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Robust Quadratic Assignment Problem with Uncertain Locations

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We consider a generalization of the classical quadratic assignment problem, where coordinates of locations are uncertain and only upper and lower bounds are known for each coordinate. We develop a mixed integer linear programming model as a robust counterpart of the proposed uncertain model. A key challenge is that, since the uncertain model involves nonlinear objective function of the uncertain data, classical robust optimization approaches cannot be applied directly to construct its robust counterpart. We exploit the problem structure to develop exact solution methods and present some computational results.

Keywords: *Uncertainty modeling, Robustness and sensitivity analysis, Facilities planning and design, Quadratic assignment problem, Non-linear integer programming.*

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1. Introduction

Koopmans and Beckmann [17] introduced quadratic assignment problem (QAP) in 1957 as a mathematical model for the location of a set of indivisible economical activities. Its standard version deals with choosing an optimal way to assign *n* facilities to *n* locations to minimize the total material handling cost, given all distances between locations and the amount of material flow between each pair of facilities. Numerous applications of the QAP are discussed in Burkard et al. [5]. Some applications are: minimizing the total amount of connections between the components in a backboard wiring (Steinberg [21]), assigning a new facility to serve a given set of clients (Francis and White [12]), scheduling problems (Geoffrion and Graves [13]), economic problems (Heffley [14, 15]), designing typewriter keyboards (Pollatschek et al. [18]) and many other applications. However, the most well-known and popular application of QAP is the facility layout problem. For example, Dickey and Hopkins [8] used QAP toassign buildings in a university campus, Elshafei [10] applied it in a hospital planning and Bos [4] used it in a forest parks problem.

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The QAP is one of the hardest problems in combinatorial optimization (see Burkard et al. [5] and Cela [6]). The famous NP-hard Traveling Salesman Problem (TSP) is a special case of QAP. Sahni and Gonzalez [19] showed that it is not possible to obtain in polynomial time even a constant-factor approximate solution for the QAP, unless P=NP.

In deterministic optimization, it is assumed that input data (flows between facilities and distances between locations in QAP) are precisely known in advance. Although this assumption can be true in some applications, it is not realistic in many others. For example, we may know estimation of flows between facilities which will be possibly different from their realizations. As another example, in the problem of minimizing the amount of connections between components in a backboard wiring (Steinberg [21]), because of implementation errors in assigning electrical elements to exact locations, the coordinates of the locations can be uncertain. Another example of uncertain locations in QAP is the problem of assigning facilities to locations when the locations are not built yet and all we know is the (large) regions in which they are located. Here, we consider a generalization of the classical quadratic assignment problem, where coordinates of locations are uncertain and only upper and lower bounds are known for each coordinate. For coordinates of locations only an interval estimate (uncertainty interval) is available, and they can take on any value from the corresponding uncertainty interval, regardless of the values of other coordinates of locations.

Solving optimization problems with some nominal data may cause infeasible or suboptimal solutions, or both, in the presence of data uncertainty (Ben-Tal and Nemirovski [1]). Therefore, mathematical modelers are concerned with uncertainty in constraints or objective function coefficients. Note that in our problem, coordinates of locations and consequently distances between locations are uncertain, which just affects the objective function coefficients and optimality of solutions. Sensitivity analysis, stochastic programming (Dantzig [7]), fuzzy programming (Zadeh [25]) and recently robust optimization (Soyster [20], Ben-Tal and Nemirovski [1], Bertsimas and Sim [2, 3], and El-Ghaoui et al. [9]) are some different approaches to handle the data uncertainty in optimization models.

Here, we propose interval and budgeted data uncertainty for the coordinates of locations in QAP, and therefore nonlinearly correlated distances between locations. It is worth mentioning that, since the objective function of the original uncertain problem is not an affine function of uncertain data, classical robust optimization approaches cannot be applied directly to construct its robust counterpart. Here, we develop mixed integer programming models to address this type of uncertainty.

The remainder of our work is organized as follows. In Section 2, we present notations and the problem statement. In Section 3, we develop a method to find the worst-case scenario and robust value of the objective function for a given feasible solution. In Section 4, we present a robust counterpart of uncertain QAP and linearize it. This is followed by computational experiments in Section 5, to consider the quality of solutions. Finally, we present some concluding remarks in Section 6.

2. Notation and Problem Statement

In this section, we first present notations and the problem statement of classical QAP. Then, we formulate the problem with uncertainty.

2.1. Classical QAP

In the standard version of QAP, it is assumed that there are n facilities to be assigned to n locations, in order to minimize the total material handling volume. Let $N = \{1, \ldots, n\}$. For any $i, j, r, s \in N$, $f_{ij} \geq 0$, the amount of flow from facility i to j and $d_{rs} \geq 0$, the distances between the locations r and s are given. For each facility i and location r, a decision variable L_{ir} is as follows:

$$L_{ir} = \begin{cases} 1, & \text{if facility } i \text{ is assigned to location } r, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

For any feasible solution $L \in \{0,1\}^{n \times n}$, each location must be assigned exactly to one facility, and similarly each facility must be located exactly in one location. Therefore, assignment constraints should be satisfied as follows:

$$\sum_{r=1}^{n} L_{ir} = 1, \quad \forall i \in N, \text{ and } \sum_{i=1}^{n} L_{ir} = 1, \quad \forall r \in N. \tag{2}$$

Let P be the set of all feasible assignments. Thus,

$$P = \left\{ L \in \{0,1\}^{n \times n} : \sum_{r=1}^{n} L_{ir} = 1, \quad \forall i \in N, \sum_{i=1}^{n} L_{ir} = 1, \quad \forall r \in N \right\}. \tag{3}$$

Given an $n \times n$ distance matrix $d = (d_{rs})$ and an assignment $L = (L_{ir}) \in P$, let $\langle d, L \rangle$ denote the corresponding cost of the assignment,

$$\langle d, L \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{r=1}^{n} \sum_{s=1}^{n} f_{ij} d_{rs} L_{ir} L_{js} = \sum_{r=1}^{n} \sum_{s=1}^{n} d_{rs} f_{rs}^{L},$$

where, for each pair r and s of locations, we have

$$f_{rs}^{L} = \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij} d_{rs} L_{ir} L_{js} \tag{4}$$

as the flow between locations r and s in the assignment L. For a given distance matrix d, the *classical* QAP is stated as follows.

Problem QAP(d). Minimize $\{\langle d, L\rangle \mid L \in P\}$.

Let $Opt(d)$ be an optimal solution to Problem QAP (d), and $F^*(d)$ be its optimal objective value.

2.2. Uncertain QAP

Problem QAP (d) is valid as long as the values of flows and distances are known precisely. Otherwise, this model cannot be applied to obtain a precise solution. In fact, if only the estimation of data with most likely intervals is available, then the result is not reliable.

In this section, we consider an uncertain QAP model with interval coordinates of location and develop a method to find the worst-case scenario and robust value of the objective function for a given feasible solution. Without loss of generality, we assume that the flows are deterministic. Suppose that $\tilde{x}_r$ and $\tilde{y}_r$ (coordinates of location r in the 2 dimensional plane) are independent uncertain parameters, where $\tilde{x}_r \in [x_r, x_r + \hat{x}_r]$ and $\tilde{y}_r \in [y_r, y_r + \hat{y}_r]$. To normalize the uncertainty set, $\tilde{x}_r$ and $\tilde{y}_r$ can be written as $x_r + \xi_r^x \hat{x}_r$ and $y_r + \xi_r^y \hat{y}_r$, respectively, where ξ_r^x and ξ_r^y can take on any value from $[0,1]$ independent of each other. The ξ^x and ξ^y are called perturbation vectors.

2.2.1. Uncertainty Sets

To address data uncertainty, in the robust optimization approach, it is assumed that uncertain parameters of the model (e.g., distances between locations) are (affine) functions of some perturbation factors (e.g., coordinates of locations). Contrary to stochastic programming, in robust optimization, no probability distribution is assigned to uncertain data. Depending on the set that perturbation factors belong to, various robust optimization approaches have been developed. For example, Soyster [20] used box uncertainty set, which is the easiest but also the most pessimistic approach. In contrast, Ben-Tal and Nemirovski [1] and El-Gaoui et al. [9] used ellipsoidal uncertainty sets to reduce the conservativeness of Soyster's model. Using ellipsoidal uncertainty sets results in conic quadratic programming models, when the original model is a linear programming model. Thus, this approach cannot be directly applied to mixed integer programming (MIP) or mixed integer non-linear programming (MINLP) models.

Later, Bertsimas and Sim [2, 3] introduced a new robust optimization approach that uses budgeted uncertainty set, which has the same complexity of the original model and adjustable conservativeness.

Since QAP is an MINLP model (and its linear equivalents are MIP models), we consider box and budgeted uncertainty sets and develop robust counterparts of QAP considering these types of uncertainty sets.

Box Uncertainty

This type of uncertainty was considered by Soyster [20] for the first time to develop robust counterparts in linear programming. It is also referred as interval data. In this type of uncertainty, it is assumed that ξ_r^x and ξ_r^y are independent uncertain variables in $[0,1]$. Let U_b^{ξ} be the box uncertainty set for ξ. Then,

$$U_b^{\xi} = \{(\xi^x, \xi^y) : \xi^x, \xi^y \in \Re^n, 0 \le \xi_r^x, \xi_r^y \le 1, \forall r \in N\}. \tag{5}$$

Box uncertainty is the most conservative model in the robust optimization literature and uses the worst case optimization approach.

Here, we use rectilinear distances between locations. Therefore, for any perturbation vector $\xi = (\xi^x, \xi^y)$, a mapping $d(\xi) : \Re^{2n} \to \Re_+^{n\times n}$ is defined as follows. For all $r, s = 1\ldots, n$,

$$d_{rs}(\xi) = \left| x_r - x_s + \xi_r^x \hat{x}_r - \xi_s^x \hat{x}_s \right| + \left| y_r - y_s + \xi_r^y \hat{y}_r - \xi_s^y \hat{y}_s \right|. \tag{6}$$

Thus, in the case of box uncertainty for ξ, the uncertainty set for distance matrix d is

$$U_b^d = \left\{ d \in \Re^{n\times n} : \exists \xi \in U_b^{\xi} \text{ such that } d = d(\xi) \right\}, \tag{7}$$

which is not a box uncertainty set.

Budgeted Uncertainty

Bertsimas and Sim [2, 3] introduced this type of uncertainty to construct robust counterparts for the LP and MILP models. Budgeted uncertainty is less conservative than box uncertainty and its conservativeness is adjustable. In this type of uncertainty, in addition to the box constraints (5), there is a budget Γ for the absolute sum of the perturbation elements ξ_r^x and ξ_r^y. Let U_Γ^{ξ} be the budgeted uncertainty set for ξ. Then,

$$U_\Gamma^{\xi} = \{\xi \in U_b^{\xi} : \sum_{r=1}^{n} (\xi_r^x + \xi_r^y) \le \Gamma\}, \tag{8}$$

where $\Gamma \in [0, 2n]$ is a given protection level and the conservativeness of the model can be adjusted by changing it. While $\Gamma = 0$ gives the most optimistic solution (deterministic problem with nominal data), $\Gamma = 2n$ generates the most pessimistic one just like box uncertainty set. In fact, a budgeted uncertainty set with protection level $\Gamma = 2n$ equals the box uncertainty set. Since box uncertainty set is a special case of budgeted uncertainty with $\Gamma = 2n$, in the remainder of our work it is assumed that U is a budgeted uncertainty set. In the case of budgeted uncertainty for ξ, the uncertainty set for distance matrix d is

$$U_\Gamma^d = \left\{ d \in \Re^{n\times n} : \exists \xi \in U_\Gamma^\xi \text{ such that } d = d(\xi) \right\}. \quad (9)$$

For any assignment $L \in P$, the value

$$Z(L) = \max_{d\in U_\Gamma^d} \langle d, L\rangle = \max_{\xi\in U_\Gamma^\xi} \langle d(\xi), L\rangle \quad (10)$$

is called the *worst-case or robust cost* for L. A maximizer of (10) is called a *worst-case scenario* for L, which is denoted by d_L^*. The Corresponding *worst-case perturbation* vector is presented by ξ_L^*, where $d_L^* = d(\xi_L^*)$. The robust QAP is formulated as follows.

Problem RQAP. Minimize $\left\{ Z(L) : L \in P \right\}$.

Let Z^* and L^* be the optimal objective value and the optimal solution for Problem RQAP, respectively.

Let $\overline{d} = d(0)$ be the nominal value of distance matrix. The difference between $Z(L)$ and $\langle \overline{d}, L\rangle$ (i.e., the deterministic cost) is called *robustness cost* of assignment L and represented by $RC(L)$. Thus,

$$RC(L) = Z(L) - \langle \overline{d}, L\rangle. \quad (11)$$

Note that the uncertainty set is an approximation of the true uncertainty in the data. Thus, for a given feasible solution L, it is possible that the realized total cost be greater than its robust value with respect to the approximated uncertainty set. The probability of this event is called *violation probability* of assignment L and is given as follows:

$$\text{Violation Probability of Assignment } L = \Pr\left\{ \langle d, L\rangle > Z(L) \right\}. \quad (12)$$

For example, suppose that the real uncertainty set is a box uncertainty set and we have modeled it by a budgeted uncertainty set. It is clear that if up to $\lfloor \Gamma \rfloor$ of the ξ_j change within their bounds, and up to one ξ_j changes by $(\Gamma - \lfloor \Gamma \rfloor)$, then the realized reliability of solution x will be greater than or equal to its robust reliability. Otherwise, realized reliability can be less than robust reliability of solution L. Bertsimas and Sim [3] discussed by details on violation probability and presented some analytical estimates of it. In Section 5, we calculate this probability empirically by the Monte Carlo simulation.

3. A Linear Programming (LP) Formulation to Find Worst-case Scenario

An important step in developing the robust counterpart models is finding the worst-case realization from uncertainty set for a given solution $L \in P$. In our problem, worst-case scenario can be obtained by solving (11), which is maximizing $\langle d(\xi), L\rangle$ over uncertainty set. Thus, let us first explore some properties of $\langle d(\xi), L\rangle$, worst-case scenario and budgeted uncertainty set.

Proposition 1. $\langle d(\xi), L\rangle$ is a piece-wise linear convex function of ξ.

Proof. For each pair of locations r and s, it can be easily seen from (6) that $d_{rs}(\xi)$ is a piecewise linear convex function of ξ. Therefore, $\langle d(\xi), L\rangle$, which is a nonnegative linear combination of $d_{rs}(\xi)$, is also a piecewise linear convex function of ξ. □

Proposition 2. For budgeted uncertainty set with integer Γ and box uncertainty set, there exists a worst-case scenario ξ^* with all entries equal to 0 or 1.

Proof. From proposition (1), $\langle d(\xi), L\rangle$ is a convex function of ξ. On the other hand, budgeted and box uncertainty sets are convex (in fact, they are polytopes). Consequently, there exists an extreme point of the uncertainty set which maximizes (11). Since any extreme point ξ^* of the box uncertainty or budgeted uncertainty with integer Γ has all entries equal to 0 or 1, the result follows. □

So far, for a given assignment $L \in P$, we know that finding $Z(L)$, the robust cost of L, requires to solve nonconvex optimization model (11). In the sequel, we propose a convex equivalent model to (11), by exploiting the structure of the model (11) and using the convex hull of hypergraph (i.e., the set of points lying on or below the graph) of $\langle d(\xi), L\rangle$.

For a given assignment $L \in P$, it is known from proposition (2) that in the worst-case scenario, for each location r, the uncertain coordinates will be at extreme points x_r or $x_r + \hat{x}_r$, and y_r or $y_r + \hat{y}_r$. Therefore, it is enough to investigate the distance between locations *r* and *s*, when they are at their extreme coordinates. For each pair of locations r and s, the notations below express these extreme distances:

$$
\begin{aligned}
&d^x_{\underline{rs}} = |x_r - x_s|, \quad d^x_{\overline{r}\underline{s}} = |(x_r + \hat{x}_r) - x_s|, \\
&d^x_{\underline{r}\overline{s}} = |x_r - (x_s + \hat{x}_s)|, \quad d^x_{\overline{rs}} = |(x_r + \hat{x}_r) - (x_s + \hat{x}_s)|, \\
&d^y_{\underline{rs}} = |y_r - y_s|, \quad d^y_{\overline{r}\underline{s}} = |(y_r + \hat{y}_r) - y_s|, \\
&d^y_{\underline{r}\overline{s}} = |y_r - (y_s + \hat{y}_s)|, \; d^y_{\overline{rs}} = |(y_r + \hat{y}_r) - (y_s + \hat{y}_s)|.
\end{aligned}
\tag{13}
$$

Note that (13) requires $8 \times \binom{n}{2} = 4 \times n(n-1)$ enumerations. Now, we are ready to define $h(L,\xi)$ as a concave counterpart of $\langle d(\xi), L\rangle$, which is equal to $\langle d(\xi), L\rangle$ at the extreme points of uncertainty set. In other words, we want to define $h(L,\xi)$ in such a way that its hypergraph is equal to the convex hull of hypergraph of $\langle d(\xi), L\rangle$. Let

$$
h(L,\xi) = \sum_{r=1}^{n}\sum_{s=1}^{n}\left[h^x_{rs}(\xi^x) + h^y_{rs}(\xi^y)\right] f^L_{rs}, \tag{14}
$$

where,

$$
\begin{aligned}
h^x_{rs}(\xi^x) \;=\; & 0.5(|\xi^x_r + \xi^x_s - 1| - \xi^x_r - \xi^x_s + 1) d^x_{\underline{rs}} \\
+\; & 0.5(-|\xi^x_r + \xi^x_s - 1| - \xi^x_r + \xi^x_s + 1) d^x_{\underline{r}\overline{s}} \\
+\; & 0.5(-|\xi^x_r + \xi^x_s - 1| + \xi^x_r - \xi^x_s + 1) d^x_{\overline{r}\underline{s}} \\
+\; & 0.5(|\xi^x_r + \xi^x_s - 1| + \xi^x_r + \xi^x_s - 1) d^x_{\overline{rs}},
\end{aligned}
\tag{15}
$$

and

$$
\begin{aligned}
h^y_{rs}(\xi^y) \;=\; & 0.5(|\xi^y_r + \xi^y_s - 1| - \xi^y_r - \xi^y_s + 1) d^y_{\underline{rs}} \\
+\; & 0.5(-|\xi^y_r + \xi^y_s - 1| - \xi^y_r + \xi^y_s + 1) d^y_{\underline{r}\overline{s}} \\
+\; & 0.5(-|\xi^y_r + \xi^y_s - 1| + \xi^y_r - \xi^y_s + 1) d^y_{\overline{r}\underline{s}} \\
+\; & 0.5(|\xi^y_r + \xi^y_s - 1| + \xi^y_r + \xi^y_s - 1) d^y_{\overline{rs}}.
\end{aligned}
\tag{16}
$$

For simplicity, we can represent $h^x_{rs}(\xi^x)$ and $h^y_{rs}(\xi^y)$ in a different way as follows:

$$
\begin{aligned}
h^x_{rs}(\xi^x) &= (-D^x_{rs}|\xi^x_r + \xi^x_s - 1| + B^x_{rs}\xi^x_r + B^x_{sr}\xi^x_s + C^x_{rs}), \\
h^y_{rs}(\xi^y) &= (-D^y_{rs}|\xi^y_r + \xi^y_s - 1| + B^y_{rs}\xi^y_r + B^y_{sr}\xi^y_s + C^y_{rs}),
\end{aligned}
\tag{17}
$$

where the constant coefficients B, C and D are computed from extreme distances (13) as follows:

$$
\begin{aligned}
&C^x_{rs} = 0.5(d^x_{\overline{r}\underline{s}} + d^x_{\underline{rs}} + d^x_{\underline{r}\overline{s}} - d^x_{\overline{rs}}), \quad C^y_{rs} = 0.5(d^y_{\overline{r}\underline{s}} + d^y_{\underline{rs}} + d^y_{\underline{r}\overline{s}} - d^y_{\overline{rs}}), \\
&B^x_{rs} = 0.5(d^x_{\overline{r}\underline{s}} - d^x_{\underline{rs}} - d^x_{\underline{r}\overline{s}} + d^x_{\overline{rs}}), \quad B^y_{rs} = 0.5(d^y_{\overline{r}\underline{s}} - d^y_{\underline{rs}} - d^y_{\underline{r}\overline{s}} + d^y_{\overline{rs}}), \\
&D^x_{rs} = 0.5(d^x_{\overline{r}\underline{s}} - d^x_{\underline{rs}} + d^x_{\underline{r}\overline{s}} - d^x_{\overline{rs}}), \quad D^y_{rs} = 0.25(d^y_{\overline{r}\underline{s}} - d^y_{\underline{rs}} + d^y_{\underline{r}\overline{s}} - d^y_{\overline{rs}}).
\end{aligned}
\tag{18}
$$

Note that

$$\sum_{r=1}^{n}\sum_{s=1}^{n} B_{sr}^{x}\xi_{s}^{x} f_{rs}^{L} = \sum_{r=1}^{n}\sum_{s=1}^{n} B_{rs}^{x}\xi_{r}^{x} f_{sr}^{L}, \text{ and } \sum_{r=1}^{n}\sum_{s=1}^{n} B_{sr}^{y}\xi_{s}^{y} f_{rs}^{L} = \sum_{r=1}^{n}\sum_{s=1}^{n} B_{rs}^{y}\xi_{r}^{y} f_{sr}^{L}.$$
(19)

Thus, by using coefficients B, C and D, and identities (14), (17) and (19), $h(L,\xi)$ can be rewritten as follows:

$$\begin{aligned} h(L,\xi) = &\sum_{r=1}^{n}\left[\left(\sum_{s=1}^{n} B_{rs}^{x}(f_{rs}^{L}+f_{sr}^{L})\right)\xi_{r}^{x} + \left(\sum_{s=1}^{n} B_{rs}^{y}(f_{rs}^{L}+f_{sr}^{L})\right)\xi_{r}^{y}\right] \\ &+\sum_{r=1}^{n}\sum_{s=1}^{n}(C_{rs}^{x}+C_{rs}^{y})f_{rs}^{L} \qquad (20) \\ &+\sum_{r=1}^{n}\sum_{s=1}^{n}(D_{rs}^{x}\,|\,\xi_{r}^{x}+\xi_{s}^{x}-1\,|+D_{rs}^{y}\,|\,\xi_{r}^{y}+\xi_{s}^{y}-1\,|)f_{rs}^{L}. \end{aligned}$$

Proposition 3. For any assignment $L \in P$, $h(L,\xi)$ is a piecewise linear concave function of ξ. Furthermore, in the case of integral Γ, for any extreme scenario $\xi \in U_{\Gamma}^{\xi}$, where all entries of ξ are 0 or 1, $h(L,\xi) = \langle d(\xi), L\rangle$.

Proof. For each pair of locations r and s, from (17) the coefficient of $|\,\xi_{r}^{x}+\xi_{s}^{x}-1\,|$ in $h_{rs}^{x}(\xi^{x})$ is $-D_{rs}^{x}$, which is nonpositive. (All possible cases for D_{rs}^{x} are evaluated in Table 1.) Similarly, the coefficient of $|\,\xi_{r}^{y}+\xi_{s}^{y}-1\,|$ in $h_{rs}^{y}(\xi^{y})$ is $-D_{rs}^{y}$, which is similarly nonpositive. Using these facts, it is concluded that $h_{rs}^{x}(\xi^{x})$ and $h_{rs}^{y}(\xi^{y})$ are concave functions of the ξ^{x} and the ξ^{y}. Consequently, $h(L,\xi)$, being a positive linear combination of them, is also a concave function of ξ.

To prove the second part of the proposition, it is enough to show that for each pair of locations r and s, when their coordinates are at their extreme values, it holds that $h_{rs}^{x}(\xi^{x})+h_{rs}^{y}(\xi^{y}) = d_{rs}(\xi)$. This can be verified by evaluating $h_{rs}^{x}(\xi^{x})$ and $h_{rs}^{y}(\xi^{y})$ in (15) and (16) for extreme values of ξ_{r}^{x} and ξ_{r}^{y} for each pair of locations r and s. □

Table 1. Evaluating D_{rs}^{x} in all possible cases

Possible cases	D_{rs}^{x}
$x_r \geq x_s + \widehat{x}_s$	0
$x_s \geq x_r + \widehat{x}_r$	0
$x_r + \widehat{x}_r \leq x_s + \widehat{x}_s$ and $x_s \leq x_r < x_s + \widehat{x}_s$	$\widehat{x}_r$
$x_r + \widehat{x}_r > x_s + \widehat{x}_s$ and $x_s \leq x_r < x_s + \widehat{x}_s$	$x_s + \widehat{x}_s - x_r$
$x_s + \widehat{x}_s \leq x_r + \widehat{x}_r$ and $x_r \leq x_s < x_r + \widehat{x}_r$	$\widehat{x}_s$
$x_s + \widehat{x}_s > x_r + \widehat{x}_r$ and $x_r \leq x_s < x_r + \widehat{x}_r$	$x_r + \widehat{x}_r - x_s$

Proposition 4. For any feasible assignment $L \in P$ and integer Γ, $\langle d(\xi_L^*), L\rangle = h(L, \xi_L^*)$, where ξ_L^* is the worst-case scenario for assignment L.

Proof. From proposition (3), for any assignment $L \in P$ and integer Γ, the maximum of $h(L,\xi)$ can be found by solving an LP model. Then, there exists an extreme point ξ_L^* of the uncertainty set, which maximizes $h(L,\xi)$. On the other hand, by proposition (3), there exists an extreme point of the uncertainty set which maximizes $\langle d(\xi), L\rangle$. Since $\langle d(\xi), L\rangle$ and $h(L,\xi)$ are equal at extreme points of the uncertainty set (as proved in proposition 3), it holds that $\langle d(\xi_L^*), L\rangle = h(L, \xi_L^*)$. □

4. Formulations of Problem RQAP

In the robust optimization approach to solve Problem RQAP, it is desired to minimize the robust value (worst case on the uncertainty set) of the objective function (i.e., total material handling volume) over all feasible assignments $L \in P$. Thus, Problem RQAP will be

$$\min_{L\in P}\left\{\max_{\xi\in U_\Gamma^\xi}\langle d(\xi), L\rangle\right\}. \tag{21}$$

The model (21) is not tractable, since the inner maximization problem is not a convex optimization problem. It is worth mentioning that, since $\langle d(\xi), L\rangle$ is not an affine function of uncertain data, ξ, classical robust optimization approaches cannot be applied directly to construct robust counterpart of the problem. Here, we develop a mixed integer linear programming (MILP) model to find a robust solution for this problem. Let us replace the inner maximization problem in model (21) by the LP model (22):

$$\max_{\xi \in U_\Gamma^\xi} \langle d(\xi), L\rangle = \alpha^*(L) + \sum_{r=1}^{n}\sum_{s=1}^{n}(C_{rs}^x + C_{rs}^y) f_{rs}^L, \tag{22}$$

where $\alpha^*(L)$ is the optimal value of the model (25) below. This replacement is valid, since from Proposition 3, for any feasible solution $L \in P$ and integer Γ, we have

$$\langle d(\xi_L^*), L\rangle = \max_{\xi \in U_\Gamma^\xi} \langle d(\xi), L\rangle = \max_{\xi \in U_\Gamma^\xi} h(L,\xi) = h(L,\xi_L^*). \tag{23}$$

On the other hand, from (20), $h(L,\xi_L^*)$ can be written as

$$h(L,\xi_L^*) = \sum_{r=1}^{n}\sum_{s=1}^{n}(C_{rs}^x + C_{rs}^y) f_{rs}^L + \alpha^*(L), \tag{24}$$

where,

$$\begin{aligned}
\alpha^*(L) = \max_{\xi \in U_\Gamma^\xi} &\left\{ \sum_{r=1}^{n} \left[\begin{array}{l} \left(\sum_{s=1}^{n} B_{rs}^x (f_{rs}^L + f_{sr}^L)\right)\xi_r^x + \\ \left(\sum_{s=1}^{n} B_{rs}^y (f_{rs}^L + f_{sr}^L)\right)\xi_r^y - \sum_{s=1}^{n}(t_{rs}^x + t_{sr}^y) \end{array} \right] \right\} \\
s.t. \quad & -t_{rs}^x \le D_{rs}^x f_{rs}^L (\xi_r^x + \xi_s^x - 1) \le t_{rs}^x, \quad \forall r,s \in N \\
& -t_{rs}^y \le D_{rs}^y f_{rs}^L (\xi_r^y + \xi_s^y - 1) \le t_{rs}^y, \quad \forall r,s \in N \\
& 0 \le t_{rs}^x, t_{rs}^y \quad \forall r,s \in N \\
& 0 \le \xi_r^y, \xi_r^x, \quad \forall r \in N.
\end{aligned} \tag{25}$$

Clearly, $t_{rs}^{x*} = D_{rs}^x f_{rs}^L \,|\, \xi_r^{x*} + \xi_s^{x*} - 1 \,|$ and $t_{rs}^{y*} = D_{rs}^y f_{rs}^L \,|\, \xi_r^{y*} + \xi_s^{y*} - 1 \,|$ at the optimal solution.

Note that, for a given assignment $L \in P$, model (25) is an LP with $2n^2 + 2n$ continuous variables and $4n^2$ linear constraints. In this section, we propose a robust counterpart for (21), which is a MILP model. First, we develop the quadratic MIP model to be given as (26a)-(26g). Then, we use an available linearization technique to make a MILP equivalent of the model (26a)-(26g). Recall from equation (4) that f_{rs}^L is a quadratic function of decision variables L.

Proposition 4. The model (21) is equivalent to

$$\min_{L\in P} \quad \sum_{r=1}^{n}\sum_{s=1}^{n}\left[(C_{rs}^{x}+C_{rs}^{y})f_{rs}^{L}+u_{rs}^{x}+u_{rs}^{y}\right]+\Gamma z+\sum_{r=1}^{n}(p_{r}^{x}+p_{r}^{y}) \tag{26a}$$

$$s.t. \quad \sum_{s=1}^{n}u_{rs}^{x}+p_{r}^{x}+z\geq\sum_{s=1}^{n}B_{rs}^{x}(f_{rs}^{L}+f_{sr}^{L}), \quad \forall r\in N, \tag{26b}$$

$$\sum_{s=1}^{n}u_{rs}^{y}+p_{r}^{y}+z\geq\sum_{s=1}^{n}B_{rs}^{y}(f_{rs}^{L}+f_{sr}^{L}), \quad \forall r\in N, \tag{26c}$$

$$-D_{rs}^{x}f_{rs}^{L}\leq u_{rs}^{x}\leq D_{rs}^{x}f_{rs}^{L}, \quad \forall i,r,s\in N, \tag{26d}$$

$$-D_{rs}^{y}f_{rs}^{L}\leq u_{rs}^{y}\leq D_{rs}^{y}f_{rs}^{L}, \quad \forall i,r,s\in N, \tag{26e}$$

$$z\,,\,p_{r}^{x}\,,\,p_{r}^{y}\,,\geq 0, \quad \forall r,s\in N\,. \tag{26f}$$

Proof. For a given assignment $L\in P$, the dual model of (25) can be written as

$$\begin{aligned}\beta^{*}(L)=Min \quad & \Gamma z+\sum_{r=1}^{n}(p_{r}^{x}+p_{r}^{y})+\sum_{r=1}^{n}\sum_{s=1}^{n}(u_{rs}^{x}+u_{rs}^{y})\\ s.t. \quad & (26b)-(26g).\end{aligned} \tag{27}$$

Since the uncertainty set is a polytope, from strong duality theorem it is concluded that $\alpha^{*}(L)=\beta^{*}(L)$. Then,

$$\langle d(\xi_{L}^{*}),L\rangle=h(L,\xi_{L}^{*})=\sum_{r=1}^{n}\sum_{s=1}^{n}(C_{rs}^{x}+C_{rs}^{y})f_{rs}^{L}+\beta^{*}(L). \tag{28}$$

Consequently, from proposition (4) and equation (28), robust counterpart of the model (21) can be written as model (26a)-(26g) and the proof is complete. □

Various MIP models were proposed by researchers to linearize the classical QAP (see Kaufmann and Broeckx [16], Frieze and Yadegar [11], Xia and Yuan [22], Xia [23, 24], and Zhang et al. [26]). Kaufmann and Broeckx [16] linearized QAP by adding n^2 continuous nonnegative variables and n^2 linear constraints. Although this approach has the smallest number of variables and constraints, LP relaxation of the Kaufman-Broeckx linearization gives a poor QAP bound and Yuan [22], Xia [23, 24] tightened Kaufman and Broeckx's formulation.

To linearize the model (26a)-(26g), we use the same ideas in the Kaufmann-Broeckx linearization of the classical QAP. Therefore, we develop the MILP model (29) as a robust linear counterpart of uncertain QAP:

$$
\begin{aligned}
\min_{L\in P}\quad & \left\{\sum_{r=1}^{n}\sum_{s=1}^{n}(C_{rs}^{x}+C_{rs}^{y})w_{rs}+\Gamma z+\sum_{r=1}^{n}(p_{r}^{x}+p_{r}^{y})+\sum_{r=1}^{n}\sum_{s=1}^{n}(u_{rs}^{x}+u_{rs}^{y})\right\} \\
s.t.\quad & \sum_{s=1}^{n}u_{rs}^{x}+p_{r}^{x}+z\ge\sum_{s=1}^{n}B_{rs}^{x}(w_{rs}+w_{sr}),\quad \forall r\in N \\
& \sum_{s=1}^{n}u_{rs}^{y}+p_{r}^{y}+z\ge\sum_{s=1}^{n}B_{rs}^{y}(w_{rs}+w_{sr}),\quad \forall r\in N \\
& -D_{rs}^{x}w_{rs}\le u_{rs}^{x}\le D_{rs}^{x}w_{rs},\quad -D_{rs}^{y}w_{rs}\le u_{rs}^{y}\le D_{rs}^{y}w_{rs},\quad \forall r,s\in N \\
& \sum_{j=1}^{n}f_{ij}L_{js}-f_{i}^{max}(1-L_{ir})\le w_{rs},\quad \forall i,r,s\in N \\
& \sum_{i=1}^{n}f_{i}^{min}L_{ir}\le w_{rs},\quad \forall r\ne s\in N \\
& \sum_{r=1}^{n}f_{ii}L_{ir}\le w_{rr},\quad \forall r\in N \\
& z,\ p_{r}^{x},\ p_{r}^{y},w_{rs}\ge 0,\quad \forall r,s\in N,
\end{aligned}
\tag{29}
$$

where $f_i^{max}=max_{j\in N}\{f_{ij}\}$ and $f_i^{min}=min_{j\in N}\{f_{ij}\}$, for all $i\in N$. This model has n^2 binary, n^2+2n+1 nonnegative continuous and $2n^2$ unrestricted continuous variables. Also, it has n^3+5n^2+3n linear constraints.

Since we use rectilinear distances between locations we obviously have $d_{rs}(\xi)=d_{sr}(\xi)$. Therefore, we can exploit symmetry property of distances to reduce the number of variables and constraints in the model (29). Due to symmetric distances, without loss of generality, we can assume that $f_{ij}=f_{ji}$. Let $f_{ij}^{'}=f_{ij}+f_{ji}$, $f_i^{'max}=max_{j\in N}\{f_{ij}^{'}\}$ and $f_i^{'min}=min_{j\in N}\{f_{ij}^{'}\}$, for all $i\in N$. Then, an improved version of (29) is given below:

$$
\begin{aligned}
\min_{L \in P} \quad & \left\{ \sum_{r=1}^{n-1} \sum_{s=r+1}^{n} (C_{rs}^{x} + C_{rs}^{y}) w_{rs} + \Gamma z + \sum_{r=1}^{n} (p_r^x + p_r^y) + \sum_{r=1}^{n} \sum_{s=1, s \neq r}^{n} (u_{rs}^x + u_{rs}^y) \right\} \\
s.t. \quad & \sum_{s=1, s\neq r}^{n} u_{rs}^x + p_r^x + z \geq \sum_{s=1}^{r-1} B_{rs}^x w_{sr} + \sum_{s=r+1}^{n} B_{rs}^x w_{rs}, \quad \forall r \in N \\
& \sum_{s=1, s\neq r}^{n} u_{rs}^y + p_r^y + z \geq \sum_{s=1}^{r-1} B_{rs}^y w_{sr} + \sum_{s=r+1}^{n} B_{rs}^y w_{rs}, \quad \forall r \in N \\
& -D_{rs}^x w_{rs} \leq 2u_{rs}^x \leq D_{rs}^x w_{rs}, \quad -D_{rs}^y w_{rs} \leq 2u_{rs}^y \leq D_{rs}^y w_{rs}, \quad \forall r, s > r \in N \\
& -D_{rs}^x w_{sr} \leq 2u_{rs}^x \leq D_{rs}^x w_{sr}, \quad -D_{rs}^y w_{sr} \leq 2u_{rs}^y \leq D_{rs}^y w_{sr}, \quad \forall r, s < r \in N \\
& \sum_{j=1}^{n} f_{ij}^{'} L_{js} - f_i^{'max} (1 - L_{ir}) \leq w_{rs}, \quad \forall i, r, s > r \in N \\
& \sum_{i=1}^{n} f_i^{'min} L_{ir} \leq w_{rs}, \quad \forall r, s > r \in N \\
& z, p_r^x, p_r^y, w_{rs} \geq 0, \quad \forall r, s > r \in N .
\end{aligned} \tag{30}
$$

The MIP model (30) which is equivalent to (29) has n^2 binary, $0.5n^2 + 1.5n + 1$ nonnegative continuous and $2n^2 - 2n$ unrestricted continuous variables. Also, it has $0.5n^3 + 4n^2 - 0.5n$ linear constraints. In the next section, we experimentally examine the proposed approach and the model (30) on some random instances.

5. Experimental Results

We generated 20 uncertain QAPs of size 8, and solved model (30) for each instance by tuning Γ from 0 to 12 (totally 260 models were solved). Although Γ can potentially vary from 0 to $2 \times 8 = 16$ in these instances, but since the results for $\Gamma > 12$ were the same as the ones for $\Gamma = 12$ we just considered the values from $\Gamma = 0$ to $\Gamma = 12$. In randomly generated instances, f_{ij}, $x_r + \hat{x}_r$ and $y_r + \hat{y}_r$ were integer values in [0,10000]. Also, x_r and y_r were random integers in $[0, x_r + \hat{x}_r]$ and $[0, y_r + \hat{y}_r]$, respectively. The experiments were conducted on a PC with a processor Intel Core Duo 2.66 GHz and with 2.99 GB of RAM by using CPLEX 12.2, (default parameters) interfaced with C++. After solving these 260 instances, we used the homogeneous Monte Carlo simulation to generate coordinates of locations (100,000 simulation runs for each instance) to analyze the quality and robustness of solutions in the presence of uncertainty. The results are summarized in Table 2.

Table 2. Summary of experimental results

Γ	time	Z^*	$Z_{average}$	$Z_{0.05}$	Z_{max}	P
0	19.6	1,076	1,395	1,624	1,849	0.91155
1	27.8	1,335	1,378	1,580	1,786	0.51717
2	30.5	1,523	1,379	1,573	1,774	0.12409
3	34	1,675	1,378	1,568	1,765	0.00543
4	34	1,796	1,376	1,567	1,766	0.00004
5	34.9	1,890	1,374	1,565	1,763	0
6	37.6	1,960	1,376	1,565	1,762	0
7	36.8	2,010	1,374	1,564	1,768	0
8	38.4	2,041	1,373	1,563	1,766	0
9	38.7	2,061	1,373	1,562	1,764	0
10	38.9	2,071	1,374	1,564	1,768	0
11	40.2	2,076	1,374	1,564	1,767	0
12	38.3	2,077	1,373	1,563	1,766	0

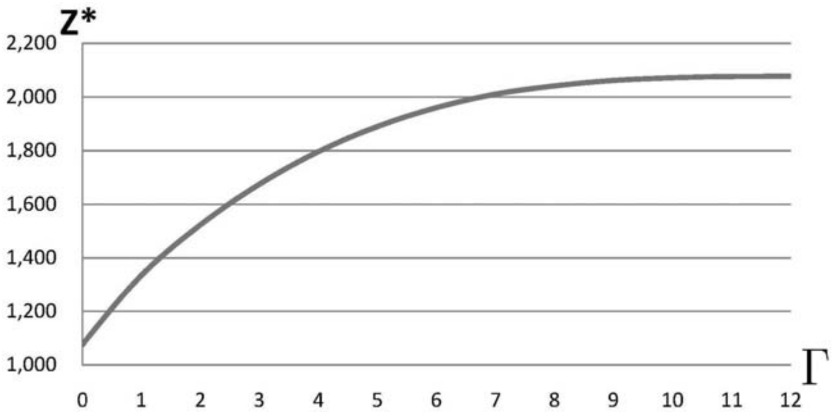

Figure 1. Diagram of Z^* with respect to Γ

In Table 2, time, Z^*, $Z_{average}$ and Z_{max} are respectively time in seconds spend to solve the model, optimal robust objective value, average objective value in simulation and maximum objective value in simulation. Also, $Z_{0.05}$ is the value for which frequency of greater objective value than this is 5%, and P is the violation probability (the percentage of instances in simulation with objective values greater than robust cost Z^*), calculated empirically. The values in Table 2 are the average of 20 instances. In the Figures 1-4, diagrams of Z^*, $Z_{average}$, $Z_{0.05}$ and Z_{max} with respect to the protection level, Γ, are depicted.

Since solving the classical deterministic QAP is NP-hard (see Sahni and Gonzalez [19]), our proposed formulations for Problem RQAP are also NP-hard. Although, for small size instances, we can hope to find the exact solution of Problem RQAP (as seen in Table 2, takes less than 1 minute to solve the MILP equivalent of RQAP with *n*=8), but for large instances, heuristics may be needed.

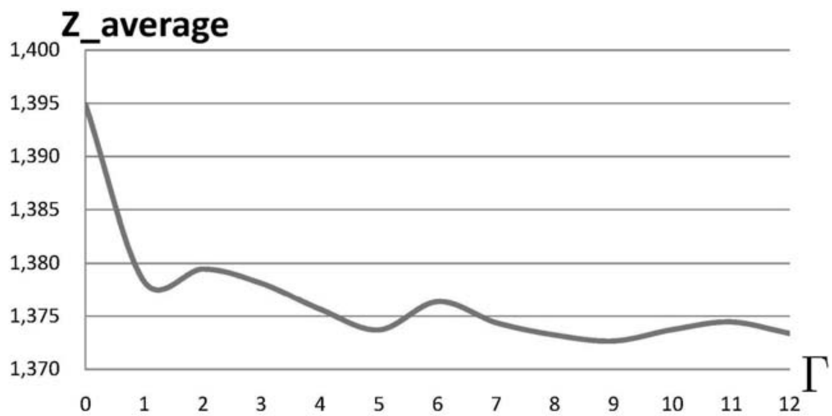

Figure 2. Diagram of $Z_{average}$ with respect to Γ

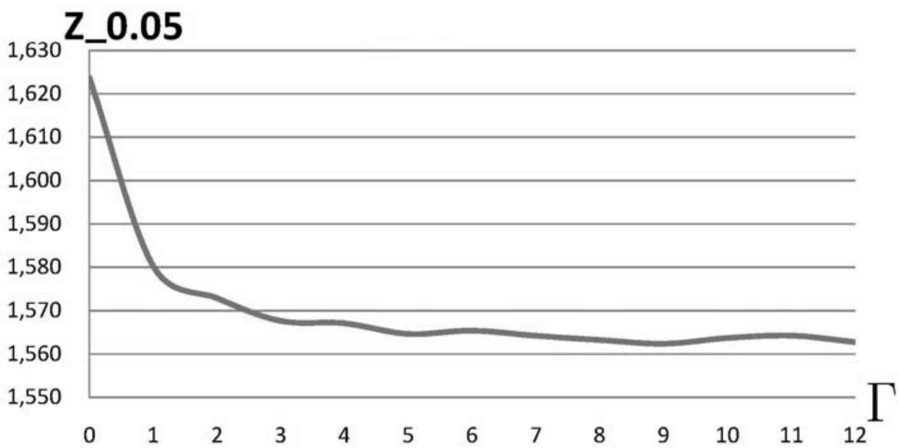

Figure 3. Diagram of $Z_{0.05}$ with respect to Γ

Table 2 and Fig. 1 show that Z^* increases by increasing the protection level Γ, as expected, since increasing Γ results in conservative solutions. In the worst case scenario for the robust solution, all coordinates of locations are at their extreme (lower or upper) values. Although in the worst case scenario of solution for $\Gamma = 0$, all coordinates of locations are at their lower values, for $\Gamma > 0$, Γ coordinates of locations are not necessarily at their upper values. Therefore, after some value, increasing Γ does not change the solution. In our instances, this value turned to be12.

$Z_{average}$, $Z_{0.05}$ and Z_{max} in figures 1-3 and violation probability P in Table 2 have similar trends. In general, they decrease by increasing the protection level Γ.

The numerical results in Table 2 and Figures 1 and 4 suggest that the model builder can use our proposed robust counterpart of Problem RQAP to find more reliable solutions by trading off between optimal robust objective value and violation probability. For example, to find a solution with zero violation probability, we can set $\Gamma = 5$ which causes 76% increase in Z^*. On the other hand, it results in 1.5%, 3.6% and 4.7% reductions in $Z_{average}$, $Z_{0.05}$ and Z_{max}, respectively. Note that Z^* does not provide useful information, because it is just the optimal robust value in the case of assumed protection level Γ for budgeted uncertainty. Therefore, if the underlying uncertainty is the same (for example, box uncertainty), to compare the quality of optimal solutions for different protection levels Γ, it makes sense to use $Z_{average}$, $Z_{0.05}$ or Z_{max} as the judging criteria.

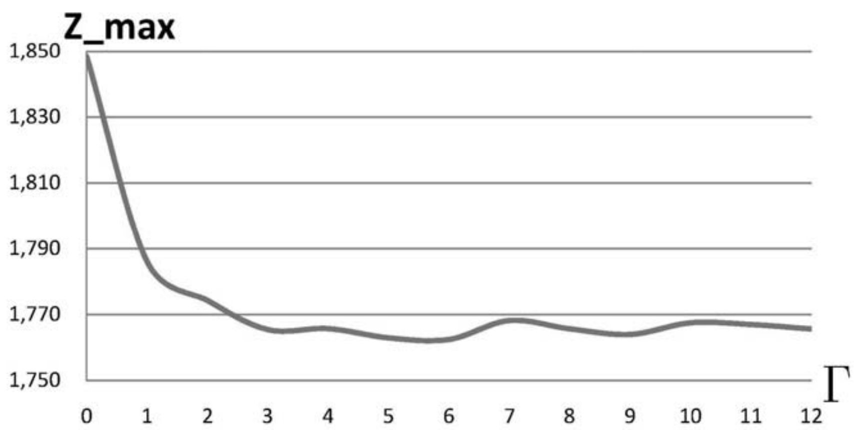

Figure 3. Diagram of Z_{max} with respect to Γ

6. Conclusions

We considered quadratic assignment problem with interval and budgeted uncertainty in coordinates of locations and developed robust counterpart of the problem. It is worth mentioning that, since the objective function of the original uncertain problem is not an affine function of uncertain data, classical robust optimization approaches cannot be applied directly to construct its robust counterpart. We developed a linear mixed integer programming model to find the robust its solution of this problem. The main contributions of our work are:

(1) A linear programming model to find the worst-case scenario and robust cost of a given feasible assignment.

(2) A mathematical programming formulation of the robust QAP.

(3) Line arizations of the robust QAP to allow for the use of CPLEX or other solvers.

(4) Illustrative experimental results.

The main conclusions are:

(1) The robust QAP is as hard as the classical QAP. Although for small size instances (for example, $n = 8$), exact solution can be obtained by MIP solvers like CPLEX, heuristics should be developed to find good solutions for moderately large instances.

(2) Modelers can make trade-off between violation probability, robustness cost, average cost, maximum cost or desired quantiles (in the simulation) to get a solution with acceptable risk.

(3) For moderately small values of protection levels, we can expect to have favorable solutions. There is no need to make the most conservative decisions (with the largest protection level) to obtain robust solutions.

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