

“Research Note”

A New Infeasible Interior-Point Algorithm with Full Nesterov-Todd Step for Semi-Definite Optimization

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We present a new full Nesterov and Todd step infeasible interior-point algorithm for semi-definite optimization. The algorithm decreases the duality gap and the feasibility residuals at the same rate. In the algorithm, we construct strictly feasible iterates for a sequence of perturbations of the given problem and its dual problem. Every main iteration of the algorithm consists of a feasibility step and some centering steps. We show that the algorithm converges and finds an approximate solution in polynomial time. A numerical study is made for the numerical performance. Finally, a comparison of the obtained results with those by other existing algorithms is made.

Keywords: *Infeasible interior-point algorithm, Semi-definite optimization, Full Nesterov-Todd step, Polynomial time complexity.*

Manuscript received on 20/6/2012 and accepted for publication on 9/3/2013.

1. Introduction

Here, we use the matrix inner product $A \bullet B = \text{Tr}(A^T B)$. Let S^n, S_+^n and S_{++}^n denote the cone of symmetric, symmetric positive semi-definite and symmetric positive definite $n \times n$ matrices, respectively, $\|\cdot\|_F$ and $\|\cdot\|_2$ denote the Frobenius norm and the spectral norm for matrices, respectively. For any $Q \in S_{++}^n$, let the expression $Q^{\frac{1}{2}}$ denote its symmetric square root. For any $V \in S^n$, let $\lambda_{\min}(V)$ and $\lambda_{\max}(V)$ denote the smallest eigenvalue and largest eigenvalue of V , respectively.

We consider the semidefinite optimization (SDO) problem in the standard form,

$$\begin{aligned} \min & C \bullet X \\ \text{s. t.} & A_i \bullet X = b_i, i = 1, 2, \dots, m \\ & X \succeq 0, \end{aligned} \quad (P)$$

where $A_i \in S^n, b = (b_1, b_2, \dots, b_m)^T \in R^m, C \in S^n$ and $X \succeq 0$ means that X is positive semi-definite. Moreover, the matrices A_i are linearly independent. The dual problem of (P) is given by

$$\begin{aligned} & \max b^T y \\ & \text{s. t. } \sum_{i=1}^m y_i A_i + S = C, \quad (D) \\ & \quad \quad \quad S \succcurlyeq 0, \end{aligned}$$

with $y \in R^m$ and $S \in S^n$.

In 1984, Karmarkar [5] proposed a polynomial-time algorithm, the so-called interior-point methods (IPMs) for solving linear optimization (LO) problems. This method was extended to SDO, with an important contribution made by Nesterov and Todd [13, 17]. For a comprehensive study of IPMs for SDO, we refer to [2, 4, 20, 15]. We assume that (P) and (D) satisfy the interior-point condition (IPC), i.e.; there exist X^0, y^0 and S^0 such that

$$A_i \bullet X^0 = b_i, \quad i = 1, 2, \dots, m, \quad \sum_{i=1}^m y_i^0 A_i + S^0 = C, \quad X^0 \succ 0, S^0 \succ 0,$$

where $X^0 \succ 0$ means that X^0 is positive definite.

It is well known that finding an optimal solution of (P) and (D) is equivalent to solving the following system:

$$\begin{aligned} A_i \bullet X &= b_i, \quad i = 1, 2, \dots, m, \quad X \succcurlyeq 0, \\ \sum_{i=1}^m y_i A_i + S &= C, \quad S \succcurlyeq 0, \\ XS &= 0. \end{aligned} \quad (1)$$

The basic idea of primal-dual IPM is to replace the third equation in (1), the so-called complementarity condition for (P) and (D), by the parameterized equation $XS = \mu E$, with $\mu > 0$, where E is the n by n identity matrix. Thus, one may consider

$$\begin{aligned} A_i \bullet X &= b_i, \quad i = 1, 2, \dots, m, \quad X \succcurlyeq 0, \\ \sum_{i=1}^m y_i A_i + S &= C, \quad S \succcurlyeq 0, \\ XS &= \mu E. \end{aligned} \quad (2)$$

For each $\mu > 0$, the parameterized system (2) has a unique solution $(X(\mu), y(\mu), S(\mu))$, see [9, 14], which is called a μ -center of (P) and (D). The set of μ -centers is said to be the central path of (P) and (D). The central path converges to the solution pair of (P) and (D) as μ reduces to zero [14].

A natural way to define a search direction for an IPM is to follow the Newton approach and linearize the third equation in (2) by replacing X, y and S with $X + \Delta X, y + \Delta y$ and $S + \Delta S$ respectively. This leads to

$$\begin{aligned} A_i \bullet \Delta X &= 0, \quad i = 1, 2, \dots, m, \\ \sum_{i=1}^m \Delta y_i A_i + \Delta S &= 0, \\ S \Delta X + X \Delta S &= \mu E - XS. \end{aligned} \quad (3)$$

The system (3) can be rewritten as

$$\begin{aligned}
A_i \bullet \Delta X &= 0, \quad i = 1, 2, \dots, m, \\
\sum_{i=1}^m \Delta y_i A_i + \Delta S &= 0, \\
\Delta X + X \Delta S S^{-1} &= \mu S^{-1} - X.
\end{aligned} \tag{4}$$

It is clear that ΔS is symmetric due to the second equation in (4). However, a crucial observation is that ΔX is not necessarily symmetric, because $X \Delta S S^{-1}$ may not be symmetric. Several researchers have proposed methods for symmetrizing the third equation in (4) such that the resulting new system had a unique symmetric solution. Among them, we consider the symmetrization scheme yielding the Nesterov-Todd (NT)-direction [13, 18]. In the NT-scheme, we replace the term $X \Delta S S^{-1}$ in the third equation of (4) by $P \Delta S P^T$ and obtain

$$\begin{aligned}
A_i \bullet \Delta X &= 0, \quad i = 1, 2, \dots, m, \\
\sum_{i=1}^m \Delta y_i A_i + \Delta S &= 0, \\
\Delta X + P \Delta S P^T &= \mu S^{-1} - X,
\end{aligned} \tag{5}$$

where

$$P := X^{\frac{1}{2}} (X^{\frac{1}{2}} S X^{\frac{1}{2}})^{-1} X^{\frac{1}{2}} [= S^{-\frac{1}{2}} (S^{\frac{1}{2}} X S^{\frac{1}{2}})^{\frac{1}{2}} S^{-\frac{1}{2}}]. \tag{6}$$

Let $D = P^{\frac{1}{2}}$. Then, D can be used to scale X and S to the same matrix V defined by [2]

$$V := \frac{1}{\sqrt{\mu}} D^{-1} X D^{-1} = \frac{1}{\sqrt{\mu}} D S D = \frac{1}{\sqrt{\mu}} (D^{-1} X S D)^{\frac{1}{2}}. \tag{7}$$

Note that the matrices D and V are symmetric and positive definite. If we define

$$\begin{aligned}
\bar{A}_i &:= \frac{1}{\sqrt{\mu}} D A_i D, \quad i = 1, 2, \dots, m, \\
D_X &:= \frac{1}{\sqrt{\mu}} D^{-1} \Delta X D^{-1}, \\
D_S &:= \frac{1}{\sqrt{\mu}} D \Delta S D,
\end{aligned} \tag{8}$$

then the NT-search directions can be written as the solution of the following system:

$$\begin{aligned}
\bar{A}_i \bullet D_X &= 0, \quad i = 1, 2, \dots, m, \\
\sum_{i=1}^m \Delta y_i \bar{A}_i + D_S &= 0, \\
D_X + D_S &= V^{-1} - V.
\end{aligned} \tag{9}$$

A crucial observation is that the right-hand side of the third equation in (9) equals the negative gradient of the logarithm barrier function $\Psi(V)$, i.e.,

$$D_X + D_S = -\nabla \Psi(V),$$

where

$$\Psi(V) = \sum_{i=1}^n \psi(\lambda_i(V)), \quad \psi(t) = \frac{t^2 - 1}{2} - \log(t).$$

Here, we replace $\psi(t)$ by $\psi(t) = \frac{1}{2}(t - 1)^2$, which leads to

$$\begin{aligned} \bar{A}_i \bullet D_X &= 0, & i = 1, 2, \dots, m, \\ \sum_{i=1}^m \Delta y_i \bar{A}_i + D_S &= 0, \\ D_X + D_S &= E - V. \end{aligned} \tag{10}$$

The new search directions D_X and D_S are obtained by solving (10), and so ΔX and ΔS can be computed via (8). Then, the new iterates are given by

$$X^+ = X + \Delta X, y^+ = y + \Delta y, S^+ = S + \Delta S.$$

Note that this can be seen as a search direction induced by a kernel function $\psi(t)$ as described in [1, 3, 7, 19] for SDO; these methods are called feasible IPMs. However, this kernel function does not satisfy some of the properties kernel functions; in particular, $\lim_{t \rightarrow 0^+} \psi(t) = +\infty$. These methods start with a strictly feasible interior point and maintain feasibility during the process. However, a problem is how to find an initial feasible interior point. For this problem, the so-called infeasible IPMs (IIPMs) methods. These methods start with an arbitrary positive point (not necessarily feasible) and feasibility is reached as progress is made to the optimal solution. The first IIPMs were proposed by Lustig [11]. Global convergence was shown by Kojima et al. [9], whereas Zhang [21] proved an $O(n^2L)$ iteration bound for IIPMs under certain conditions for LO and extended it to SDO [22]. In 2009, Mansouri and Roos [12] proposed the first full-Newton step IIPM for the SDO problem, which is an extension of the work for LO by Roos [16]. Liu and Sun [10] adopted the basic analysis used in [12] to the SDO problem based on the kernel function $\psi(t) = \frac{1}{2}(t - 1)^2$.

Here, we develop a different analysis of the mentioned works for the full-Newton step IPMs and IIPMs. We provide search directions and show that the iteration bound coincides with the best known bound for IIPMs, while tendering a simple analysis. By a numerical example, we investigate the advantage of our algorithm than the existing algorithms.

The remainder of our work is organized as follows. In Section 2, after recalling some necessary results, we give a new analysis for the full-NT step, which includes a feasible condition, the effect of a full-NT step on the proximity measure and the convergence of the full-NT step. In Section 3, we first propose the perturbed problems corresponding to (P) and (D), and then give a description of the full-NT step IIPMs based on the new search directions. Furthermore, we present our algorithm. In subsection 3.3, we analyze the feasibility step used in the algorithm. This includes a feasible condition and the effect of full-NT step on the proximity function after parameter updating. We derive the complexity bound of the algorithm in subsection 3.6. Finally, the conclusion and some remarks are given in Section 4.

2. A New Analysis of Full-NT Step

A primal-dual pair (X, S) is called an ε -solution of (P) and (D), if $\text{Tr}(XS) \leq \varepsilon$. Assume that a pair (X^0, S^0) with $X^0 \succ 0$ and $S^0 \succ 0$ is given ‘close to’ $(X(\mu), S(\mu))$, for some $\mu = \mu^0$, in the sense of the proximity measure $\sigma(X^0, S^0; \mu^0)$. This quantity is defined as follows:

$$\sigma(X, S; \mu) := \sigma(V) := \|E - V\|_F, \tag{11}$$

where V is defined by (7). Due to the first two equations of (10), D_X and D_S are orthogonal. Using the third equation of (10), we obtain

$$\|D_X + D_S\|_F^2 = \|D_X\|_F^2 + \|D_S\|_F^2 = \|E_V\|_F^2 = \sigma(V)^2.$$

This implies that D_X and D_S are both zero if and only if $E - V = 0$. In this case, X and S satisfy $XS = \mu E$, indicating that X and S are the μ -centers.

2.1. Some Basic Results

Here, we recall some useful results. Let $0 \leq \alpha \leq 1$. Define

$$X^\alpha = X + \alpha \Delta X, S^\alpha = S + \alpha \Delta S.$$

We recall two useful lemmas in [2], which will be used later.

Lemma 1. (Lemma 6.1 in [2]) Suppose that $X > 0$ and $S > 0$. If

$$\det(X^\alpha S^\alpha) > 0, \quad \forall 0 \leq \alpha \leq \bar{\alpha},$$

then $X^{\bar{\alpha}} > 0$ and $S^{\bar{\alpha}} > 0$.

Lemma 2. (Lemma A.1 in [2]) Let $Q \in S_{++}^n$, and let $M \in R^{n \times n}$ be skew-symmetric. Then, $\det(Q + M) > 0$. Moreover, if the eigenvalues of $Q + M$ are real, then

$$0 < \lambda_{\min}(Q) \leq \lambda_{\min}(Q + M) \leq \lambda_{\max}(Q + M) \leq \lambda_{\max}(Q).$$

The next lemma gives some upper bounds for the 2-norm and the Frobenius norm of D_{XS} , where

$$D_{XS} := \frac{1}{2}(D_X D_S + D_S D_X).$$

Note that D_{XS} is a symmetric matrix.

Lemma 3. (lemmas 6.2 and 7.3 in [2]) Let $D_X \in S^n$ and $D_S \in S^n$ be such that $\text{Tr}(D_X D_S) = 0$. Then,

$$\|D_{XS}\|_2 \leq \frac{1}{4} \|D_X + D_S\|_F^2, \quad \|D_{XS}\|_F \leq \frac{\sqrt{2}}{4} \|D_X + D_S\|_F^2.$$

This implies that

$$\|D_{XS}\|_2 \leq \frac{1}{4} \sigma(V)^2, \quad \|D_{XS}\|_F \leq \frac{\sqrt{2}}{4} \sigma(V)^2. \quad (12)$$

The next lemma gives a fundamental property about the proximity measure $\sigma(V)$.

Lemma 4. We have

$$1 - \sigma(V) \leq \lambda_i(V) \leq 1 + \sigma(V), \quad i = 1, 2, \dots, n,$$

where $\lambda_i(V)$ is the i th eigenvalue of V .

Proof. Since $E - V$ is a symmetric matrix, we have

$$\sigma(V)^2 = \text{Tr}((E - V)^2) = \sum_{i=1}^n \lambda_i((E - V)^2) = \sum_{i=1}^n (1 - \lambda_i(V))^2,$$

□

and this proves the lemma.

2.2. Properties of The Full-NT Step

Using (8), we obtain

$$\begin{aligned} X^+ &= X + \Delta X = \sqrt{\mu}D(V + D_X)D, \\ S^+ &= S + \Delta S = \sqrt{\mu}D^{-1}(V + D_S)D^{-1}. \end{aligned}$$

Therefore,

$$X^+S^+ = \mu D(V + D_X)(V + D_S)D^{-1}.$$

The last matrix is similar to $\mu(V + D_X)(V + D_S)$. Thus, we have

$$X^+S^+ \sim \mu(V + D_X)(V + D_S).$$

To simplify the notation, in the sequel we let

$$M := (D_X V - V D_X) + \frac{1}{2}(D_X D_S - D_S D_X),$$

where M is skew-symmetric. From the third equation in (10), we obtain, by multiplying both sides from the left by V ,

$$V D_X + V D_S = V - V^2. \quad (13)$$

Now, we may write, using (13),

$$\begin{aligned} (V + D_X)(V + D_S) &= V^2 + V D_S + D_X V + D_X D_S \\ &= V - V D_X + D_X V + D_X D_S. \end{aligned}$$

By subtracting and adding $\frac{1}{2}D_S D_X$ to the last expression, we get

$$(V + D_X)(V + D_S) = V + \frac{1}{2}(D_X D_S + D_S D_X) + \frac{1}{2}(D_X D_S - D_S D_X) + (D_X V - V D_X)$$

$$= V + D_{XS} + M.$$

Therefore,

$$X^+ S^+ \sim \mu(V + D_{XS} + M). \quad (14)$$

We want the new iterates be positive definite. We call the NT step is strictly feasible, if $X^+ > 0$ and $S^+ > 0$. The next lemma gives a sufficient condition for strict feasibility of the full NT step.

Lemma 5. *Let $X > 0$ and $S > 0$. Then, the iterates (X^+, y^+, S^+) are strictly feasible if $V + D_{XS} > 0$.*

Proof. Consider $\alpha, 0 \leq \alpha \leq 1$, and define

$$X^\alpha = X + \alpha \Delta X, y^\alpha = y + \alpha \Delta y, S^\alpha = S + \alpha \Delta S.$$

Considering Lemma 1, it suffices to show that the determinant of $X^\alpha S^\alpha$ is positive, for all $0 \leq \alpha \leq 1$. We may write

$$\begin{aligned} \frac{X^\alpha S^\alpha}{\mu} &\sim (V + \alpha D_X)(V + \alpha D_S) \\ &= V^2 + \alpha(VD_S + D_X V) + \alpha^2 D_X D_S \\ &= V^2 + \alpha(VD_X + VD_S) + \alpha(D_X V - VD_X) + \alpha^2 D_X D_S. \end{aligned}$$

Using (13), we get

$$\begin{aligned} \frac{X^\alpha S^\alpha}{\mu} &\sim V^2 + \alpha(V - V^2) + \alpha(D_X V - VD_X) + \alpha^2 D_X D_S \\ &= (1 - \alpha)V^2 + \alpha V + \alpha(D_X V - VD_X) + \alpha^2 D_X D_S. \end{aligned}$$

By subtracting and adding $\frac{\alpha^2}{2} D_S D_X$ to the right-hand side of the above expression, we obtain

$$\frac{X^\alpha S^\alpha}{\mu} \sim (1 - \alpha)V^2 + \alpha(1 - \alpha)V + \alpha^2(V + D_{XS}) + \alpha(\alpha M + (1 - \alpha)(D_X V - VD_X)).$$

The matrix $\alpha M + (1 - \alpha)(D_X V - VD_X)$ is skew-symmetric, for $0 \leq \alpha \leq 1$. Lemma 2 therefore implies that the determinant of $X^\alpha S^\alpha$ is positive if the symmetric matrix $(1 - \alpha)V^2 + \alpha(1 - \alpha)V + \alpha^2(V + D_{XS})$ is positive definite. The latter is true for $0 \leq \alpha \leq 1$, because $V + D_{XS} > 0$ and V and V^2 are positive definite. Thus, $\det(X^\alpha S^\alpha) > 0$. In addition, since by assumption, $X^0 = X > 0$ and $S^0 = S > 0$, Lemma 1 implies that $X^1 = X^+ > 0$ and $S^1 = S^+ > 0$. This completes the proof. \square

Corollary 6. *The new iterates (X^+, S^+) are strictly feasible, if*

$$\|D_{XS}\|_2 < \lambda_{\min}(V).$$

Proof. With $V \in S^n$ and $D_{XS} \in S^n$, we have

$$\begin{aligned}\lambda_i(V + D_{XS}) &\geq \lambda_{\min}(V) - |\lambda_{\max}(D_{XS})| \\ &\geq \lambda_{\min}(V) - \|D_{XS}\|_2, \quad i = 1, 2, \dots, n.\end{aligned}$$

By Lemma 5, X^+ and S^+ are strictly feasible, if $V + D_{XS} > 0$, and this holds if $\|D_{XS}\|_2 < \lambda_{\min}(V)$. The proof is complete. \square

Lemma 7. *Let $\sigma(V)$ be defined as (11) and $X, S > 0$. If $\sigma(V) < 2\sqrt{2} - 2$, then the full NT step is strictly feasible.*

Proof. From (12) and Lemma 4, we have

$$\|D_{XS}\|_2 \leq \frac{1}{4}\sigma(V)^2 \text{ and } 1 - \sigma(V) \leq \lambda_{\min}(V).$$

By Corollary 6, the full NT step is strictly feasible, if

$$\|D_{XS}\|_2 < \lambda_{\min}(V).$$

This last inequality holds, if

$$\frac{1}{4}\sigma(V)^2 < 1 - \sigma(V),$$

which leads to $\sigma(V) < 2\sqrt{2} - 2$. This completes the proof. \square

The next lemma gives the effect of full NT step on the duality gap.

Lemma 8. *If $\sigma(V) < 2\sqrt{2} - 2$, then*

$$\text{Tr}(X^+S^+) < (2\sqrt{2} - 1)n\mu.$$

Proof. Using (14), the skew-symmetry of M , $\text{Tr}(D_X D_S) = 0$ and Lemma 4, we have

$$\begin{aligned}\text{Tr}(X^+S^+) &= \mu\text{Tr}(V + D_{XS} + M) = \mu\text{Tr}(V + D_{XS}) \\ &= \mu(\text{Tr}(V) + \text{Tr}(D_{XS})) = \mu\text{Tr}(V) = \mu \sum_{i=1}^n \lambda_i(V) \\ &\leq n\mu\lambda_{\max}(V) \leq n\mu(1 + \sigma(V)) < n\mu(2\sqrt{2} - 1),\end{aligned}$$

which proves the lemma. \square

We denote the NT scaling of $\frac{X^+S^+}{\mu}$ by $(V^+)^2$. So, from (14), we have

$$(V^+)^2 \sim \frac{X^+S^+}{\mu} \sim V + D_{XS} + M \quad (15)$$

Lemma 9. The following holds:

$$\lambda_{\min}(V^+) \geq \sqrt{1 - \sigma(V) - \frac{1}{4}\sigma(V)^2}.$$

Proof. Using (15), we get

$$\lambda_{\min}((V^+)^2) = \lambda_{\min}(V + D_{XS} + M).$$

Since M is skew-symmetric, Lemma 2 implies that

$$\begin{aligned} \lambda_{\min}((V^+)^2) &\geq \lambda_{\min}(V + D_{XS}) \\ &\geq \lambda_{\min}(V) - \|D_{XS}\|_2 \\ &\geq 1 - \sigma(V) - \frac{1}{4}\sigma(V)^2, \end{aligned}$$

Where the last inequality follows by (12) and Lemma 4. □

The following lemma describes the effect of a full-NT step on the proximity measure.

Lemma 10. Let $X, S > 0$ and $\mu > 0$. Moreover, let $\sigma(V) < 2\sqrt{2} - 2$. Then,

$$\sigma(X^+, S^+; \mu) \leq \frac{\sigma(V) + \frac{\sqrt{2}}{4}\sigma(V)^2}{1 + \sqrt{1 - \sigma(V) - \frac{1}{4}\sigma(V)^2}}$$

Proof. Using (15), we have

$$\begin{aligned} \sigma(X^+, S^+; \mu) &= \|E - V^+\|_F \\ &= \|(E - (V^+)^2)(E + V^+)^{-1}\|_F \\ &\leq \frac{1}{1 + \lambda_{\min}(V^+)} \|E - (V^+)^2\|_F. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \|E - V - D_{XS} - M\|^2 &= \sum_{i=1}^n (\lambda_i(V + D_{XS} + M) - 1)^2 \\ &= \sum_{i=1}^n ((\lambda_i(V + D_{XS} + M))^2 - 2\lambda_i(V + D_{XS} + M) + 1) \\ &= \sum_{i=1}^n (\lambda_i(V + D_{XS} + M))^2 - 2\sum_{i=1}^n \lambda_i(V + D_{XS} + M) + n \\ &= \text{Tr}((V + D_{XS} + M)^2) - 2\text{Tr}(V + D_{XS} + M) + n, \end{aligned}$$

where the last equality is true by $(\lambda_i(V + D_{XS} + M))^2 = \lambda_i((V + D_{XS} + M)^2)$, for each i . Using the skew-symmetry of M , we obtain $\text{Tr}(V + D_{XS} + M) = \text{Tr}(V + D_{XS})$ and

$$\text{Tr}((V + D_{XS} + M)^2) = \text{Tr}((V + D_{XS})^2 + (V + D_{XS})M + M(V + D_{XS}) - MM^T).$$

Since $(V + D_{XS})M + M(V + D_{XS})$ is skew-symmetric, we obtain

$$\text{Tr}((V + D_{XS} + M)^2) = \text{Tr}((V + D_{XS})^2 - MM^T) \leq \text{Tr}((V + D_{XS})^2),$$

where the inequality follows since the matrix MM^T is positive semidefinite. Hence,

$$\begin{aligned} \|E - V - D_{XS} - M\|^2 &\leq \text{Tr}((V + D_{XS})^2) - 2\text{Tr}(V + D_{XS}) + n \\ &= \text{Tr}((E - (V + D_{XS}))^2) = \|E - V - D_{XS}\|^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \sigma(X^+, S^+; \mu) &\leq \frac{1}{1 + \lambda_{\min}(V^+)} \|E - V - D_{XS}\|_F \\ &\leq \frac{\sigma(V) + \frac{\sqrt{2}}{4}\sigma(V)^2}{1 + \sqrt{1 - \sigma(V) - \frac{1}{4}\sigma(V)^2}}, \end{aligned}$$

where the last inequality follows from Lemma 9, the triangle inequality, (11) and (12). This completes the proof. \square

The following corollary guarantees the convergence of the full NT step.

Corollary 11. *If $\sigma(V) := \sigma(X, S; \mu) \leq \frac{1}{2}$, then $\sigma(X^+, S^+; \mu) \leq \frac{4}{5}\sigma(V)$.*

3. A Full NT Step IIPM

In the case of an infeasible method, we call the triplet (X, y, S) an ε -solution of (P) and (D) if the norms of the residual vectors $(r_p)_i = b_i - A_i \bullet X, i = 1, 2, \dots, m$, and $r_d = C - \sum_{i=1}^m y_i A_i - S$ do not exceed ε , and also $X \bullet S \leq \varepsilon$. In what follows, we present an infeasible-start algorithm that generates an ε -solution of (P) and (D) , if it exists, or establishes that no such solution exists.

3.1. The Perturbed Problems

We assume (P) and (D) have an optimal solution (X^*, y^*, S^*) with $X^* \bullet S^* = 0$. As usual for IIPMs, we start the algorithm with $(X^0, y^0, S^0) = \xi(E, 0, E)$ and $\mu^0 = \xi^2$, where ξ is a positive number such that

$$X^* + S^* \preceq \xi E. \quad (16)$$

The initial values of the primal and dual residual vectors are $(r_p^0)_i = b_i - A_i \bullet X^0, i = 1, 2, \dots, m$, and $r_d^0 = C - \sum_{i=1}^m y_i^0 A_i - S^0$. In general, $(r_p^0)_i \neq 0, i = 1, 2, \dots, m$, and $r_d^0 \neq 0$. The iterates generated by the algorithm will be infeasible for the (P) and (D) , but they will be feasible for perturbed versions of (P) and (D) as given below. For any $\nu, 0 < \nu \leq 1$, consider the perturbed problem as

$$\begin{aligned}
& \min (C - \nu r_d^0) \bullet X \\
& \text{s. t. } b_i - A_i \bullet X = \nu (r_p^0)_i, \quad i = 1, 2, \dots, m, \quad (P_\nu) \\
& \quad X \succeq 0,
\end{aligned}$$

and its dual as

$$\begin{aligned}
& \max \sum_{i=1}^m (b - \nu (r_p^0)_i) y_i \\
& \text{s. t. } C - \sum_{i=1}^m y_i A_i - S = \nu r_d^0 \quad (D_\nu) \\
& \quad S \succeq 0.
\end{aligned}$$

Note that if $\nu = 1$, then $X = X^0$ and $(y, S) = (y^0, S^0)$ yield strictly feasible solutions of (P_ν) and (D_ν) , respectively. We conclude that if $\nu = 1$, then (P_ν) and (D_ν) are strictly feasible, which means that both perturbed problems (P_ν) and (D_ν) satisfy IPC. More generally, one has the following result (Lemma 4.1 in [12]).

Lemma 12. *Let the original problems (P) and (D) be feasible. Then, for each ν satisfying $0 < \nu \leq 1$, the perturbed problems (P_ν) and (D_ν) are strictly feasible.*

Assuming that (P) and (D) are both feasible, it follows from Lemma 12 that the problems (P_ν) and (D_ν) satisfy IPC, for each $0 < \nu \leq 1$. Then, their central paths exist, meaning that the system

$$b_i - A_i \bullet X = \nu (r_p^0)_i, \quad i = 1, 2, \dots, m, \quad (17)$$

$$C - \sum_{i=1}^m y_i A_i - S = \nu r_d^0, \quad (18)$$

$$XS = \mu E, \quad (19)$$

has a unique solution, for any $\mu > 0$. For $0 < \nu \leq 1$ and $\mu = \nu \mu^0$, we denote this unique solution in the sequel by $(X(\mu, \nu), y(\mu, \nu), S(\mu, \nu))$, where $X(\mu)$ is the μ -center of (P_ν) and $(y(\mu), S(\mu))$ is the μ -center of (D_ν) . By taking $\nu = 1$, one has $(X(1), y(1), S(1)) = (X^0, y^0, S^0) = (\xi E, 0, \xi E)$ and $X^0 S^0 = \mu^0 E$. Hence, X^0 is the μ^0 -center of the perturbed problem (P_1) and (y^0, S^0) is the μ^0 -center of the perturbed problem (D_1) .

3.2. Description of the Full-NT Step IIPM

are the μ -centers of (P_ν) and (D_ν) , respectively. We measure proximity of the iterate (X, y, S) to the μ -centers of the perturbed problems (P_ν) and (D_ν) by the quantity $\sigma(X, S; \mu)$ as defined by (11).

Initially, we have $\sigma(X, S; \mu) = 0$. In the sequel, we assume that at the start of each iteration, just before the μ - and ν -update, $\sigma(X, S; \mu) \leq \tau$, where τ is a positive threshold value. This certainly holds at the start of the first iteration, since we then have $\sigma(X, S; \mu) = 0$.

Now, we describe a main iteration of our algorithm. The algorithm begins with an infeasible interior-point (X, y, S) such that (X, y, S) is feasible for the perturbed problems (P_ν) and (D_ν) , with $\mu = \nu \mu^0$ and such that $X \bullet S \leq (2\sqrt{2} - 1)n\mu$ and $\sigma(X, S; \mu) \leq \tau$. We reduce ν to $\nu^+ = (1 - \theta)\nu$, with $\theta \in (0, 1)$, and find new iterate (X^+, y^+, S^+) that is feasible for the perturbed problems (P_{ν^+}) and (D_{ν^+}) such that $\sigma(X^+, S^+; \mu^+) \leq \tau$. Every iteration consists of a feasibility step, a μ -update and a few

centering steps. First, we find a new point (X^f, y^f, S^f) which is feasible for the perturbed problems with $v^+ := (1 - \theta)v$. Then, μ is decreased to $\mu^+ := (1 - \theta)\mu$. Generally, there is no guarantee that $\sigma(X^f, S^f; \mu^+) \leq \tau$. So, a few centering steps is applied to produce a new point (X^+, y^+, S^+) such that $\sigma(X^+, S^+; \mu^+) \leq \tau$. This process is repeated until the algorithm terminates. We now summarize the steps of the algorithm as Algorithm 1 below.

Algorithm 1: A Full NT Step IIPM.

Input:

accuracy parameter $\varepsilon > 0$,

barraier update parameter $\theta, 0 < \theta < 1$,

threshold parameter $0 < \tau \leq \frac{1}{2}$.

begin

$X := \xi E; y := 0; S := \xi E; \mu := \mu^0 = \xi^2; v = 1;$

while $\max(X \bullet S, \|r_p\|_F, \|r_d\|_F) > \varepsilon$ **do**

begin

feasibility step:

solve (23) and obtain

$$(X, y, S) := (X, y, S) + (\Delta^f X, \Delta^f y, \Delta^f S);$$

μ – update:

$$\mu := (1 - \theta)\mu;$$

centering step:

while $\sigma(X, S; \mu) > \tau$ **do**

begin

solve (10) and obtain

$$(X, y, S) := (X, y, S) + (\Delta X, \Delta y, \Delta S)$$

end

end

end.

3.3. Analysis of the Feasibility Step

First, we describe the feasibility step in details. The analysis will follow in the sequel. Suppose that we have strictly feasible iterate (X, y, S) for (P_v) and (D_v) . This means that (X, y, S) satisfies (17) and (18) with $\mu = v\xi^2$. We need displacements $\Delta^f X$, $\Delta^f y$ and $\Delta^f S$ such that

$$X^f := X + \Delta^f X, \quad y^f := y + \Delta^f y, \quad S^f := S + \Delta^f S, \quad (20)$$

are feasible for (P_{v^+}) and (D_{v^+}) . One may easily verify that (X^f, y^f, S^f) satisfies (17) and (18), with v replaced by v^+ , only if the first two equations in the following system are satisfied.

$$\begin{aligned} A_i \bullet \Delta^f X &= \theta v (r_p^0)_i, \quad i = 1, 2, \dots, m, \\ \sum_{i=1}^m \Delta^f y_i A_i + \Delta^f S &= \theta v r_d^0 \\ \Delta^f X + P \Delta^f S P^T &= \mu S^{-1} - X. \end{aligned} \quad (21)$$

The third equation is inspired by the third equation in the system (5) that we used to define search directions for the feasible case.

According to (21), after the feasibility step, the iterates satisfy the affine equations in (17) and (18), with v replaced by v^+ . The hard part in the analysis will be to guarantee that X^f, S^f are positive definite and to guarantee that the new iterate satisfies $\sigma(X^f, S^f; \mu^+) \leq \frac{1}{2}$.

Let (X, y, S) denote the iterate at the start of an iteration with $X \bullet S \leq (2\sqrt{2} - 1)n\mu$ and $\sigma(X, S; \mu) \leq \tau$. This is certainly true at the start of the first iteration, because $X^0 \bullet S^0 = n\mu^0$ and $\sigma(X^0, S^0; \mu^0) = 0$. Defining

$$D_X^f := \frac{1}{\sqrt{\mu}} D^{-1} \Delta^f X D^{-1}, \quad D_S^f := \frac{1}{\sqrt{\mu}} D \Delta^f S D, \quad (22)$$

with D as defined in (6), one can easily check that the system (21), which defines the search directions $\Delta^f X$, $\Delta^f y$ and $\Delta^f S$, by considering $\psi(t) = \frac{1}{2}(t-1)^2$, can be expressed in terms of the scaled search directions D_X^f and D_S^f as follows:

$$\begin{aligned} \widehat{A}_i \bullet D_X^f &= \frac{1}{\sqrt{\mu}} \theta v (r_p^0)_i, \quad i = 1, 2, \dots, m, \\ \sum_{i=1}^m \frac{\Delta^f y_i}{\sqrt{\mu}} \widehat{A}_i + D_S^f &= \frac{1}{\sqrt{\mu}} \theta v D r_d^0 D, \\ D_X^f + D_S^f &= E - V, \end{aligned} \quad (23)$$

where $\widehat{A} = D A_i D$. To get the search directions $\Delta^f X$ and $\Delta^f S$ in the original X and S spaces, we use (22), which gives

$$\Delta^f X = \sqrt{\mu} D D_X^f D, \quad \Delta^f S = \sqrt{\mu} D^{-1} D_S^f D^{-1}.$$

The new iterates are obtained by taking a full step, as given by (20). Hence, we have

$$\begin{aligned} X^f &= X + \Delta^f X = \sqrt{\mu} D (V + D_X^f) D, \\ S^f &= S + \Delta^f S = \sqrt{\mu} D^{-1} (V + D_S^f) D^{-1}. \end{aligned} \quad (24)$$

Using (24) and the third equation of (23), we get

$$\begin{aligned} X^f S^f &\sim \mu (V + D_X^f) (V + D_S^f) \\ &= \mu (V^2 + V D_S^f + D_X^f V + D_X^f D_S^f) \\ &= \mu (V - V D_X^f + D_X^f V + D_X^f D_S^f) \\ &= \mu (V + D_{XS}^f + M^f), \end{aligned} \quad (25)$$

where $M^f := (D_X^f V - V D_X^f) + \frac{1}{2} (D_X^f D_S^f - D_S^f D_X^f)$ and $D_{XS}^f = \frac{1}{2} (D_X^f D_S^f + D_S^f D_X^f)$. Note that M^f is skew-symmetric and D_{XS}^f is symmetric.

The proof of the next lemma is similar to the proof of Lemma 5, and is thus omitted.

Lemma 13. *Let $X, S > 0$. Then, the iterate (X^f, y^f, S^f) is strictly feasible, if $V + D_{XS}^f > 0$.*

In the sequel, we denote

$$w(V) := \frac{1}{2} \sqrt{\|D_X^f\|_F^2 + \|D_S^f\|_F^2},$$

which implies $\|D_X^f\|_F \leq 2w(V)$ and $\|D_S^f\|_F \leq 2w(V)$. Moreover,

$$\|D_{XS}^f\|_F \leq \|D_X^f\|_F \|D_S^f\|_F \leq \frac{1}{2} (\|D_X^f\|_F^2 + \|D_S^f\|_F^2) = 2w(V)^2, \quad (26)$$

$$|\lambda_i(D_{XS}^f)| \leq \|D_{XS}^f\|_F \leq 2w(V)^2, \quad i = 1, 2, \dots, n. \quad (27)$$

We proceed by deriving an upper bound for $\sigma(X^f, S^f, \mu^+)$. Recall from (11) that

$$\sigma(X^f, S^f; \mu^+) := \sigma(V^f) = \|E - V^f\|_F, \quad \text{where} \quad (V^f)^2 = \frac{1}{\mu^+} D^{-1} X^f S^f D. \quad (28)$$

Lemma 14. *Let $V + D_{XS}^f > 0$ and $\mu^+ = (1 - \theta)\mu$. Then,*

$$\sigma(V^f) \leq \frac{\sigma(V) + 2w(V)^2 + \theta\sqrt{n}}{1 - \theta + \sqrt{(1 - \theta)(1 - \sigma(V) - 2w(V)^2)}}$$

Proof. Using (25) and (28), we get

$$(V^f)^2 \sim \frac{V + D_{XS}^f + M^f}{1 - \theta}, \quad (29)$$

and have

$$\lambda_{\min}((V^f)^2) = \frac{1}{1 - \theta} \lambda_{\min}(V + D_{XS}^f + M^f).$$

Since M^f is skew-symmetric, Lemma 2 implies that

$$\begin{aligned} \lambda_{\min}((V^f)^2) &\geq \frac{1}{1 - \theta} \lambda_{\min}(V + D_{XS}^f) \\ &\geq \frac{1}{1 - \theta} (\lambda_{\min}(V) - \|D_{XS}^f\|_F) \\ &\geq \frac{1}{1 - \theta} (1 - \sigma(V) - 2w(V)^2), \end{aligned} \quad (30)$$

where the last inequality follows from (27) and Lemma 4. Using (28), (29) and the triangle inequality, we get

$$\sigma(V^f) = \|E - V^f\|_F = \|(E + V^f)^{-1} (E - (V^f)^2)\|_F$$

$$\begin{aligned}
&\leq \frac{1}{1 + \lambda_{\min}(V^f)} \| E - (V^f)^2 \|_F \\
&= \frac{1}{1 + \lambda_{\min}(V^f)} \| E - \frac{V + D_{XS}^f + M^f}{1 - \theta} \|_F \\
&\leq \frac{1}{(1-\theta)(1+\lambda_{\min}(V^f))} (\sigma(V) + \| \theta E + D_{XS}^f + M^f \|_F). \tag{31}
\end{aligned}$$

Since $\theta E + D_{XS}^f$ is symmetric and M^f is skew-symmetric, we have

$$\begin{aligned}
\| \theta E + D_{XS}^f + M^f \|_F^2 &= \sum_{i=1}^n \lambda_i(\theta E + D_{XS}^f + M^f)^2 \\
&= \sum_{i=1}^n (\theta + \lambda_i(D_{XS}^f + M^f))^2 \\
&\leq (\sqrt{\sum_{i=1}^n \theta^2} + \sqrt{\sum_{i=1}^n \lambda_i(D_{XS}^f + M^f)^2})^2 \\
&= (\theta\sqrt{n} + \sqrt{\text{Tr}(D_{XS}^f + M^f)^2})^2 \\
&\leq (\theta\sqrt{n} + \| D_{XS}^f \|_F)^2 \leq (\theta\sqrt{n} + 2w(V)^2)^2. \tag{32}
\end{aligned}$$

Substituting (30) and (32) into (31), the result follows. \square

Because we need to have $\sigma(V^f) \leq \frac{1}{2}$, by Lemma 14, it suffices to have

$$\frac{\sigma(V) + 2w(V)^2 + \theta\sqrt{n}}{1 - \theta + \sqrt{(1-\theta)(1-\sigma(V) - 2w(V)^2)}} \leq \frac{1}{2}. \tag{33}$$

At this stage, we let

$$\tau = \frac{1}{8}, \quad \theta = \frac{3\alpha}{10\sqrt{n}}, \quad \alpha \leq 1. \tag{34}$$

The left-hand side of (33) is monotonically increasing with respect to $2w(V)^2$, then for $n \geq 1$ and $\sigma(V) \leq \tau$, one can verify that

$$w(V) \leq \frac{1}{2\sqrt{2}} \Rightarrow \sigma(V^f) \leq \frac{1}{2}. \tag{35}$$

3.4. Upper Bound for $w(V)$

In this subsection, we intend to find an upper bound for $w(V)$, to be able to set a default value for θ . For this purpose, consider the system (23). By eliminating D_S^f from the system (23), we have

$$\begin{aligned}\widehat{A}_i \bullet D_X^f &= \frac{1}{\sqrt{\mu}} \theta v (r_p^0)_i, \quad i = 1, 2, \dots, m, \\ -\sum_{i=1}^m \frac{\Delta^f y_i}{\sqrt{\mu}} \widehat{A}_i + D_X^f &= (E - V) - \frac{1}{\sqrt{\mu}} \theta v D r_d^0 D.\end{aligned}\quad (36)$$

After some manipulations (for more details, see [8]), we get

$$\|D_X^f\|_F^2 + \|D_S^f\|_F^2 \leq 2\sigma(V)^2 + \frac{3\theta^2}{\xi^2 \lambda_{\min}(V)^2} (\text{Tr}(X + S))^2.$$

Lemma 15 (Lemma 5.14 in [12]). *Let (X, y, S) be feasible for the perturbed problems (P_v) and (D_v) and let $(X^0, y^0, S^0) = (\xi E, 0, \xi E)$ and (X^*, y^*, S^*) be as defined by (16). Then,*

$$v\xi \text{Tr}(X + S) \leq S \bullet X + vn\xi^2. \quad (37)$$

Corollary 16. *Using the same notation as in Lemma 15, we have*

$$\text{Tr}(X + S) \leq (1 + (1 + \sigma(V))^2)n\xi. \quad (38)$$

Proof. Using the inequality (37) and $\mu = v\xi^2$, we get

$$\text{Tr}(X + S) \leq \text{Tr}\left(\frac{XS}{\mu}\right)\xi + n\xi. \quad (39)$$

Using $XS \sim \mu V^2 = D^{-1}XSD$ and Lemma 4, we deduce that

$$\text{Tr}\left(\frac{XS}{\mu}\right) = \text{Tr}(V^2) = \sum_{i=1}^n \lambda_i(V)^2 \leq \sum_{i=1}^n (1 + \sigma(V))^2 = n(1 + \sigma(V))^2,$$

which by substituting into (39), the result easily follows. \square

Finally, using (38) and Lemma 4, we obtain

$$\|D_X^f\|_F^2 + \|D_S^f\|_F^2 \leq 2\sigma(V)^2 + \frac{3\theta^2 n^2}{(1 - \sigma(V))^2} (1 + (1 + \sigma(V))^2)^2. \quad (40)$$

3.5. Value for θ

At this stage, we choose $\tau = \frac{1}{8}$. Since $\sigma(V) \leq \tau = \frac{1}{8}$ and the right-hand side of (40) is monotonically increasing in $\sigma(V)$, we have

$$\|D_X^f\|_F^2 + \|D_S^f\|_F^2 \leq \frac{1}{32} + 3\left(\frac{145}{56}\right)^2 \theta^2 n^2.$$

Using $\theta = \frac{3\alpha}{10\sqrt{n}}$, the above inequality becomes

$$\|D_X^f\|_F^2 + \|D_S^f\|_F^2 \leq \frac{1}{32} + 3\left(\frac{145}{56}\right)^2 \times \frac{9\alpha^2 n}{100}. \quad (41)$$

From (35) we know that $w(V) \leq \frac{1}{2\sqrt{2}}$ is needed in order to have $\sigma(V^f) \leq \frac{1}{2}$. Due to (41), this will hold, if

$$\frac{1}{32} + 3\left(\frac{145}{56}\right)^2 \times \frac{9\alpha^2 n}{100} \leq \frac{1}{2}.$$

If we take

$$\alpha = \frac{1}{2\sqrt{n}}, \quad (42)$$

then the above inequality is satisfied.

3.6. Complexity Analysis

We have seen that if at the start of an iteration the iterate satisfies $\sigma(x, s; \mu) \leq \tau$, with $\tau = \frac{1}{8}$, then after the feasibility step, with θ as defined in (34) and α as in (42), the iterate is strictly feasible and satisfies $\sigma(X^f, S^f; \mu^+) \leq \frac{1}{2}$.

After the feasibility step, we perform a few centering steps in order to get the iterate (X^+, y^+, S^+) satisfying $\sigma(X^+, S^+; \mu^+) \leq \tau$. By Corollary 11, after k centering steps, we will have the iterate (X^+, y^+, S^+) still feasible for (P_{v^+}) and (D_{v^+}) satisfying

$$\sigma(X^+, S^+; \mu^+) \leq \frac{1}{2} \left(\frac{4}{5}\right)^k.$$

From this, one easily deduces that $\sigma(X^+, S^+; \mu^+) \leq \tau$ will hold after at most

$$\frac{\log_2 2\tau}{\log_2 0.8} \quad (43)$$

centering steps. According to (43), and since $\tau = \frac{1}{8}$, at most seven centering steps suffice to get the iterate (X^+, y^+, S^+) that satisfies $\sigma(X^+, S^+; \mu^+) \leq \tau$. So, each main iteration consists of at most eight so-called inner iterations.

In each main iteration, both the duality gap and the norms of the residual vectors are reduced by the factor $1 - \theta$. Hence, the total number of main iterations is bounded above by

$$\frac{1}{\theta} \log \frac{\max\{n\xi^2, \|r_p^0\|, \|r_d^0\|\}}{\varepsilon}.$$

Due to (34), (42) and the fact that we need at most eight inner iterations per main iteration, we may state the main result of the paper.

Theorem 17. *If (P) and (D) are feasible and $\xi > 0$ is such that $X^* + S^* \leq \xi E$, for some optimal solution X^* of (P) and (y^*, S^*) of (D), then after at most*

$$\frac{160}{3} n \log \frac{\max\{n\xi^2, \|r_p^0\|_F, \|r_d^0\|_F\}}{\varepsilon}$$

inner iterations, the algorithm finds an ε -optimal solution of (P) and (D).

Example 1. *To illustrate an application of the algorithm, let us consider the problems (P) and (D) with the following data:*

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 & -1 \\ 0 & -1 & 1 & -1 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & -2 & 2 & 0 \\ 0 & 2 & 1 & 0 & 2 \\ -2 & 1 & -2 & 0 & 1 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 2 & 2 & -1 & -1 & 1 \\ 2 & 0 & 2 & 1 & 1 \\ -1 & 2 & 0 & 1 & 0 \\ -1 & 1 & 1 & -2 & 0 \\ 1 & 1 & 0 & 0 & -2 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 & 3 & -3 & 1 & 1 \\ 3 & 5 & 3 & 1 & 2 \\ -3 & 3 & -1 & 1 & 2 \\ 1 & 1 & 1 & -3 & -1 \\ 1 & 2 & 2 & -1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix}.$$

The starting point, as usual for IIPMs, is $(X^0, y^0, S^0) = (\xi E, 0, \xi E)$ with $\xi = 1$ and we obtain a primal-dual optimal solution as follows:

$$X^* = \begin{bmatrix} 0.0714 & -0.0718 & 0.0167 & 0.0650 & -0.1580 \\ -0.0718 & 0.0725 & -0.0182 & -0.0603 & 0.1674 \\ 0.0167 & -0.0182 & 0.0103 & -0.0085 & -0.0770 \\ 0.0650 & -0.0603 & -0.0085 & 0.1486 & 0.0060 \\ -0.1580 & 0.1674 & -0.0770 & 0.0060 & 0.6017 \end{bmatrix}, \quad y^* = \begin{bmatrix} 0.8584 \\ 1.0937 \\ 0.7832 \end{bmatrix},$$

$$S^* = \begin{bmatrix} 1.4334 & 0.5749 & -0.0290 & -0.4044 & 0.2167 \\ 0.5749 & 1.0954 & 0.3395 & 0.2167 & -0.1125 \\ -0.0290 & 0.3395 & 1.1877 & 0.2167 & 0.0477 \\ -0.4044 & 0.2167 & 0.2167 & 0.2835 & -0.1415 \\ 0.2167 & -0.1125 & 0.0477 & -0.1415 & 0.0959 \end{bmatrix}.$$

With a tolerance $\varepsilon = 10^{-3}$, the algorithm reaches this solution in 182 iterations, taking 1809.358530 seconds. Now, a comparison of the results obtained by our proposed algorithm and the ones obtained by three other algorithms can be seen in Table 1. It can be seen that the required iteration number and the needed time by our proposed algorithm are preferred to the ones required by the other algorithms.

Table 1. Comparative results

| Method | No. of iterations | Time (in seconds) |
|------------------------|-------------------|-------------------|
| Our proposed algorithm | 182 | 1809.35 |
| Algorithm in [12] | 229 | 2581.33 |
| Algorithm in [10] | 369 | 5765.32 |
| Algorithm in [6] | 369 | 3830.86 |

4. Conclusion

We presented a full NT step infeasible interior-point algorithm for semi-definite optimization. The centering steps in [10, 12] and also the feasibility step in [12] were induced by the classic logarithm barrier function. We used a kernel function to induce both the centering and the feasibility steps and analyzed the algorithm based on these search directions, giving an analysis different from the one in [10, 12]. The resulting complexity coincides with the best known iteration bound for IIPMs, while the practical performance of the algorithm is better than the existing algorithms.

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