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A semidefinite relaxation scheme for quadratically constrained quadratic problems with an additional linear constraint

M. Salahi

Semidefinite optimization relaxations are among the widely used approaches to find global optimal or approximate solutions for many nonconvex problems. Here, we consider a specific quadratically constrained quadratic problem with an additional linear constraint. We prove that under certain conditions the semidefinite relaxation approach enables us to find a global optimal solution of the underlying problem in polynomial time.

Keywords: *Quadratically constrained quadratic problems, Semidefinite optimization, Relaxation, Interior-point methods.*

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1. Introduction

Quadratic optimization has received much attention in the literature and it is a fundamental problem in optimization theory and practice; see Nocedal and Wright [5], Polik and Terlaky [6], Shor [8] and Yakubovich [11]. This problem appears in many disciplines such as economic equilibrium, combinatorial optimization, numerical partial differential equations, and general nonlinear programming. Recently, there were several results on quadratic optimization. Among these, using semidefinite optimization (SDO) relaxations, it has been shown that global optimal solution can be found in polynomial time using interior point algorithms, see de Klerk [3], Salahi [7], Sturm and Zhang [10], Ye and Zhang [12], and references therein.

Here, we consider minimizing a quadratic function subject to a quadratic equality constraint with an additional linear inequality constraint as follows:

$$\begin{aligned} \min \quad & x^{\mathrm{T}}Qx - 2g^{\mathrm{T}}x + f \\ \text{s.t.} \quad & ||x||^2 = \beta, \\ & a^{\mathrm{T}}x \geq (\text{or} =) c, \end{aligned} \tag{1}$$

where $Q \in S^{n\times n}$ (space of symmetric $n \times n$ matrices), $g \in R^n, f \in R, a \in R^n, c \in R$. Moreover, we assume that the feasible region of (1) is nonempty and for the case of linear inequality constraint, it has a strictly feasible point called x_0.

Obviously, (1) is not a convex problem and classical quadratic optimization algorithms do not necessarily find a global optimal solution; see Nocedal and Wright [5]. Semidefinite optimization (SDO) relaxations are one of the widely used approaches to find approximate or global optimal solutions of such problems; see Fortin and Wolkowicz [4], Polik and Terlaky [6], Salahi [7], Sturm and Zhang [10], and Ye and Zhang [12]. Here we show that the SDO relaxations enables, us to find a global optimal solution of (1), under certain conditions, in polynomial time.

2. SDO Relaxation Approach

Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran. Email:salahim@guilan.ac.ir

SDO is an extension of linear optimization and in its standard primal form is given by:

$$\begin{aligned} &\min C \bullet X \\ &\text{s.t.}\ \ A_i \bullet X = b_i,\ i = 1,\dots,m, \\ &X \succcurlyeq 0_{n\times n}, \end{aligned}$$

where $A \in R^{n\times n}$, $b \in R^m$, $A \bullet B = \text{Trace}(AB^{\mathrm{T}})$ and $X \succeq 0_{n\times n}$ means that X is positive semidefinite; see Alizadeh [1] and Ben-Tal and Nemorovski [2]. The dual of the primal SDO is given by

$$\begin{aligned} &\max b^{\mathrm{T}} y \\ &\sum_{i=1}^{m} y_i A_i \succcurlyeq C, \end{aligned}$$

where $A \succeq B$ means $A - B$ is positive semidefinite. The homogenized version of (1) is:

$$\begin{aligned} &\min x^{\mathrm{T}} Q x - 2 t g^{\mathrm{T}} x + f t^2 \\ &x^{\mathrm{T}} x = \beta t^2, \\ &a^{\mathrm{T}} t x \geq (\text{or } =) c t^2, \\ &t^2 = 1. \end{aligned} \tag{2}$$

It is easy to check that if $(t, x^{\mathrm{T}})^{\mathrm{T}}$ is a solution of (2), then $\frac{x}{t}$ is a solution of (1). Now, we may write (2) as follows:

$$\begin{aligned} &\min M_0 \bullet \hat{X} \\ &M_1 \bullet \hat{X} = 0, \\ &M_2 \bullet \hat{X} \geq (\text{or } =) 0, \\ &M_3 \bullet \hat{X} = 1, \end{aligned} \tag{3}$$

where $\hat{X} = \begin{bmatrix} t^2 & t x^{\mathrm{T}} \\ t x & x x^{\mathrm{T}} \end{bmatrix}$ and

$$M_0 = \begin{bmatrix} f & -g^{\mathrm{T}} \\ -g & Q \end{bmatrix},\ M_1 = \begin{bmatrix} -\beta & 0_{1\times n} \\ 0_{n\times 1} & I_n \end{bmatrix},\ M_2 = \begin{bmatrix} -\mathrm{c} & \frac{1}{2} a^{\mathrm{T}} \\ \frac{1}{2} a & 0_{n\times n} \end{bmatrix},\ M_3 = \begin{bmatrix} 1 & 0_{1\times n} \\ 0_{n\times 1} & 0_{n\times n} \end{bmatrix}.$$

One can see that $\hat{X}$ is a positive semidefinite matrix. Then, let us relax it to a general positive semidefinite matrix. The relaxed problem becomes:

$$\begin{aligned} &\min\ \ M_0 \bullet X \\ &M_1 \bullet X = 0, \\ &M_2 \bullet X \geq (\text{or } =) 0, \\ &M_3 \bullet X = 1, \\ &X \succeq 0_{(n+1)\times(n+1)}, \end{aligned} \tag{4}$$

where $X = \begin{bmatrix} X_{00} & x_0^{\mathrm{T}} \\ x_0 & \bar{X} \end{bmatrix}$. Following the duality notion introduced in the introduction, the dual of (4) is given by

$$\begin{aligned} &\max\ \ y_3 \\ &Z = M_0 - y_1 M_1 - y_2 M_2 - y_3 M_3, \\ &Z \succeq 0_{(n+1)\times(n+1)},\ y_2 \geq 0 \end{aligned} \tag{5}$$

(or free for the quality case).

In the sequel, first we discuss conditions under which both (4) and (5) are solvable, and then give the optimal solution for the original problem (1).

Theorem 2.1. Suppose x_0 be a feasible solution for (1) (strictly feasible for the case of linear inequality) and $v \in \text{null}\,(a^{\mathrm{T}})$ with $x_0^{\mathrm{T}} v \neq 0$. Then problems (4) and (5) satisfy the Slater regularity conditions. Therefore, they are both solvable and the duality gap is zero.

Proof. Let $X = \begin{bmatrix} 1 & u^{\mathrm{T}} \\ u & \varepsilon I_{\mathrm{n}} + uu^{\mathrm{T}} \end{bmatrix}$, where $u = x_0 + \alpha v$ and ε is a positive constant such that $n\varepsilon||v||^2 < ({x_0}^{\mathrm{T}} v)^2$. Obviously, from the Schur complement theorem, $X \succeq 0_{(n+1)\times(n+1)}$ and $M_2 \bullet X \geq (\text{or} =) 0$, $M_3 \bullet X = 1$. Moreover, to have $M_1 \bullet X = 0$ is equivalent to having

$$n\varepsilon + \alpha^2 ||v||^2 + 2\alpha x_0^T v = 0.$$

This definitely holds for appropriately chosen α. For the dual problem (5), by choosing $y_1 < \lambda_{\min}(Q)$ and y_2 a small positive number ($y_2 = 0$ for the case of equality), and y_3 a sufficiently small negative number, Z will be positive definite, which implies the Slater regularity of (5).

The following lemma is crucial for constructing a solution of the original problem from the solution of (4); see Strum and Zhang [10].

Lemma 2.1. Let X be a symmetric positive semi definite matrix of rank r and G be an arbitrary symmetric matrix with $G \bullet X \geq 0$. Then, there exists a rank one decomposition of matrix X such that

$$X = \sum_{i=1}^{r} x_i x_i^T$$

and $x_i^{\mathrm{T}} G x_i \geq 0$, $\forall\, i = 1, \dots, r$. If in particular $G \bullet X = 0$, then $x_i^{\mathrm{T}} G x_i = 0$, $\forall\; i = 1, \dots, r$.

Theorem 2.2. Suppose that for some t, $\beta + tc < 0$ and $1 + \frac{t^2||a||^2}{4(\beta + tc)} > 0$. Then, the SDO relaxation (4) gives a global optimal solution of (1) in polynomial time.

Proof. Suppose that X^* (of rank r) and $(y_1^*, y_2^*, y_3^*, Z^*)$ are optimal solutions for (4) and (5), respectively. By Lemma 2.1, we have

$$X^* = \sum_{i=1}^{r} x_i^* (x_i^*)^T, \tag{6}$$

for which $(x_j^*)^{\mathrm{T}} M_1 x_j^* = 0$, $\forall\, j = 1, \dots, r$. Now since for some t, $\beta + tc < 0$ and $1 + \frac{t^2||a||^2}{4(\beta + tc)} > 0$, then by the Schur Complement theorem, $M_1 + tM_2 \succeq 0_{(n+1)\times(n+1)}$. If for the linear inequality case we have $M_2 \bullet X^* = 0$, then from $(M_1 + tM_2) \bullet X^* = 0$, we further have $(\mathrm{x}_{\mathrm{i}}^*)^{\mathrm{T}} (M_1 + tM_2) x_i^* = 0, \forall\, i = 1, \dots, r$. Moreover, since $(x_i^*)^{\mathrm{T}} M_1 x_i^* = 0, \forall\, i = 1, \dots, r$, then $(x_i^*)^{\mathrm{T}} M_2 x_i^* = 0$, $\forall\, i = 1, \dots, r$.

Thus, for at least a $k, 1 \leq k \leq r$, we have

$$x_k^* = \begin{bmatrix} t_k^* \\ \bar{x}_k^* \end{bmatrix},$$

where $t_k^* \neq 0$; otherwise from $(x_k^*)^{\mathrm{T}} M_1 x_k^* = 0$, we have $\bar{x}_k^* = 0_{n\times 1}$, since $(x_k^*)^{\mathrm{T}} M_1 x_k^* = ||\bar{x}_k^*||^2$. Thus $\mathrm{x}_{\mathrm{k}}^* = 0$, which is a contradiction. Furthermore, by the com plementarity condition, $X^* \bullet Z^* = 0$. Thus we have $(x_k^*)^{\mathrm{T}} Z^* x_k^* = 0$. This further implies that

$$Z^* \bullet \left(\begin{bmatrix} 1 \\ \frac{\bar{x}_k^*}{t_k^*} \end{bmatrix} [1 \ \frac{(\bar{x}_k^*)^T}{t_k^*}] \right) = 0. \tag{7}$$

One can easily check that

$$M_1 \bullet \left(\begin{bmatrix} 1 \\ \frac{\bar{x}_k^*}{t_k^*} \end{bmatrix} [1 \ \frac{(\bar{x}_k^*)^T}{t_k^*}] \right) = 0, M_2 \bullet \left(\begin{bmatrix} 1 \\ \frac{\bar{x}_k^*}{t_k^*} \end{bmatrix} [1 \ \frac{(\bar{x}_k^*)^T}{t_k^*}] \right) = 0, M_3 \bullet \left(\begin{bmatrix} 1 \\ \frac{\bar{x}_k^*}{t_k^*} \end{bmatrix} [1 \ \frac{(\bar{x}_k^*)^T}{t_k^*}] \right) = 1.$$

Therefore, $\begin{bmatrix} 1 \\ \frac{\bar{x}_k^*}{t_k^*} \end{bmatrix} [1 \ \frac{(\bar{x}_k^*)^{\mathrm{T}}}{t_k^*}]$ is an optimal solution for (4) and since (4) is a relaxation of (3), then $\frac{\bar{x}_k^*}{t_k^*}$ is optimal for (1).

However, if for the problem having linear inequality constraint we have $M_2 \bullet X^* > 0$, then $y_2^* = 0$. Now, since $M_1 + tM_2 \succeq 0_{(n+1)\times(n+1)}$ for some t, satisfies conditions discussed before, then from

$$(x_i^*)^T (M_1 + tM_2) x_i^* \geq 0, \qquad \forall i = 1, \dots, r,$$

we have $t(x_i^*)^{\mathrm{T}} M_2 x_i^* = 0, \forall\, i = 1, \dots, r$. Therefore, $tM_2 \bullet X^* \geq 0$. Since $M_2 \bullet X^* > 0$, this implies $t \geq 0$. Therefore, if $t < 0$, then we cannot have $M_2 \bullet X^* > 0$. Now, suppose that $M_2 \bullet X^* > 0$ and $t > 0$ satisfying conditions discussed before. To have $\beta + tc < 0$, we need to have $c < 0$. Therefore, for $c < 0$ and conditions on the statement of the theorem for some t, we have

$$M_1 + tM_2 \succeq 0_{(n+1)\times(n+1)}.$$

Thus, $(x_i^*)^{\mathrm{T}} M_2 x_i^* \geq 0, \forall\, i = 1, \dots, r$. Now, for at least one k we have

$$x_k^* = \begin{bmatrix} t_k^* \\ \bar{x}_k^* \end{bmatrix},$$

where $t_k^* \neq 0$ as before. The rest of the proof is given as before. Finally, since an SDO problem can be solved in polynomial time using the interior point algorithms, then we can find a global optimal solution of (1) polynomialy under conditions stated in the theorem; see Alizadeh [1] and Sturm [10].

Now, let us present a small example for problem (1) with linear inequality constraint with the following randomly generated data:

$$Q=\begin{pmatrix} 1.9003 & 0.9932 & 1.2223 & 0.8917 & 0.9492 \\ 0.9932 & 0.9129 & 0.8104 & 1.7569 & 0.7976 \\ 1.2223 & 1.8104 & 1.8436 & 1.6551 & 0.9894 \\ 0.8917 & 1.7569 & 1.6551 & 0.8205 & 0.9035 \\ 0.9492 & 0.7976 & 0.9894 & 0.9035 & 0.2778 \end{pmatrix}, \ g=-\begin{pmatrix} 0.2028 \\ 0.1987 \\ 0.6038 \\ 0.2722 \\ 0.1988 \end{pmatrix}, \beta=1,$$

$$a=[0.0153,0.7468,0.4451,0.9318,0.4660]^T, \ c=0.91, \ f=0.$$

The solution obtained by SeDuMi from solving (4) is

$$X^*=\begin{pmatrix} 1.0000 & -0.2760 & 0.2175 & -0.3376 & 0.8252 & 0.2856 \\ -0.2760 & 0.0762 & -0.0600 & 0.0932 & -0.2277 & -0.0788 \\ 0.2175 & -0.0600 & 0.0473 & -0.0734 & 0.1795 & 0.0621 \\ -0.3376 & 0.0932 & -0.0734 & 0.1140 & -0.2786 & -0.0964 \\ 0.8252 & -0.2277 & 0.1795 & -0.2786 & 0.6810 & 0.2357 \\ 0.2856 & -0.0788 & 0.0621 & -0.0964 & 0.2357 & 0.0816 \end{pmatrix}.$$

Furthermore, by applying Lemma 2.1 we have the following optimal solution of original problem:

$$x^*=[-0.2760 \quad 0.2175 \quad -0.3376 \quad 0.8252 \quad 0.2856]^T.$$

3. Conclusions

We proved that a global optimal solution of an indefinite quadratic minimization problem with one quadratic equality constraint and one linear equality or inequality constraint could be found in polynomial time by an SDO relaxation under certain conditions.

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