [\left(b_{i\alpha}^{L} + b_{i\alpha}^{U}\right)/2],\\ \sum_{j=i}^{p}[\left(a_{ij\alpha}^{L} + a_{ij\alpha}^{U}\right)/2]x_j \ge [\left(b_{i\alpha}^{L} + b_{i\alpha}^{U}\right)/2],\\ and \quad x_j \ge 0\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l}\sum_{j=i}^{p} a_{ij}\, x_j \le b_i,\\ \sum_{j=i}^{p} a_{ij}\, x_j \ge b_i,\\ and \quad x_j \ge 0,\end{array}\right\} \Rightarrow \ x \in Q_2, \ \ hence \quad Q_1 = Q_2$$

Now, suppose that $x^0 \in Q_1$ is an optimal feasible solution for fuzzy FPP. Then for all $x \in$ w have

$$\left.\frac{[c]x^0}{[d]x^0} \ge \frac{[c]x}{[d]x}\right\},$$

$$\Rightarrow \frac{\sum_{j=1}^{p}[c_j]x_j^0}{\sum_{j=1}^{p}[d_j]x_j^0} \geq \frac{\sum_{j=1}^{p}[c_j]x_j}{\sum_{j=1}^{p}[d_j]x_j}\Bigg\}$$

$$\Rightarrow \frac{\sum_{j=1}^{p}\left[c_{j\alpha}^L, c_{j\alpha}^U\right]x_j^0}{\sum_{j=1}^{p}\left[d_{j\alpha}^L, d_{j\alpha}^U\right]x_j^0} \geq \frac{\sum_{j=1}^{p}\left[c_{j\alpha}^L, c_{j\alpha}^U\right]x_j}{\sum_{j=1}^{p}\left[d_{j\alpha}^L, d_{j\alpha}^U\right]x_j}\Bigg\}$$

$$\Rightarrow \frac{\sum_{j=1}^{p}\left[\frac{1}{2}\left(c_{j\alpha}^L + c_{j\alpha}^U\right)\right]x_j^0}{\sum_{j=1}^{p}\left[\frac{1}{2}\left(d_{j\alpha}^L + d_{j\alpha}^U\right)\right]x_j^0} \geq \frac{\sum_{j=1}^{p}\left[\frac{1}{2}\left(c_{j\alpha}^L + c_{j\alpha}^U\right)\right]x_j}{\sum_{j=1}^{p}\left[\frac{1}{2}\left(d_{j\alpha}^L + d_{j\alpha}^U\right)\right]x_j}\Bigg\}$$

$$\Rightarrow \frac{\sum_{j=1}^{p}c_j x_j^0}{\sum_{j=1}^{p}d_j x_j^0} \geq \frac{\sum_{j=1}^{p}c_j x_j}{\sum_{j=1}^{p}d_j x_j}$$

Hence, we conclude that $x^0 \in Q_1$ is an optimal feasible solution for the classic FPP. Hence, the given fuzzy and classic form of FPP are equivalent.

5. Numerical example

Let us consider a fuzzy FPP with PQFNs as:

$$\begin{aligned}
&Max\,Z = \frac{(1\quad 2\quad 5\quad 8\quad 9)x_1 + (1\quad 2\quad 3\quad 4\quad 7)x_2}{(1\quad 2\quad 4\quad 8\quad 9)x_1 + (1\quad 1.5\quad 2\quad 2.5\quad 4)x_2 + (.25\quad .5\quad 1\quad 1.5\quad 3)} \\
&Subject\ to,\ (1\quad 2\quad 3\quad 4\quad 6)x_1 + (1\quad 2\quad 5\quad 8\quad 9)x_2 \leq (12\quad 14\quad 15\quad 16\quad 18) \\
&\qquad\qquad (1\quad 2\quad 4\quad 8\quad 9)x_1 + (1\quad 2\quad 3\quad 4\quad 6)x_2 \leq (6\quad 8\quad 10\quad 12\quad 15) \\
&and \qquad x_1, x_2 \geq 0.
\end{aligned}$$

On applying close interval approximation, the reduced FPP is:

$$\begin{aligned}
&Max\,Z = \frac{[2,8]x_1 + [2,4]x_2}{[2,8]x_1 + [1.5,2.5]x_2 + [.5,2.5]} \\
&Subject\ to,\ [2,4]x_1 + [2,8]x_2 \leq [14,16] \\
&\qquad\qquad [2,8]x_1 + [2,4]x_2 \leq [8,12] \\
&and \qquad x_1, x_2 \geq 0.
\end{aligned}$$

Using modified simplex method for fuzzy coefficients without converting the above FPP to a crisp problem, obtain the results in Tables 1-4.

Table 1. Initial simplex table.

Initial Simplex Table with IBFS = {α_3, α_4}			c_j	[2, 8]	[2, 4]	[-1, 1]	[-1, 1]
			d_j	[2, 8]	[1.5, 2.5]	[-1, 1]	[-1, 1]
d_B	c_B	X_B	B	y_1	Y_2	y_3	y_4
[-1, 1]	[-1, 1]	x_3	[14,16]	[2, 4]	[2, 8]	[0, 2]	[-1, 1]
[-1, 1]	[-1,1]	x_4	[8 ,12]	[2, 8]*	[2, 4]	[-1, 1]	[0, 2]
Z_N = [-3, 3]	$\Delta_j^N = \sum c_B x_j - c_j \rightarrow$			[-12, 2]	[-8, 2]	[-3, 3]	[-3, 3]
Z_D =[-2.5, 4.5]	$\Delta_j^D = \sum d_B x_j - d_j \rightarrow$			[-12, 2]	[-6.5, 2.5]	[-3, 3]	[-3, 3]
$Z = Z_N / Z_D$	$\Delta_j = Z_D \Delta_j^N - Z_N \Delta_j^D \rightarrow$			[-50.5, 40.5]	[-34, 28]	[-19.5, 19.5]	[-19.5, 19.5

Table 2. First simplex table

First Simplex Table with BFS = {α_3, α_1}			c_j	[2, 8]	[2, 4]	[-1, 1]	[-1, 1]
			d_j	[2, 8]	[1.5, 2.5]	[-1, 1]	[-1, 1]
d_B	c_B	X	B	Y_1	Y_2	y_3	Y_4
[-1, 1]	[-1,1]	X	$\left[\frac{76}{10}, \frac{104}{10}\right]$	$\left[-\frac{16}{10}, \frac{16}{10}\right.$	$\left[0, \frac{64}{10}\right]^*$	$\left[\frac{-2}{10}, \frac{22}{10}\right]$	$\left[\frac{-18}{10}, \frac{6}{10}\right.$
[2, 8]	[2, 8]	X	$\left[\frac{16}{10}, \frac{24}{10}\right]$	$\left[\frac{4}{10}, \frac{16}{10}\right]$	$\left[\frac{4}{10}, \frac{8}{10}\right]$	$\left[\frac{-2}{10}, \frac{2}{10}\right]$	$\left[0, \frac{4}{10}\right]$
$Z_N = \left[\frac{74}{10}, \frac{126}{10}\right.$	$\Delta_j^N = \sum c_B x_j - c_j \rightarrow$			$\left[\frac{-64}{10}, \frac{64}{10}\right.$	$\left[\frac{-48}{10}, \frac{48}{10}\right]$	$\left[\frac{-28}{10}, \frac{28}{10}\right.$	$\left[\frac{-18}{10}, \frac{38}{10}\right.$
Z_D = $\left[\frac{79}{10}, \frac{141}{10}\right]$	$\Delta_j^D = \sum d_B x_j - d_j \rightarrow$			$\left[\frac{-64}{10}, \frac{64}{10}\right.$	$\left[\frac{-33}{10}, \frac{53}{10}\right]$	$\left[\frac{-28}{10}, \frac{28}{10}\right.$	$\left[\frac{-18}{10}, \frac{38}{10}\right.$
$Z = Z_N / Z_D$	$\Delta_j = Z_D \Delta_j^N - Z_N \Delta_j^D$			$\left[\frac{-496}{25}, \frac{49}{2}\right.$	$\left[\frac{-1803}{50}, \frac{80}{5}\right.$	$\left[\frac{-399}{25}, \frac{39}{2}\right.$	$\left[\frac{-748}{50}, \frac{84}{5}\right.$

Table 3. Second simplex table

Second Simplex Table with BFS = {α_2, α_1}			c_j	[2, 8]	[2, 4]	[-1, 1]	[-1, 1]
			d_j	[2, 8]	[1.5, 2.5]	[-1, 1]	[-1, 1]
d_B	c_B	X_B	b	Y_1	Y_2	y_3	Y_4
[-1.5, 2.5]	[2, 4]	x_2	$\left[\frac{19}{8}, \frac{26}{8}\right]$	$\left[\frac{-1}{2}, \frac{1}{2}\right]$	$\left[0, 2\right]$	$\left[\frac{-1}{16}, \frac{11}{16}\right]$	$\left[\frac{-9}{16}, \frac{3}{16}\right]$
[2, 8]	[2, 8]	x_1	$\left[\frac{-7}{40}, \frac{4}{5}\right]$	$\left[\frac{3}{10}, \frac{17}{10}\right]$	$\left[\frac{-2}{5}, \frac{2}{5}\right]$	$\left[\frac{-37}{80}, \frac{7}{80}\right]$	$\left[\frac{3}{80}, \frac{47}{80}\right]^*$
$Z_N = \left[\frac{81}{10}, \frac{597}{40}\right]$	$\Delta_j^N = \sum c_B x_j - c_j \rightarrow$			$\left[\frac{-56}{10}, \frac{56}{10}\right]$	$\left[\frac{-16}{5}, \frac{16}{5}\right]$	$\left[\frac{-11}{5}, \frac{11}{5}\right]$	$\left[\frac{-4}{5}, \frac{16}{5}\right]$
$Z_D = \left[\frac{701}{160}, \frac{1659}{160}\right]$	$\Delta_j^D = \sum d_B x_j - d_j \rightarrow$			$\left[\frac{-107}{20}, \frac{97}{20}\right]$	$\left[\frac{-11}{5}, \frac{11}{5}\right]$	$\left[\frac{-141}{80}, \frac{241}{80}\right]$	$\left[\frac{-33}{40}, \frac{128}{40}\right]$
$Z = Z_N / Z_D$	$\Delta_j = Z_D \Delta_j^N - Z_N \Delta_j^D \rightarrow$			$\left[\frac{102793}{800}, \frac{14884}{800}\right.$	$\left[\frac{6299}{400}, \frac{683}{40}\right.$	$\left[\frac{-28069}{1280}, \frac{964}{128}\right.$	$\left[\frac{-56564}{3200}, \frac{-2570}{3200}\right.$

Table 4. Third simplex table

Third Simplex Table with BFS = $\{\alpha_2, \alpha_4\}$			c_j	[2, 8]	[2, 4]	[-1, 1]	[-1, 1]
			d_j	[2, 8]	[1.5, 2.5]	[-1, 1]	[-1, 1]
d_B	c_B	B	B	Y_1	Y_2	Y_3	y_4
[-1.5, 2.5]	[2,4]	2	$\left[\frac{791}{400}, \frac{1609}{400}\right]$	$\left[\frac{-37}{50}, \frac{97}{50}\right]$	$\left[\frac{-12}{25}, \frac{62}{25}\right]$	$\left[\frac{-101}{200}, \frac{181}{200}\right]$	$\left[\frac{-141}{200}\right.$, [illegible]
[-1,1]	[-1,1]	4	$\left[\frac{-14}{25}, \frac{64}{25}\right]$	$\left[\frac{24}{25}, \frac{136}{25}\right]$	$\left[\frac{-32}{25}, \frac{32}{25}\right]$	$\left[\frac{-37}{25}, \frac{7}{25}\right]$	$\left[\frac{3}{25}, \frac{47}{25}\right]$
$Z_N = \left[\frac{2567}{400}, \frac{4633}{400}\right]$	$\Delta_j^N = \sum c_B x_j - c_j \rightarrow$			$\left[\frac{-537}{50}, \frac{243}{50}\right]$	$\left[\frac{-94}{25}, \frac{94}{25}\right]$	$\left[\frac{-397}{200}, \frac{637}{200}\right]$	$\left[\frac{-517}{200}\right.$, [illegible]
$Z_D = \left[\frac{3543}{800}, \frac{7657}{800}\right]$	$\Delta_j^D = \sum d_B x_j - d_j \rightarrow$			$\left[\frac{-1067}{100}, \frac{211}{100}\right]$	$\left[\frac{-63}{25}, \frac{63}{25}\right]$	$\left[\frac{-733}{400}, \frac{745}{400}\right]$	$\left[\frac{-893}{400}\right.$, [illegible]
$Z = Z_N / Z_D$	$\Delta_j = Z_D\Delta_j^N - Z_N\Delta_j^D \rightarrow$			$\left[\frac{-744717}{40000}, \frac{2179917}{40000}\right]$	$\left[\frac{-80879}{5000}\right.$, [illegible]	$\left[\frac{-147057}{2000}\right.$, [illegible]	$\left[\frac{611}{16}\right.$, [illegible]

From Table 4, we see that $\Delta_j \geq 0$ for all value of j. Hence, the optimality conditions are satisfied and the optimal solution is

$$x_1 = 0 \quad x_2 = \left[\tfrac{791}{400}, \tfrac{1609}{400}\right] = [(791+1609)/800] = [2400/800] = 3$$

$$x_3 = 0$$

$$x_4 = \left[\tfrac{-14}{25}, \tfrac{64}{25}\right] = [(-14+64)/50] = [50/50] = 1$$

and $Z = Z_N / Z_D = \left[\frac{2567}{400}, \frac{4633}{400}\right] / \left[\frac{3543}{800}, \frac{7657}{800}\right]$

$$= \left[\tfrac{2(2567x2)}{3543+7657}, \tfrac{2(4633x2)}{3543+7657}\right] = \left[\tfrac{10268}{11200}, \tfrac{18532}{11200}\right] = \left[\tfrac{10268+18532}{2(11200)}\right] = \tfrac{28800}{22400} = \frac{9}{7}$$

6. Conclusion

We developed a modified simplex method for solving fuzzy fractional programs without converting the fuzzy coefficients to crisp values. Although the calculations involved in the method can be lengthy, our method can be more efficient than the existing ones. If Z_D turns out to be equal to one, then our model reduces to a linear programming by our proposed simplex method without needing to convert it to crisp values.

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