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A Benders' Decomposition Based Solution Method for Solving User Equilibrium Problem: Deterministic and Stochastic Cases

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Traffic assignment problem is an important problem for analyzing and optimizing a transportation network to find optimal flows. This study presents a new formulation based on a generalized Benders' decomposition approach to solve the user equilibrium problems, in deterministic and stochastic cases. The new approach decomposes the problem into a master problem and a sub-problem. The former is a nonlinear and the latter is a linear programming problem. Iteratively, the master problem is solved and its outputs are used to solve the sub-problem by forming appropriate cuts and adding them to the master problem to be used in the next iteration. Based on the convergence of Benders' decomposition, the iterative process is terminated in a finite number of steps. Some numerical examples are worked through and comparisons are made with other methods.

Keywords: *Benders' Decomposition, Benders' cut, stochastic user equilibrium, Traffic assignment.*

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1. Introduction

Network design problems are central to a large number of contexts including transportation, telecommunication, computer and power systems. The idea is to establish a network of links (roads, optimal fibers, electric lines, etc.) that enables the flow of commodities (people, data packets, electricity, etc.) in order to satisfy some demand characteristics. By paying attention to the importance of travel time in urban journeys of big cities, there is particularly a huge degree of interest in urban network design problems; thus, professional allocation has been of special importance in the past two decades. One of the most important problems on the analysis and optimization of transportation networks is the traffic assignment that finds an optimal flow in a network.

Decomposition technique is a general approach for solving large scale problems, in which the problem is broken to some smaller ones so that, by solving each separately (either in parallel or sequentially), the solution of the main problem is achieved. Indeed, decomposing a large scale problem to some smaller ones is an old idea and several methods of this kind have been proposed and their applications have been extended in different areas. Regarding the importance of the associated problems, several solution methodologies are available for network design. These include purely heuristic methods and optimal implicit enumerations. Among the successful solution approaches, Benders' decomposition was found to be popular for application. The basic idea behind the method

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is to decompose the problem into two simpler parts. In the first part, a master problem develops a relaxed version of the problem and obtains values for a subset of the variables. In the second part, a sub-problem is to obtain the values for the remaining variables while keeping the first ones fixed which are used to generate cuts for the master problem. The master problem and the sub-problem are solved iteratively until no more cuts can be generated ([5], [7], [10], [12], [15], [16], [18], [20], [25]). After reviewing the use of Benders' decomposition in Section 2, in Section 3, different traffic assignment problems are explained. Section 4 is devoted to a new formulation of the deterministic user equilibrium (UE) model for application of Benders' decomposition. The technique is presented by determining its related master and sub-problems and a case study is also given. In Section 5, first a new formulation of stochastic user equilibrium (SUE) is presented. Then, an application of Benders' decomposition for finding the solution with a numerical example is explained. Finally, Section 6 gives our concluding remarks.

2. Benders' Decomposition

Benders' decomposition is a classical solution approach for combinatorial optimization problems based on partition and delayed constraint generation. This method was originally purposed by J. F. Benders in 1962 for solving large scale combinatorial optimization problems [2] and then several extensions were proposed. One of the most important ones was presented by Geoffrion [12] who proposed a "generalized Benders' decomposition" approach. He used nonlinear duality theory and extended the Benders' method to the case where the sub-problem was convex. This development enabled the application of the Benders' decomposition to a whole new set of problems, particularly those in which a joint problem was generally nonconvex but could be made convex by fixing one set of variables. Examples of successful application of this methodology to mixed-integer problems are abundant. Also, there are a number of applications; for instance, the seminal paper by Geoffrion and Graves on multi commodity distribution network design [9] and the extension presented by Cordea ([4], [8]) on the same problem can be mentioned. Other applications include the locomotive and car assignment problems [11], large scale water resource management problem [9], two stage stochastic linear problem and robust shortest path problem ([9], [17]).

The method partitions the model to be solved into two simpler problems named master and sub-problem. Indeed, summarizing Benders' decomposition, first the relaxed master problem is solved to obtain a lower bound on the optimal values of the objective function of the initial problem, and then, the sub-problem uses inputs of the master problem to form an approximate cut and adds it to the master problem in the next iteration. Also, by solving the sub-problem, an upper bound is found for the initial problem. During the iterative process, by adding a new constraint to the master problem, the optimal value of its objective function can only increase or stay the same. On the other hand, in each iteration, by solving a sub-problem, the upper bound of objective function of the initial problem can only decrease or stay the same. As soon as the lower and upper bounds of the initial problem are sufficiently close, the iterative process can be terminated with a sufficiently small tolerance. Based on the convergence theorem of Benders' decomposition method, the algorithm achieves the optimal solution after a finite number of iterations ([2], [4], [8], [21]).

3. Traffic Assignment Problem and the UE Principle

The traffic assignment problem is a fundamental transportation problem concerned with the distribution of travel demands to routes in a traffic network. As a mathematical model, the problem is commonly represented by a discrete graph in which each link is associated with a travel cost function and the demands are associated, more than the travel cost, with the number of trips which are given by an origin-destination (O-D) matrix.

The number of travels that can be made on the streets or junctions in an urban environment is equivalent to the user's personal decisions in a special time interval. The problems such as arriving from an origin to a destination at what time and what route in a crowded network depend on the user's decisions. The prediction of flow in the network is an important problem with which urban transportation network designers are faced. So, traffic assignment is a problem to find the amounts of flows in networks.

The classical assumption for the models regarding the distribution is the user equilibrium (UE) principle. This principle, due to Wardrop [9] in 1952, is stated as follows:

"The journey times on all the routes which are actually used are equal and less than those which are experienced by a single vehicle on any unused route."

Characteristic features of the situation described by the UE principle are that all travels have prefect information about travel costs and are uniform in the sense that they have the same travel cost perception. Based on each of these behavioral assumptions, models may present reasonable approximation of the actual traffic situation. However, if there is lack of information among the travelers about the shortest routs or if travelers have different preferences and perceive travel costs differently, it is then natural to assume that traffic flows do not satisfy the user equilibrium conditions [9].

4. Basic User Equilibrium Model

Consider a transportation network $G = (N, A)$ where N denotes the set of nodes and each directed link $a \in A$ is associated with a generalized travel cost $t_a(f_a)$, which represents the disutility of using link a as a function of its flow f_a ; This cost may include several additive components, the most important of which is perhaps the travel time on the link. It is assumed that t_a is positive and is a strictly increasing function of the flow on link a. This function is represented as follows:

$$t_a = t_0^a \left[1 + \beta \left(\frac{f_a}{C'} \right)^{\alpha} \right], \quad (1)$$

where t_0^a is travel time in zero link, C'_a is practical capacity and α and β are the model parameters that are usually set to be $\alpha = 4$ and $B = 0.15$ [24].

For certain pairs of origins and destinations $(p, q) \in C$, where

$$C \subset N \times N$$

, there is a given positive demand d_{pq} of flows. For each O-D pair (p, q), the set of simple routes from p to q is denoted by r_{pq} (a set which, in general, is not known explicitly) and the flow on route from p to q is denoted by h_{pqr}. By defining a link-route incidence matrix (δ_{pqra}), $\delta_{pqra} = 1$ if route $r \in R_{pq}$ contains link α and 0, otherwise. The user equilibrium traffic assignment problem can be formulated as the following convex program (here, f denotes the vector of link flows):

$$Min \quad T(f)=\sum_{a\in A}\int_0^{f_a} t_a(s)ds$$
$$s.t. \quad \sum_{r\in R_{pq}} h_{pqr} = d_{pq} \quad , \quad \forall (p,q)\in C$$
$$\sum_{(p,q)\in C}\sum_{r\in R_{pq}} \delta_{pqa} h_{pqr} = f_a , \quad \forall a \epsilon A \tag{2}$$
$$h_{pqr} \geq 0 \quad , \quad r\in R_{pq} , \quad (p,q) \in C.$$

It should be reminded that, in this model, the objective function is not known explicitly ([9], [13]). To avoid calculation of $T(f)$ (since it is too time consuming), the interval $[0,f_a]$ is divided into n equal subintervals, each having length $\frac{f_a}{n}$, and the scaling points t_i, $i=1,2,\dots,n$, to have

$$\int_0^{f_a} t_a(s)ds \approx \sum_{i=1}^{n} t_a(t_i)\frac{f_a}{n} \tag{3}$$

4.1. Benders' decomposition method for solving UE problem

Considering (1) and (2), the initial problem (2) can be rewritten as follows [25]:

$$p(h,f):$$
$$Min \quad \sum_{i=1}^{n} t_a^0 \,[1+\beta(\tfrac{(3i-2)f_a}{2nC_a})^{\alpha}\,]\frac{f_a}{n}$$
$$\text{s. t :} \quad \sum_{r\in R_{pq}} h_{pqr} = d_{pq} \quad , \quad \forall (p,q)\in C \tag{4}$$
$$\sum_{(p,q)\in C}\sum_{r\in R_{pq}} \delta_{pqra} h_{pqr} = f_a , \quad \forall a \epsilon A$$
$$h_{pqr} \geq 0 \quad ,\forall\, r\in R_{pq}, (p,q)\in C.$$

The problem $p(h,f)$ is consisted of two variables h_{pqr} and f_a.

As explained in Section 2, to apply the Benders' decomposition approach, first the $p(h,f)$ problem should be divided into two related master problem and sub-problem. The initial master model $M(h,f,0)$ can be stated as follows:

$$M(h,f,m=0):$$
$$Min \quad \sum_{i=1}^{n} t_a^0 \,[1+\beta(\tfrac{(3i-2)f_a}{2nC_a})^{\alpha}\,]\frac{f_a}{n} \tag{5}$$
$$\text{s. t:} \quad f_a \geq 0 , \quad \forall a\in A.$$

By solving [5], the variables f_a are obtained somehow optimally. Then, by replacing these values in the linear part of $p(h,f)$, the problem contains only the variables h_{pqr}.

Now, by introducing two dual variables w_{pq} and π_α corresponding to the first and second constraint sets in (4), respectively, the dual formulation of the linear part of $p(h,f)$ can be set up as:

$$\text{S}(w,\pi|h,f): Max \quad \sum_{(p,q)\in C} w_{pq}d_{pq} + \sum_{a\in A}\pi_a f_a$$
$$\text{s .t. } (\text{wA} + \pi\Delta) \quad \leq 0 , \tag{6}$$
$$w_{pq}, \pi_a, free, \quad \forall (p,q)\in C, \quad a\epsilon A,$$

where the coefficients of the variables h_{pqr} in the first and second sets of constraints $p(h,f)$ are represented by A and Δ matrices, respectively, and the vectors w and π respectively contain the variables w_{pq} and π_a.

When the optimal values of the variables w and π are obtained by solving the sub-problem $S(w, \pi \mid h, f)$, a Benders' cut can be built by replacing these variables in the objective function of the problem (6) as follows:

$$\sum_{(p,q)\in C} wpq\, d_{pq} + \sum_{a\epsilon A} \pi_a f_a \leq m, \tag{7}$$

where w_{pq} and π_α are parameters and m and f_a are continuous variables. Then, by adding this Benders' cut to the initial problem $M(h,f,0)$, the following regular master problem $M(h,f,m)$ can be obtained. Note that the optimal dual variables w and π are given an extra index $t \in B$ for each Benders' cut related to the following iterations:

$$\begin{aligned} &Min \quad \textstyle\sum_{a\epsilon A} \sum_{i=1}^{n} t_a^0 \left[1 + \beta \left(\frac{(3i-2)f_a}{2n.C_a{}'}\right)^{\alpha}\right]. \frac{f_a}{n} + m \\ &\text{s .t.} \quad \textstyle\sum_{(p,q)\in C} w_{pqt} d_{pq} + \sum_{a\epsilon A} \pi_{at} f_a \leq m, \quad \forall t \in B, \quad (p,q) \in C \\ &f_a \geq 0, \quad m \geq 0, \quad \forall a\epsilon A. \end{aligned} \tag{8}$$

Indeed, here m is the least amount of the objective function of the sub-problem. Continuing this procedure iteratively and adding a new cut to each iteration cause the solution to get closer to the optimal solution. This procedure is stopped when the obtained upper and lower bounds for the initial objective function are close enough.

4.2. Case study and model implementation

A famous urban transportation network model is the Alsop and Charlsworth model ([1], [3]). The model contains 5 origins, say A, C, D, E and G, 5 destinations, say A, B, D, E and F, and 2B links with 6G nodes. In this model, as shown in Figure 1, origins and destinations are respectively shown by rectangles and circles and the connecting streets between junctions are shown by arrows [1].

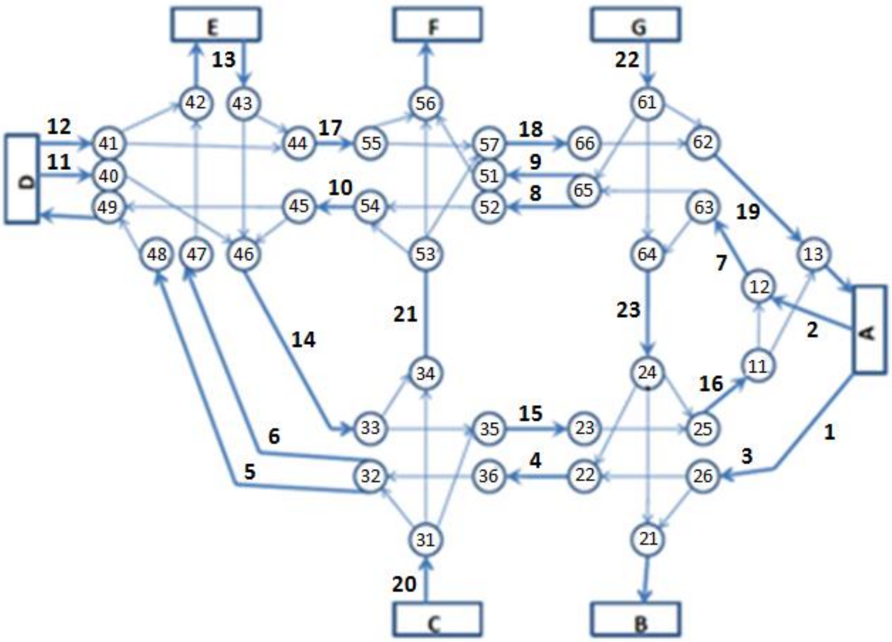

Figure1. The Alsop and Charlesworth network model

The corresponding values of the network travel requests are presented in Table 1 for $\alpha = 4$ and $\beta = 0.15$.

Table 1. Travel requests for the Alsop and Charlesworth model

Origin	Destination					Total
	A	B	D	E	F	
A	-	250	700	30	200	1180
C	40	20	200	130	900	1290
D	400	250	-	50	100	800
E	300	130	30	-	20	480
G	550	450	170	60	20	1250

Also, the travel times in zero link and the practical capacities of the links are presented in Table 2, in which the times are in minutes.

The model was solved by Benders' decomposition method as mentioned above using the MATLAB 7.8 solver. The algorithm stopped with 45 steps in 166 seconds. The obtained optimal values of the network link flows are shown in Table 3.

Table 2. Travel free time and practical capacities of the links

Link	Travel free time	Practical capacity	Link	Travel free time	Practical capacity
1	0	400	13	0	440
2	0	320	14	20	640
3	10	640	15	15	520
4	15	640	16	10	580
5	20	360	17	10	340
6	20	370	18	15	340
7	10	360	19	10	300
8	15	370	20	0	560
9	15	340	21	15	640
10	10	440	22	0	720
11	0	400	23	15	640
12	0	360			

Table 3. Optimal value of network link flows by Benders' decomposition method

Link	Flow	Link	Flow	Link	Flow
1	990.7697	9	220	17	537.86
2	537.14	10	577.14	20	375.82
3	970.77	11	412.09	19	608.71
4	432.86	12	407.91	20	129
5	522.86	13	520.04	21	927.96
6	242.04	14	734.18	22	125
7	569.23	15	764.18	23	859.20
8	507.14	16	1081.30		

Also, the optimal route flows and their travel times are presented in Table 4. However, these values that are obtained with approximation could supply the network demand.

Table 4. Optimal values of network link flows for UE

(Origin- Destination)	Route	Flow
(A,B)	1-3 2-7-23	245.3060 1.4899
(A,D)	1-3-4-5 2-7-8-10 2-7-23-4-5	345.192 345.4179 2.0943
(A,E)	1-3-4-6 2-7-8-10-12 2-7-23-4-6	23.2393 0.0105 0.1449
(A,F)	2-7-9 2-7-23-4-6-13-17 1-3-4-5-12-17	185.1943 0.0009 9.6284
(C,A)	20-15-16 20-21-18-19 20-6-13-17-18-19	41.1204 0.0002 0.0002
(C,B)	20-15-16-7-23 20-21-18-19-1-3	0.3103 18.2270

	20-21-18-19-2-7-23	0.0001
(C,D)	20-5 20-21-10 20-15-16-7-8-10	0.0001 165.2806 34.5883
(C,E)	20-6 20-21-10-12 20-5-12	0.0003 130.6696 0.0118
(C,F)	20-21 20-6-13-17 20-15-16-7-9	866.4691 32.8049 0.0001
(D,A)	12-17-18-19 11-14-15-16 11-14-21-18-19	4.2589 398.6697 0.0025
(D,B)	11-14-15-16-7-23 12-17-18-19-1-3 12-17-18-19-2-7-23	12.2987 235.9942 1.4320
(D,E)	12	50.081
(D,F)	12-17 11-14-21 11-14-15-16-7-9	102.1407 0.0002 0.0007
(E,A)	13-17-18-19 13-14-15-16 13-14-21-18-19	7.9939 291.3877 0.0007
(E,B)	13-14-15-16-7-23 13-17-18-19-2-7-23 13-17-18-19-1-3	2.4920 122.9348 0.0007
(E,D)	13-14-21-10 13-14-15-16-7-8-10 13-17-18-19-1-3-4-5	0.7512 27.8201 0.0001
(E,F)	13-17 13-14-21 13-14-15-16-7-9	18.5937 0.0001 2.3578
(G,A)	22-19 22-23-16 22-8-10-15-16	31.6173 450.4278 0.0001
(G,B)	22-23 22-19-1-3 22-19-2-7-23	60.0299 0.0004 365.9059
(G,D)	22-8-10 22-23-4-5 22-19-1-3-4-5	100.7519 0.0007 27.8730
(G,E)	22-8-10-12 22-23-4-6 22-23-4-5-12	54.9218 0.0020 0.0009
(G,F)	22-9 22-19-1-3-4-5-12-7 22-23-4-6-13-7	19.0575 0.0001 0.0001

5. Stochastic User Equilibrium SUE Model

A consequence of the difference in travel cost perception is that the routes with higher costs than the least-cost routes are also utilized (it is still natural to assume that a more costly route has less probability to be chosen by a traveler than a less costly one). In order to allow for variations in the traveler's perception of travel cost, one could extend the basic UE model to include randomness in the travel cost function. The probability of choosing a specific route to give an actual cost then depends on this randomness [14].

Damberg and Sheffi ([9], [24]) extended the user equilibrium principle to the principle of Stochastic User Equilibrium (SUE) stated as follows:

"*In a stochastic user equilibrium network, no user believes s/he can improve his/her travel time by unilaterally changing routes*."

In a stochastic user equilibrium model, the deterministic traffic model is extended by including random components in the travel cost functions to account for the variations in the traveler's perception of travel cost. So, the logit-based stochastic user equilibrium model by Fisk in 1980, is as follows:

$$p(h,f):$$

$$Min \qquad T(f)=\frac{1}{\theta}\sum_{(p,q)\in C}\sum_{r\in R_{pq}} h_{pqr}\ln h_{pqr}+\sum_{a\epsilon A}\int_0^{f_a} t_a(s)ds$$

$$\text{s.t.} \qquad \sum_{r\in R_{pq}} h_{pqr}=d_{pq}, \qquad \forall (p,q)\in C \tag{9}$$

$$\sum_{(p,q)\in C}\sum_{r\in R_{pq}} \delta_{pqra} h_{pqr}=f_a, \qquad \forall a\epsilon A$$

$$h_{pqr}\geq 0, \qquad \forall\, r\in R_{pq},\ (p,q)\in C,$$

where, h denotes the vector of route flows, the h_{pqr}, and the parameter θ is assumed to be nonnegative, with θ being the value of user's available information [9].

In (9), Fisk defined $x\ln x$ to be zero at $x=0$; also, it is known that $\log x$ is undefined at $x=0$ and is not differentiable at this point. This fact could pose some difficulties in the solution procedure as the condition is not considered in modeling or in the proposed solution methods ([6], [9], [13]).

Taylor's expansion of $xlax$ was considered in close neighborhoods of zero. Therefore, the real point (re) called "realmin", the smallest positive floating point number in MATLAB 7.8, was the center point of the Taylor's expansion. Using these initiatives, not only a constraint that was ignored before was considered, but also the problem was posed by extending the differentiability.

5.1. SUE model in new formulation

To overcome the mentioned difficulties and also to be able to solve large problems, Benders' decomposition method was applied due to its advantages. Paying attention to the first constraint of problems (4) and (5), it is obvious that the values of the variables lie between 0 and d_{pq}; and thus to increase the approximation accuracy, it is better to decrease the interval. Accordingly, $x_{p,q,r}=\frac{h_{p\,q\,r}}{d_{p\,q}}$ is defined. Then, substituting h_{pqr} by x_{pqr} changes the interval [0, d_{pq}] to [0,1]. Here, the function

$f(h) = h \ln h$ could be reformulated with respect to the variable x as $f(x) = xd \ln(xd) = dx \ln x + (d \ln d)\, x$.

Now, considering the first two terms of the Taylor's expansion of $x \ln x$, a linear approximation of this function can be obtained and thus $f(x)$ may be reformulated as

$$f(x) = d(1 + \ln(re) + \ln d)x - d(re).$$

Applying the above reformulations, the initial problem (9) is turned into

$$\begin{aligned} & P(x,f): \\ & \text{Min Z} = \frac{1}{\theta} \sum_{(p,q)\in C} \sum_{r\in R_{pq}} d_{pq}(1 + \ln(re) + \ln(d_{pq})) + \sum_{i=1}^{n} t_a^0 (1 + \\ & \beta(\tfrac{(3i-2)f_a}{2nC_a})^{\alpha}) f_a \\ & \text{s.t.} \quad \sum_{r\in R_{pq}} x_{pqr} = 1 ; \qquad \forall (p,q) \in C \\ & \qquad \sum_{(p,q)\in C} \sum_{r\in R_{pq}} \delta_{pqra} x_{pqr} d_{pq} = f_a \qquad \forall a \epsilon A \\ & \qquad 0 \le x_{pqr} \le 1 \quad , \qquad \forall r \in R_{pq} \ , \quad (p,q) \in C. \end{aligned} \tag{10}$$

These initiatives lead to a decrease in computations as shown by our numerical examples.

5.2. Benders' decomposition for SUE problem

The problem $p(x,f)$ contains two variables x_{pqr} and f_a . To apply the Benders' decomposition, first it is necessary to divide the variables and constraints into two groups. So the nonlinear part of the objective function containing the variable f_a represents the nonlinear part of the initial problem and the variable x_{pqr} with linear part of the objective function and the two constraints represent the linear part which should be dualized.

The initial master model $M(x, f, 0)$ can be stated as

$$\begin{aligned} & M(x, f, 0) \\ & \text{Min} \quad \sum_{a\epsilon A} \sum_{i=1}^{n} t_a^0 \left[1 + \beta \left(\frac{(3i-2)f_a}{2nC_a{}'}\right)^{\alpha} \right] . \frac{f_a}{n} \\ & \text{s.t.} \quad f_a \ge 0 \ , \ \forall a \epsilon A. \end{aligned} \tag{11}$$

Note that the initial master model does not yet contain any Benders' cuts. By solving this problem, the variable f_a can be obtained for all $a \in A$. Then, by replacing these values in the linear part of $p(x,f)$, this part will only contain the variable x_{pqr}.

By introducing three dual variables w_{pq} , π_a and z_{pqr} corresponding to the three constraints, the dual formulation of the linear part of $p(x,f)$ can be written as:

$$S(w\,,\pi\,,z\,|x,f):\ Max \quad \sum_{(p,q)\in C} w_{pq} + \sum_{a\in A} \pi_a\, f_a + \sum_{(p,q)\in C} \sum_{r\epsilon R_{pq}} z_{pqr} - d(re)$$

$$s.t.\ \ (wa + \pi\,\Delta + ZI)' \ \le \frac{1}{\theta} C'\,, \qquad \forall (p,q) \in C, r \in R_{pq}\,, a\epsilon A \tag{12}$$

$$w_{pq}\,, \pi_a\,, free\,, \qquad \forall (p,q) \in C, \qquad a\epsilon A$$

$$z_{pqr} \le 0 \quad , \qquad \forall (p,q) \in C\ ,\ \ r \in R_{pq}.$$

In this problem, the coefficients of the variables x_{pqr} in the first, second and third constraints of the linear part of problem *p(x, f)* are represented by A, Δ and I matrices, respectively; also, the matrix C demonstrates the coefficients of the variables in the objective function of the problem and the vectors w, π and z respectively contain the variables $w_{pq,}$ π_α and Z_{pqr}.

In each iteration of Benders' decomposition, a constraint (Benders' cut) is built and added to problem (12). This cut is directly derived from the objective function of the above sub-problem S(w,$\boldsymbol{\pi}$,z|x,f) which is evaluated at the solution (w, π, Z). This new constraint is

$$\sum_{(p,q)\in C} w_{pq} + \sum_{a\epsilon A} \pi_a f_a + \sum_{(p,q)\in C} \sum_{r\in R_{pq}} Z_{pqr} - d(re) \le m, \tag{13}$$

where $(w_{pq}, \pi_\alpha, Z_{pqr})$ are parameters and w and f_α are the variables.

After introducing the set B of generated Benders' cut, by adding the Benders' cut to the initial problem $M(x, f, 0)$, the following regular master problem $M(x, f, m)$ is obtained (note that the optimal dual variables w, π and Z are given an extra index $b \in B$ for each bender cut):

$$\text{Min} \quad \sum_{a\epsilon A} \sum_{i=1}^{n} t_a^0 \left[1 + \beta \left(\frac{(3i-2) f_a}{2nC_a}\right)^{\alpha}\right] . \frac{f_a}{n} + \frac{m}{n}$$

$$\text{s .t.} \quad \sum_{(p,q)\in C} w_{pqb} + \sum_{a\epsilon A} \pi_{ab} f_a + \sum_{(p,q)\in C} \sum_{r\in R_{pq}} Z_{pqr} - d(re) \le m, \tag{14}$$

$$\forall b\ \in B\ ,\ \ (p,q) \in C\ ,\ \ a\epsilon A,$$

$$f_a \ge 0 \quad , \qquad \forall a\epsilon A.$$

5.3. Numerical results of a case study for SUE

The Alsop and Charleworth model presented in Section 4 was solved by applying the new mentioned formulation, and the Benders' decomposition discussed above using MATLAB 7.8 solver. The algorithm stopped after 35 iterations. To obtain the optimal path flows, problem was encountered to be unbounded. Since using the extreme rays in such situations is not always effective (specially, when the problem is large scale), to speed up the computation, the obtained optimal edge values were applied to the problem $P(h, f)$. Therefore, the problem was tested only with the h_{qr} variables. By selecting $\theta = 1$, $\alpha = 4$ and $\beta = 0.15$ (as recommended in [9]) and then minimizing this problem, the optimal network link flow was determined in 270 second with the total optimal value of 28598.46. The obtained optimal values of the network link flows are given in Table 5.

Table 5. Optimal value of network link flows for SUE

Link	Flow	Link	Flow	Link	Flow
1	1015.04	9	220	17	533.17
2	564.96	10	571.02	18	400
3	995.04	11	400	19	607.71
4	425.04	12	400	20	1290
5	528.98	13	513.17	21	922.89
6	253.16	14	730	22	1250
7	564.96	15	739.99	23	852.29
8	534.96	16	1082.23		

Also, the optimal route flows and their travel times are given in Table 6 where the times are in minutes.

Table 6. Optimal value of network link flows for SUE

(Origin- Destination)	Route	Flow	Survey time (minutes)
(A,B)	1-3 2-7-23	247.699 0.277	0.279 0.686
(A,D)	1-3-4-5 2-7-8-10 2-7-23-4-5	344.143 346.538 0.374	1.235 0.960 1.647
(A,E)	1-3-4-6 2-7-8-10-12 2-7-23-4-6	29.095 0.010 0.033	1.235 0.960 1.647
(A,F)	2-7-9 2-7-23-4-6-13-17 1-3-4-5-12-17	199.203 0.010 0.01	0.686 1.509 1.506
(C,A)	20-15-16 20-21-18-19 20-6-13-17-18-19	40.478 0.010 0.010	0.686 1.098 1.509
(C,B)	20-15-16-7-23 20-21-18-19-1-3 20-21-18-19-2-7-23	0.0425 0.936 17.973	0.961 1.372 1.784
(C,D)	20-5 20-21-10 20-15-16-7-8-10	0.021 193.669 6.410	0.549 0.681 1.647
(C,E)	20-6 20-21-10-12 20-5-12	0.010 130.075 0.010	0.549 0.687 0.549
(C,F)	20-21 20-6-13-17 20-15-16-7-9	866.425 33.511 0.106	0.412 0.823 1.377
(D,A)	12-17-18-19 11-14-15-16 11-14-21-18-19	1.822 398.933 0.012	0.961 1.235 1.647
(D,B)	11-14-15-16-7-23 12-17-18-19-1-3 12-17-18-19-2-7-23	0.073 248.441 0.278	1.921 1.235 1.647
(D,E)	12	49.996	0.051

(D,F)	12-17 11-14-21 11-14-15-16-7-9	99.4876 0.563 0.010	0.274 0.961 1.921
(E,A)	13-17-18-19 13-14-15-16 13-14-21-18-19	1.465 299.30 0.010	0.921 1.235 1.647
(E,B)	13-14-15-16-7-23 13-17-18-19-2-7-23 13-17-18-19-1-3	0.035 127.960 0.676	1.921 1.647 1.235
(E,D)	13-14-21-10 13-14-15-16-7-8-10 13-17-18-19-1-3-4-5	0.143 29.997 0.010	1.235 2.195 2.195
(E,F)	13-17 13-14-21 13-14-15-16-7-9	19.948 0.106 0.209	0.274 0.961 1.921
(G,A)	22-19 22-23-16 22-8-10-15-16	341.406 449.070 0.01	0.274 0.686 1.372
(G,B)	22-23 22-19-1-3 22-19-2-7-23	100.001 100.007 170.073	0.412 0.549 0.960
(G,D)	22-8-10 22-23-4-5 22-19-1-3-4-5	170.458 0.001 170.458	0.686 1.372 1.509
(G,E)	22-8-10-12 22-23-4-6 22-23-4-5-12	60.249 0.01 0.01	0.686 1.372 1.372
(G,F)	22-9 22-19-1-3-4-5-12-7 22-23-4-6-13-7	20.341 0.01 0.01	0.412 1.784 1.649

6. Conclusions

To obtain a least cost network for supplying origin-destination demands, first Benders' decomposition method was described, and then its application was investigated for solving the problem. After introducing the traffic assignment problem as a basic transportation problem, user equilibrium and stochastic user equilibrium models (as efficient cases for description of users' selection trajectory in urban transportation) were explained. Then, Benders' decomposition method was employed to solve these problems. Using the Taylor expansion, two steps of approximation in the master problem, namely Benders' cut and posing a suitable sub-problem, a new solution method was presented for the UE and USE problems. The new model had the capability to handle all the conditions of the UE problem directly. Additionally, the presented solution approach for the model required less computing time in comparison with other methods, specially for large scale networks. Also, as shown by numerical tests, the proposed method possessed the inherent convergence properties of Benders' decomposition.

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