Non-Lipschitz Semi-Infinite Optimization Problems Involving Local Cone Approximation

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We study the non-smooth semi-infinite programming problem with inequality constraints. First, we consider the notions of local cone approximation $\mathcal{A}$ and $\mathcal{A} -$ subdifferential. Then, we derive the Karush-Kuhn-Tucker optimality conditions under the Abadie and the Guignard constraint qualifications.

Key words: semi-infinite programming problem, constraint qualification, optimality condition, local cone approximation.

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1. Introduction

Semi-infinite programming (SIP) is an active field of research in applied mathematics. This is due to the fact that several engineering problems lead to SIP, such as robotics, mathematical physics, optimal control, transportation problems, Chebyshev approximation, etc., see [4,7,14].

Here, we use the concept of local cone approximation and the associated subdifferential in order to construct optimality conditions for semi-infinite problems with inequality constraints. The problem is defined as follows:

$$\text{SIP:} \quad \inf_{x \in M} \varphi(x)$$

with the feasible set defined as

$$M := \{x \in \mathbb{R}^n \mid \psi_j(x) \leq 0, \quad \forall j \in J\},$$

Where $J \neq \emptyset$ is an arbitrary set, not necessarily finite. All the functions $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{\pm \infty\}$ and $\psi_j : \mathbb{R}^n \to \mathbb{R} \cup \{\pm \infty\}$, for $j \in J$, are assumed to be finite and continuous at a certain point $\hat{x}$.

The Karush-Kuhn-Tucker (KKT) necessary conditions are definitely among the most important results in modern optimization theory. If the set $J$ is finite, necessary conditions of KKT type for optimality can be established under various constraint qualifications; for instance, the Guignard and the Abadie qualifications, which are denoted by GCQ and ACQ, respectively. For various CQs in smooth and nonsmooth see [3].

Necessary optimality conditions for SIP problem have been studied by several authors; e.g., see [4, 14] in the linear case, [2,4,16,17,18] in the convex case, [8] in the smooth case, and [10,11,12,20] in the locally Lipshitz case.

We observe that since ACQ and GCQ are equivalent for convex problems, the GCQ was not defined for convex SIP problems. The ACQ and related constraint qualifications for the convex SIP problem have been studied in several papers; e.g., see [13,19]. In [10] and [11] respectively introduced...

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the GCQ and the ACQ for a non-differentiable SIP problem with locally Lipschitz emerging functions.

We point out that almost all of the references cited above are restricted to differentiability, convexity or Lipschitzity assumptions. Thus, it should be useful and interesting to study optimality conditions for non-smooth and non-Lipschitz SIP problems. The first aim of our work here is to establish the KKT type necessary conditions for a SIP problem without any regularity assumptions via local cone approximation.

On the other hand, convexity is one of the most frequently used hypotheses in optimization theory. It is usually used to obtain sufficiency for conditions that are only necessary, as with the classical KKT conditions in nonlinear programming. Since the convexity assumptions are often not satisfied in real-world models, several classes of functions are defined for the purpose of weakening the limitations of convexity. Following this idea, Hanson [6] and also Graven [5] introduced the class of invex functions. In [15], one can find several extensions, references, and applications of smooth invex functions. On this line, there has been a very popular growth and applications of invexity theory to locally Lipschitz functions, with derivative replaced by the Clarke generalized gradient [3].

The KKT type sufficient optimality conditions under invexity assumption for a non-smooth SIP problem with Lipschitzian data was established in [11]. We observe that the sufficiency results stated in [11] are based on the Clarke subdifferential, and they require the convexity of Clarke generalized directional derivative. Another aim of our work is to introduce two versions of invexity via local cone approximation, and give general sufficient optimality conditions for non-Lipschitz SIP problem without requiring the convexity of the directional derivatives.

The rest of the paper is organized as follows. In Section 2, we present basic definitions as well as some preliminary results. In Section 3, we consider the Abadie and the Guignard constraint qualifications for a non-Lipschitz SIP problem. Next, these constraint qualifications are used to obtain necessary optimality conditions. In Section 4, we introduce two forms of invex functions based on the local cone approximation. Then, we apply invexity to derive sufficient optimality conditions for non-Lipschitz SIP problems.

2. Preliminaries

Given a nonempty set $S \subseteq \mathbb{R}^n$, we denote by $\overline{S}$, $\text{int}(S)$, $\text{conv}(S)$, and $\text{cone}(S)$, the closure of $S$, the interior of $S$, convex hull and convex cone (containing the origin) generated by $S$, respectively. The polar cone of $S$ is defined by

$$S^\ominus := \{ d \in \mathbb{R}^n \mid \langle x, d \rangle \leq 0, \ \forall \ x \in S \},$$

where $\langle \cdot,\cdot \rangle$ stands for the standard inner product in $\mathbb{R}^n$. Notice that $S^\ominus$ is always a closed convex cone. The bipolar theorem states that $(S^\ominus)^\ominus = \overline{\text{cone}(S)}$, where $\overline{\text{cone}(S)}$ denotes the closed convex cone of $S$ (see [9]).

**Definition 2.1.** A set-valued mapping $A:2^{\mathbb{R}^n} \times \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$ is called a local cone approximation, if to each set $S \subseteq \mathbb{R}^n$ and to each point $x \in \mathbb{R}^n$, a cone $A(S,x)$ is associated with the following properties:

- $A(S-x_0,0) = A(S,x_0)$.
- $A(T(S),T(x_0)) = T(A(S,x_0))$, for each nonsingular linear mapping $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$.
- $A(S \cap U^{x_0}, x_0) = A(S,x_0)$, for each neighborhood $U^{x_0}$ of $x_0$.
- $A(S,x_0) = \emptyset$ \quad $\forall \ x_0 \notin \overline{S}$.
\(A(S, x_0) = \mathbb{R}^n \quad \forall x_0 \in \text{int}(S)\).

\(A(S, x_0) + C \subseteq A(S, x_0)\), for each cone \(C \subseteq \mathbb{R}^n\) with \(S + C \subseteq S\).

Two important examples for local cone approximation are the contingent cone \(\mathcal{T}(S, x)\), and the interior directions cone \(\mathcal{J}(S, x)\), with the following definitions:

\(\mathcal{T}(S, x_0) := \{z \in \mathbb{R}^n | \exists t_k \downarrow 0, \exists z_k \to z, \text{ such that } x_0 + t_k z_k \in S, \forall t \in \mathbb{N}\}\),

\(\mathcal{J}(S, x_0) := \{z \in \mathbb{R}^n | \exists U^z, \exists \lambda > 0, \forall t \in 0, \lambda), \forall z^* \in U^z, x_0 + tz^* \in S\}\).

We observe that \(U^z\), above, denotes a neighborhood of \(z\).

**Definition 2.2.** Let \(A(., .)\) be a local cone approximation. If \(f\) is a function from \(\mathbb{R}^n\) to \(\mathbb{R}\) and \(x_0 \in \mathbb{R}^n\). Then

- the extended real-valued function \(f^A (x_0; .) : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{-\infty\}\), defined by
  \(f^A (x_0; v) = \inf \{r \in \mathbb{R} | (v, r) \in \Lambda(\text{epi} f, (x_0, f(x_0)))\}\),
  is called the \(\Lambda\)-directional derivative of \(f\) at \(x_0\), where \(\text{epi} f\) denotes the epigraph of \(f\), i.e.,
  \(\text{epi} f = \{(x, r) \in \mathbb{R}^n \times \mathbb{R} | f(x) \leq r\}\).
- the set
  \(\partial_A f(x_0) := \{\xi \in \mathbb{R}^n | \langle \xi, v \rangle \leq f^A (x_0; v), \forall v \in \mathbb{R}^n\}\),
  is called the \(\Lambda\)-subdifferential of \(f\) at \(x_0\).

We observe that the \(\Lambda\)-subdifferential is always a closed convex set. To get more details, we need local cone approximation, \(A(., .)\), to be convex, i.e., \(A(S, x_0)\) be convex for each \((S, x_0) \in 2^{\mathbb{R}^n} \times \mathbb{R}^n\). An important example for convex local cone approximation is the Clark tangent cone \(\mathcal{TC}(S, x)\), defined as follows:

\(\mathcal{TC}(S, x_0) := \{z \in \mathbb{R}^n | \forall x_k \rightarrow x_0, \forall t_k \downarrow 0, \exists z_k \rightarrow z, \text{ such that } x_k + t_k z_k \in S, \forall t \in \mathbb{N}\}\),

where \(x_k \rightarrow x_0\) means that \(x_k\) is a sequence in \(S\) converging to \(x_0\).

In the following theorem we summarize some important properties of the \(\Lambda\)-directional derivative and the \(\Lambda\)-subdifferential, taken from [3], which are to be used later on.

**Theorem 2.3.** Let \(A(., .)\) be a convex cone approximation. Then, the following assertions hold.

(a) The function \(v \rightarrow f^A (x_0; v)\) is convex on \(\mathbb{R}^n\).

(b) If \(f^{\text{int}(\Lambda)}(x_0; 0) = 0\), then

1) \(\partial_A f(x_0)\) is a compact set.
2) \(f^A (x_0; v) = \max \{\langle \xi, v \rangle | \xi \in \partial_A f(x_0)\}\), \(\forall v \in \mathbb{R}^n\).
3) \(\partial_{\text{int}(\Lambda)} f(x_0) = \partial_A f(x_0) = \partial_{\text{epi}} f(x_0)\).

## 3. Necessary Optimality Conditions

We first prove an important lemma.

**Lemma 3.1.** Suppose that \(\hat{x} \in \mathbb{R}^n\) is a local minimizer of SIP problem. Let \(\Lambda(., .)\) be a cone approximation with \(\text{int}(\Lambda(M, \hat{x})) \subseteq \mathcal{J}(M, \hat{x})\). Then,
\( \phi^{\text{int}}(\hat{x}; v^*) \geq 0 \), \quad \forall v^* \in \mathcal{J}(M, \hat{x}). 

**Proof:** If \((v, r) \in \mathcal{J}(\text{epi } \phi, (\hat{x}, \phi(\hat{x})))\), then by definition of \( \mathcal{J} \) we obtain:

\[
\exists U^{(v,r)}, \exists \alpha > 0, \forall t \in (0,\alpha), \forall (\bar{v}, \bar{r}) \in U^{(v,r)}: \quad (\hat{x}, \phi(\hat{x})) + t(\bar{v}, \bar{r}) \in \text{epi } \phi,
\]

where \( U^{(v,r)} \) is a neighborhood of \((v, r)\). Since \( U^{(v,r)} = U^v \times (r - \varepsilon, r + \varepsilon) \) for some neighborhood \( U^v \) of \( v \), for some \( \varepsilon > 0 \) the last relation is equivalent to:

\[
\exists \varepsilon > 0, \exists U^v, \exists \alpha > 0, \forall t \in (0,\alpha), \forall \bar{v} \in U^v, \forall \bar{r} \in (r - \varepsilon, r + \varepsilon): \quad \phi(\hat{x} + t \bar{v}) \leq \phi(\hat{x}) + t \bar{r},
\]

which yields:

\[
\exists \varepsilon > 0, \exists U^v, \exists \alpha > 0, \forall t \in (0,\alpha), \forall \bar{v} \in U^v: \quad \frac{\phi(\hat{x} + t \bar{v}) - \phi(\hat{x})}{t} \leq r - \varepsilon.
\]

Let \( \bar{v} \in \mathbb{R}^n \) be arbitrary. The above inequality, the assumption of \( \text{int}(\Lambda(M, \hat{x})) \subseteq \mathcal{J}(M, \hat{x}) \), and definition of \( \Lambda \)-directional derivative imply that

\[
\phi^{\text{int}}(\hat{x}; \bar{v}) = \inf \{ r \in \mathbb{R} \mid (\bar{v}, r) \in \text{int } \Lambda(\text{epi } \phi, (\hat{x}, f(\hat{x}))) \}
\]

\[
\geq \inf \{ r \in \mathbb{R} \mid (\bar{v}, r) \in \mathcal{J}(\text{epi } \phi, (\hat{x}, f(\hat{x}))) \}
\]

\[
\geq \inf \left\{ r \in \mathbb{R} \mid \exists \varepsilon > 0, \exists U^\bar{v}, \exists \alpha > 0, \forall t \in (0,\alpha), \forall \bar{v} \in U^\bar{v}: \frac{\phi(\hat{x} + t \bar{v}) - \phi(\hat{x})}{t} \leq r - \varepsilon \right\}
\]

\[
\geq \inf \left\{ r \in \mathbb{R} \mid \limsup_{t \to 0^+} \frac{\phi(\hat{x} + t \bar{v}) - \phi(\hat{x})}{t} < r \right\} = \limsup_{t \to 0^+} \frac{\phi(\hat{x} + t \bar{v}) - \phi(\hat{x})}{t}. \quad (1)
\]

Now, let \( v^* \in \mathcal{J}(M, \hat{x}) \). Then, there exists a sequence \( \{v_k\} \subseteq \mathbb{R}^n \) converging to \( v^* \) and a positive sequence \( \{t_k\} \subseteq \mathbb{R}_+ \) converging to zero such that \( \hat{x} + t_k v_k \in M \), for each \( k \in \mathbb{N} \). Since \( \hat{x} \) is a local minimum of \( \phi \) on \( M \), by virtue of (1) we get

\[
\phi^{\text{int}}(\hat{x}; v^*) \geq \limsup_{k \to \infty} \frac{\phi(\hat{x} + t_k v_k) - \phi(\hat{x})}{t_k} \geq 0. \quad \Box
\]

For each \( x_0 \in M \), with convention \( \bigcup_{j \in \emptyset} X_j = \emptyset \), set

\[
J(x_0): = \{ j \in J \mid \psi_j(x_0) = 0 \},
\]

\[
\mathcal{B}_A(x_0): = \bigcup_{j \in J(x_0)} \partial_A \psi_j(x_0),
\]

\[
\mathcal{D}_A(x_0): = \text{cone}(\mathcal{B}_A(x_0)).
\]

**Definition 3.2.** The SIP problem is said to satisfy the \( \Lambda \)-generalized Abadie constraint qualification (\( \Lambda \)-GACQ) at \( x_0 \in M \), if
Theorem 3.3. Let $\hat{x} \in M$ be a local minimizer of SIP problem, and $\Lambda(\ldots)$ be a convex cone approximation with $\text{int}(\Lambda(M, \hat{x})) \subseteq \mathcal{J}(M, \hat{x})$, and $\varphi^{\text{int}}\Lambda(\hat{x}; 0) = 0 = \psi^{\text{int}}\Lambda(\hat{x}; 0)$, for all $j \in J(\hat{x})$. If $\Lambda$-GACQ is satisfied at $\hat{x}$, then

$$0 \in \partial_{\Lambda} \varphi(\hat{x}) + \overline{\mathcal{D}_{\Lambda}(\hat{x})}.$$  

Proof. Let $v \in \left(\overline{\mathcal{D}_{\Lambda}(\hat{x})}\right)^{\ominus}$. Because of $\left(\overline{\mathcal{D}_{\Lambda}(\hat{x})}\right)^{\ominus} = \left(\mathcal{B}_{\Lambda}(\hat{x})\right)^{\ominus}$ and satisfaction of $\Lambda$-GACQ, we conclude that $\varphi^{\text{int}}\Lambda(\hat{x}; v) \geq 0$, in view of Lemma 3.1. By assumption of $\varphi^{\text{int}}\Lambda(\hat{x}; 0) = 0$, we thus obtain that the optimization problem

$$\min \varphi^{\text{int}}\Lambda(\hat{x}; \cdot)(v)$$  

s.t. $v \in \left(\overline{\mathcal{D}_{\Lambda}(\hat{x})}\right)^{\ominus}$

has a local solution at $\hat{v} = 0$. Since the objective function and the constraint set of the above problem are convex (see Theorem 2.3(a)), by the necessary optimality condition for convex optimization problems (e.g., see [9]), we get

$$0 \in \partial \left(\varphi^{\text{int}}\Lambda(\hat{x}; \cdot)\right)(0) + N \left(\left(\overline{\mathcal{D}_{\Lambda}(\hat{x})}\right)^{\ominus}, 0\right),$$  

where $\partial$ and $N$ respectively denote the subdifferential and the normal cone in the sense of convex analysis.

At this point, we observe that $N \left(\left(\overline{\mathcal{D}_{\Lambda}(\hat{x})}\right)^{\ominus}, 0\right) = \overline{\mathcal{D}_{\Lambda}(\hat{x})}$, by the well-known bipolar theorem. Now, regarding the definition of $\partial_{\Lambda}$, and the assumption of $\varphi^{\text{int}}\Lambda(\hat{x}; 0) = 0$, we also have that

$$\partial \left(\varphi^{\text{int}}\Lambda(\hat{x}; \cdot)\right)(0) = \{\xi \in \mathbb{R}^{n} | \varphi^{\text{int}}\Lambda(\hat{x}; w) - \varphi^{\text{int}}\Lambda(\hat{x}; 0) \geq \varphi^{\text{int}}\Lambda(\hat{x}; \cdot) \forall w \in \mathbb{R}^{n}\}$$

$$= \{\xi \in \mathbb{R}^{n} | \varphi \varphi^{\text{int}}\Lambda(\hat{x}; w) \geq \varphi^{\text{int}}\Lambda(\hat{x}; \cdot) \forall w \in \mathbb{R}^{n}\}$$

$$= \partial_{\text{int} \Lambda} \varphi(\hat{x}).$$

Hence, from (2) and Theorem 2.3(b3) we obtain:

$$0 \in \partial_{\text{int} \Lambda} \varphi(\hat{x}) + \overline{\mathcal{D}_{\Lambda}(\hat{x})} = \partial_{\Lambda} \varphi(\hat{x}) + \overline{\mathcal{D}_{\Lambda}(\hat{x})}. \qed$$

Now, we introduce another constraint qualification being weaker than $\Lambda$-GACQ.

Definition 3.4. Let $x_{0} \in M$. We say that the SIP problem satisfies in the $\Lambda$-generalized Guignard constraint qualification ($\Lambda$-GGCQ) at $x_{0}$, if

$$\left(\mathcal{B}_{\Lambda}(x_{0})\right)^{\ominus} \subseteq \overline{\text{conv}}(\mathcal{J}(M, x_{0})).$$
Owing to the definitions, it follows that $\Lambda$-GACQ implies $\Lambda$-GGCQ. Moreover, for the case that all emerging functions $\psi_j, (j \in J)$ are convex, these CQs are equivalent.

**Lemma 3.5.** Suppose that $\hat{x} \in \mathbb{R}^n$ is a local minimizer of SIP problem and $\Lambda(\cdot, \cdot)$ is a cone approximation with $\text{int}(\Lambda(M, \hat{x})) \subseteq J(M, \hat{x})$. If the mapping $v \rightarrow \varphi^{\text{int} \Lambda}(\hat{x}; v)$ is linear, then

$$
\varphi^{\text{int} \Lambda}(\hat{x}; \bar{v}) \geq 0, \quad \forall \bar{v} \in \text{conv}(J(M, \hat{x})).
$$

**Proof.** First, suppose that $v \in \text{conv}(J(M, \hat{x}))$. Then, there exist scalars $\alpha_1, \ldots, \alpha_s \geq 0$ and vectors $v_1^*, \ldots, v_s^* \in J(M, \hat{x})$ such that $\sum_{k=1}^s \alpha_k = 1$ and $v = \sum_{k=1}^s \alpha_k v_k^*$. Using the linearity of $\varphi^{\text{int} \Lambda}(\hat{x}; \cdot)$ and Lemma 3.1, we get

$$
\varphi^{\text{int} \Lambda}(\hat{x}; v) = \varphi^{\text{int} \Lambda}(\hat{x}; \sum_{k=1}^s \alpha_k v_k^*) = \sum_{k=1}^s \alpha_k \varphi^{\text{int} \Lambda}(\hat{x}; v_k^*)
$$

$$
\geq 0. \quad (3)
$$

Now, let $\bar{v} \in \text{conv}(J(M, \hat{x}))$. Then, there exists a sequence $\{v_k\}$ in $\text{conv}(J(M, \hat{x}))$ converging to $\bar{v}$. Taking into consideration the continuity of $\varphi^{\text{int} \Lambda}(\hat{x}; \cdot)$ and the validity of (3), the proof is complete. □

The proof of the following theorem is omitted, since it is similar to that of Theorem 3.3. We observe that the linearity of $v \rightarrow \varphi^{\text{int} \Lambda}(\hat{x}; v)$ obviously implies $\varphi^{\text{int} \Lambda}(\hat{x}; 0) = 0$.

**Theorem 3.6.** Let $\hat{x} \in M$ be a local minimum point of SIP, and $\Lambda(\cdot, \cdot)$ be a convex cone approximation with $\text{int}(\Lambda(M, \hat{x})) \subseteq J(M, \hat{x})$, and $\psi_j^{\text{int} \Lambda}(\hat{x}; 0) = 0$ for all $j \in J(\hat{x})$, and the mapping $v \rightarrow \varphi^{\text{int} \Lambda}(\hat{x}; v)$ is linear. If the $\Lambda$-GGCQ is satisfied at $\hat{x}$, then it holds

$$
0 \in \partial \varphi(\hat{x}) + D_{\Lambda}(\hat{x}).
$$

Now, we can formulate the main result of this section.

**Theorem 3.7.** Let $\hat{x} \in M$ be a local minimizer of SIP problem, and $\Lambda(\cdot, \cdot)$ be a convex cone approximation with $\text{int}(\Lambda(M, \hat{x})) \subseteq J(M, \hat{x})$, and $\psi_j^{\text{int} \Lambda}(\hat{x}; 0) = 0$ for all $j \in J(\hat{x})$. Furthermore, suppose that one of the following conditions holds:

1) $\Lambda$-GACQ is satisfied at $\hat{x}$, and $\varphi^{\text{int} \Lambda}(\hat{x}; 0) = 0$.

2) $\Lambda$-GGCQ is satisfied at $\hat{x}$, and the mapping $v \rightarrow \varphi^{\text{int} \Lambda}(\hat{x}; v)$ is linear.

If $D_{\Lambda}(\hat{x})$ is a closed set, then there exist a finite index set $J_0 \subseteq J(\hat{x})$, and non-negative scalars $\lambda_j > 0$, for $j \in J_0$, such that

$$
0 \in \partial \varphi(\hat{x}) + \sum_{j \in J_0} \lambda_j \partial \psi_j(\hat{x}).
$$

**Proof.** The result is an immediate consequence of Theorems 3.3 and 3.6, and the following equality for convex sets $X_\gamma, \gamma \in \Gamma$ (e.g., see [9]):
cone \left( \bigcup_{\gamma \in \Gamma} X_\gamma \right) = \{ \sum_{\gamma \in \Gamma_0} a_\gamma | \Gamma_0 \text{ is finite subset of } \Gamma, \ a_\gamma \in X_\gamma, \ \tau_{\gamma} \geq 0 \}\). □

If we choose \( A(\cdot, \cdot) = \mathcal{C}(\cdot, \cdot) \), we can see that the inclusion \( int \mathcal{C}(M, x_0) \subseteq J(M, x_0) \) holds. Moreover, regarding the associated \( A \)-directional derivatives, we have for a function \( f \) which is Lipschitzian around \( x_0 \) (e.g., see [15]), the relation \( f^{int \mathcal{C}}(x_0; 0) = 0 \) and

\[
f^{\mathcal{C}}(x_0; \nu) = \limsup_{t \downarrow 0, x \rightarrow x_0} \frac{f(x + t \nu) - f(x)}{t}, \quad \forall \nu \in \mathbb{R}^n.
\]

Thus, for the Lipschitzian semi-infinite problem (LSIP) with locally Lipschitz emerging functions \( \varphi \) and \( \psi_j \), for \( j \in J \), Theorem 3.7 can be formulated in a simple way as is proven in [10,11].

**Theorem 3.8.** Let \( \hat{x} \in M \) be a local minimizer of LSIP, and that one of the following conditions holds:

1) \( \mathcal{C} \)-GACQ is satisfied at \( \hat{x} \).
2) \( \mathcal{C} \)-GGCQ is satisfied at \( \hat{x} \), and the mapping \( \nu \rightarrow \varphi^{int \mathcal{C}}(\hat{x}; \nu) \) is linear.

If \( \mathcal{D}_{\mathcal{C}}(\hat{x}) \) is a closed set, then there exist a finite index set \( J_0 \subseteq J(\hat{x}) \) and non-negative scalars \( \lambda_j > 0 \), for \( j \in J_0 \), such that

\[
0 \in \partial_{\mathcal{C}} \varphi(\hat{x}) + \sum_{j \in J_0} \lambda_j \partial_{\mathcal{C}} \psi_j(\hat{x}).
\]

4. **Sufficient Optimality Conditions**

In this section, we investigate sufficient conditions for optimality of the SIP problem. The following concepts are to be used.

**Definition 4.1.** Let \( A(\cdot, \cdot) \) be a local cone approximation, \( f \) be a function from \( \mathbb{R}^n \) to \( \mathbb{R} \), and \( x_0 \in \mathbb{R}^n \). Then, \( f \) is said to be

- **\( A \)-subdifferential invex (\( A \)-S-invex)** at \( x_0 \), if there exists a function \( \eta: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that

\[
f(x) - f(x_0) \geq \langle \xi, \eta(x, x - x_0) \rangle, \quad \forall \xi \in \partial_A f(x_0), \ \forall x \in \mathbb{R}^n.
\]

The function \( \eta \) is said to be the kernel of \( A \)-invexity.

- **\( A \)-directional derivative invex (\( A \)-DD-invex)** at \( x_0 \), if there exists a function \( \eta: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that

\[
f(x) - f(x_0) \geq f^A(x_0; \eta(x, x_0)), \quad \forall x \in \mathbb{R}^n.
\]

The function \( \eta \) is said to be the kernel of \( A \)-DD-invexity.

**Remark 4.2.** If \( f^A(x_0; 0) = 0 \), then the \( A \)-S-invexity and the \( A \)-DD-invexity of \( f \) at \( x_0 \) are equivalent in view of Theorem 2.3(b)2.
Remark 4.3. Let \( A(\ldots) \) be a local cone approximation, and \( x_0 \in \mathbb{R}^n \). If \( f_1: \mathbb{R}^n \to \mathbb{R} \) and \( f_2: \mathbb{R}^n \to \mathbb{R} \) are \( A \)-DD-invex functions at \( x_0 \) with the same kernel \( \eta \), then it is easy to prove that

1. for each \( \lambda > 0 \), the function \( \lambda f_1 \) is \( A \)-DD-invex at \( x_0 \) with the kernel \( \eta \).
2. \( f_1 + f_2 \) is a \( A \)-DD-invex function at \( x_0 \) with the kernel \( \eta \), if

\[
(f_1 + f_2)(x_0; \eta(x_0, x)) \leq f_1(x_0; \eta(x_0, x)) + f_2(x_0; \eta(x_0, x)), \quad \forall \ x \in \mathbb{R}^n.
\]

Theorem 4.4. Let \( A(\ldots) \) be a local cone approximation, and \( \hat{x} \in M \). Assume further that there exist a finite index set \( J_0 \subseteq J(\hat{x}) \) and non-negative scalars \( \lambda_j > 0 \) such that

\[
0 \in \partial_A \left( \varphi + \sum_{j \in J_0} \lambda_j \psi_j \right)(\hat{x}).
\]

If \( \varphi + \sum_{j \in J_0} \lambda_j \psi_j \) is a \( A \)-DD-invex function at \( \hat{x} \), then \( \hat{x} \) is a global minimum for SIP problem.

Proof. We claim that \( \hat{x} \) is a global minimizer of \( \varphi + \sum_{j \in J_0} \lambda_j \psi_j \) on \( \mathbb{R}^n \). On the contrary, suppose there exists \( x^* \in \mathbb{R}^n \) such that

\[
\left( \varphi + \sum_{j \in J_0} \lambda_j \psi_j \right)(x^*) < \left( \varphi + \sum_{j \in J_0} \lambda_j \psi_j \right)(\hat{x}).
\]

Thus, in view of \( A \)-DD-invexity assumption of \( \varphi + \sum_{j \in J_0} \lambda_j \psi_j \), we have

\[
\left( \varphi + \sum_{j \in J_0} \lambda_j \psi_j \right)^A(x^*; \eta(\hat{x}, x^*)) < 0. \tag{5}
\]

On the other hand, because of

\[
0 \in \partial_A \left( \varphi + \sum_{j \in J_0} \lambda_j \psi_j \right)(\hat{x}) = \left\{ \xi \in \mathbb{R}^n \mid \langle \xi, v \rangle \leq \left( \varphi + \sum_{j \in J_0} \lambda_j \psi_j \right)^A(\hat{x}; v), \quad \forall v \in \mathbb{R}^n \right\},
\]

by taking \( v = \eta(\hat{x}, x^*) \), we obtain:

\[
0 = \langle 0, \eta(\hat{x}, x^*) \rangle \leq \left( \varphi + \sum_{j \in J_0} \lambda_j \psi_j \right)^A(x^*; \eta(\hat{x}, x^*)).
\]

This contradicts (5), and the claim is established.

Now, since \( J_0 \subseteq J(\hat{x}) \), owing to the above result, we have
\[ \varphi(x) \geq \varphi(x) + \sum_{j \in J_0} \lambda_j \psi_j(x) \geq \varphi(\hat{x}) + \sum_{j \in J_0} \lambda_j \psi_j(\hat{x}) = \varphi(\hat{x}), \quad \forall x \in M. \]

Consequently, \( \hat{x} \) is a global optimal solution of SIP problem. \( \square \)

**Remark 4.5.** Using Remark 4.3, there is no implication between the \( \Lambda \)-DD-invexity of \( \varphi \) and \( \psi_j \), for \( j \in J_0 \), and the \( \Lambda \)-DD-invexity of \( \varphi + \sum_{j \in J_0} \lambda_j \psi_j \) in Theorem 4.4.

**Theorem 4.6.** Let \( \Lambda(\ldots) \) be a local cone approximation, and \( \hat{x} \in M \). Assume further that there exist a finite index set \( J_0 \subseteq J(\hat{x}) \) and non-negative scalars \( \lambda_j > 0 \) such that

\[ 0 \in \partial_A \varphi(\hat{x}) + \sum_{j \in J_0} \lambda_j \partial_A \psi_j(\hat{x}). \]

If \( \varphi \) and \( \psi_j \), for \( j \in J_0 \), are \( \Lambda \)-S-invex functions at \( \hat{x} \) with the same kernel \( \eta \), then \( \hat{x} \) is a global minimum of SIP problem.

**Proof.** Suppose on the contrary that there exists \( x^* \in \mathbb{R}^n \) such that \( \varphi(x^*) < \varphi(\hat{x}) \). Thus, by \( \Lambda \)-S-invexity assumption on \( \varphi \), we have

\[ \langle \xi, \eta(\hat{x}, x^*) \rangle < 0 \quad \forall \xi \in \partial_A \varphi(\hat{x}). \tag{6} \]

On the other hand, because of \( 0 \in \partial_A \varphi(\hat{x}) + \sum_{j \in J_0} \lambda_j \partial_A \psi_j(\hat{x}) \), there exist \( \hat{\xi} \in \partial_A \varphi(\hat{x}) \) and \( \hat{\xi}_j \in \partial_A \psi_j(\hat{x}), \) for \( j \in J_0 \), such that

\[ \hat{\xi} + \sum_{j \in J_0} \lambda_j \hat{\xi}_j = 0. \]

Multiplying by \( \eta(\hat{x}, x^*) \), we get

\[ \langle \hat{\xi}, \eta(\hat{x}, x^*) \rangle + \sum_{j \in J_0} \lambda_j \langle \hat{\xi}_j, \eta(\hat{x}, x^*) \rangle = 0. \]

From the above equality and in virtue of (6), we obtain \( \sum_{j \in J_0} \lambda_j \langle \hat{\xi}_j, \eta(\hat{x}, x^*) \rangle > 0 \), and hence \( \langle \hat{\xi}_{j_0}, \eta(\hat{x}, x^*) \rangle > 0 \), for some \( j_0 \in J_0 \). Now, \( \Lambda \)-S-invexity of \( \psi_{j_0} \) implies

\[ \psi_{j_0}(x^*) = \psi_{j_0}(x^*) - \psi_{j_0}(\hat{x}) \geq \langle \hat{\xi}_{j_0}, \eta(\hat{x}, x^*) \rangle > 0. \]

This is a contradiction, since we assumed \( x^* \in M \). Thus, the result is established. \( \square \)

**Remark 4.7.** Since the subadditive formula is not valid for \( \Lambda \)-subdifferential, i.e., \( \partial_A (f_1 + f_2)(x_0) \not\subseteq \partial_A f_1(x_0) + \partial_A f_2(x_0) \), in general, there is no any implication between theorems 4.4 and 4.6.

**References**


