Iranian Journal of Operations Research
Vol. 5, No. 2, 2014, pp. 67-78

Generation of a Reduced First-Level Mixed Integer Programming Problem

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We introduce a new way of generating cutting planes of a mixed integer program by way of taking binary variables. Four binary variables are introduced to form quartic inequalities, resulting in a reduced first-level mixed integer program. A new way of weakening the inequalities is presented. An algorithm to carry-out the separation of the inequalities, being exponential in number, is developed. The proposed method of cut generation, separation and strengthening is compared to the Gomory, linear branching and coordinated cutting plane methods. The computational results show the proposed method to be promising while getting to be complicated as number of variables increases.

Keywords: Reduced first level, MIP, cutting planes.

Manuscript was received on 24/06/2014, revised on 30/08/2015 and accepted for publication on 17/09/2015.

1. Introduction

Mixed integer program (MIP) is a framework intended to capture discrete decisions and continuous variables. MIP is sometimes referred to as mixed integer linear program (MILP) and it is simply a special case of linear programming (LP) in which some of the decision variables are constrained to take only integer values. Resource allocation problems are inherently discrete, and therefore MIPs (Richards and How [21]) is extensively being used. Model formulations that possess tight linear relaxations are important in designing effective heuristic solution procedures for a MIP (Sherali et al. [24]). Solvability of discrete mathematical programs depends on how the model approximates the convex hull of feasible solutions within the region of optimal solution. Polyhedral approaches of discrete programming are means for approximating the convex hull of integer programs. Disaggregation is one of the many methods introduced to produce a tighter representation of MIP (see Barnhart et al. [6] and Johnson et al. [14]). Disaggregation techniques increase the

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number of constraints while preserving problem sparsity and significantly tighten the continuous relaxation of the problem.

In practice, the first level relaxation is usually the most useful. Our research will focus on the reformulation linearization technique (RLT) introduced by Sherali and Adams [23]. RLT is a framework for constructing strong linear relaxations. A lift and project cutting plane algorithm was developed for a partial first level RLT relaxation that considers one binary variable at a time and its computational results are promising (Balas et al. [4]). RLT considers the facial disjunctive program,

$$\text{Minimize}\{c^T x : x \in X \cap Y\}, \tag{1}$$

where X is a nonempty polytope, Y is the conjunction of $\hat{u}$ disjunctions given by,

$$Y = \bigcap_{u \in U} \left(\bigcup_{i \in Q_u} \{x : a_i^u x \geq b_i^u\} \right), \tag{2}$$

with $U = \{1, \cdots \hat{u}\}$, and at least one of the inequalities $a_i^u x \geq b_i^u$, for $i \in Q_u$, must be satisfied for each $u \in U$. Convex hull of feasible solutions may be recursively constructed via the following relations, as proposed by Balas [5], where $K_0 = X$:

$$K_u = conv\left[\bigcup_{i \in Q_u} \left(K_{u-1} \cap \{x : a_i^u x \geq b_i^u\}\right)\right], \text{ for } u = 1, \cdots, \hat{u}, \tag{3}$$

to obtain the relationship

$$K_0 \subseteq K_1 \subseteq \cdots \subseteq K_{\hat{u}} \equiv conv(X \cap Y). \tag{4}$$

One can represent K_u for 0 -1 MIP as follows:

$$K_u = conv[(K_{u-1} \cap \{x : x_u \leq 0\}) \cup (K_{u-1} \cap \{x : x_u \geq 1\})]. \tag{5}$$

Sherali and Adams [23] generated a hierarchy of relaxation of (4) using RLT. By this process, each constraint in K_{u-1} is multiplied by x_u and $1 - x_u$. The linear inequality, say $\mu^T x \leq \theta$, is multiplied by each variable in turn to give $(\mu^T x) x_s \leq \theta x_s$. The resulting nonlinear program is linearized using the variable substitution technique, thereby replacing each distinct product of variables by a single new variable. This yields a new convex hull of feasible solutions if applied n times and using the identity $x_s^2 = x_s$.

Consider the feasible region X which is defined in terms of binary variables $x_1, \cdots, x_n$ and bounded continuous variables $y_1, \cdots, y_m$. Consider the dth level of the RLT relaxation for $0 \leq d \leq n$ (Sherali and Adams [23]), with the bound factors of order d as

$$F_d(J_1, J_2) = \left[\prod_{j \in J_1} x_j\right]\left[\prod_{j \in J_2} \left(1 - x_j\right)\right], \ \forall J_1, J_2 \subseteq N \equiv \{1 \cdots n\}, \tag{6}$$

such that $J_1 \cap J_2 = \emptyset$ and $J_1 \cup J_2 = d$, where $F_0(\emptyset, \emptyset) = 1$. For any given d, there are $\binom{n}{d} 2^d$ such bound-factors. The lowest level of RLT relaxation is the most widely used in order to control the size of the resulting relaxation and it has proved to be effective to obtain tight lower bounds for the original problem. This new LP relaxation of the problem is stronger than the LP relaxation of the original problem. The projection of the resulting LP formulation into the space of the original x variables satisfies all simple disjunctive cuts (Laurent [16]). RLT has been extended to mixed 0- 1 LP and global optimisation (Sherali [25]).

There are three main ways of strengthening the first level RLT relaxation in the literature as outlined below

1. Let X be the $n \times n$ symmetric matrix in which $X_{ii} = x_i$, $\forall\, i$, and let $X_{ij} = y_{ij}$, $\forall\ i \neq j$. Note that $X = xx^T$. Define the augmented matrix

$$\hat{X} = \begin{pmatrix} 1 \\ x \end{pmatrix}\begin{pmatrix} 1 \\ x \end{pmatrix}^T = \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix}. \tag{7}$$

We can strengthen the relaxation by adding the constraint that makes $\hat{X}$ positive semidefinite (Bixby [8]).

2. Given two inequalities of the form $\mu^T x \le \theta$ and $\nu^T x \le \delta$, we have $(\theta - \mu^T x)(\delta - \nu^T x) \ge 0$ which results in

$$\mu^T X \nu - \theta \nu^T x - \delta \mu^T x + \theta\delta \ge 0. \tag{8}$$

3. We can add any valid inequality to Boolean quadratic polytope defined as

$$conv\left\{(x+y) \in \{0,1\}^{n+\binom{n}{2}} : y_{ij} = x_i x_j (\{i,j\} \subset N)\right\}. \tag{9}$$

There are many valid inequalities known for this polytope; see Padberg [20]. Here, we are proposing a new way of generating cutting planes for first-level RLT relaxations of mixed 0-1 programs motivated by the argument that when the degree of polynomial terms or factors is equivalent to the number of 0-1 variables, the resulting linear system will represent a polytope whose extreme points are precisely the 0-1 solutions feasible to the original problem (Sherali and Adams [23]). The remainder of our paper is arranged as follows. In Section 2, we review recent research in finding cutting planes. We develop new methods of generating cutting planes in Section 3. In Section 4, a new way of weakening the inequalities is presented and strengthening of the inequalities is carried out in Sections 5. We implement the algorithm and executes the program on numerical examples, comparing the results obtained with those of the instances in Section 6. Conclusions are drawn in Section 7.

2. Review of Recent Work on Cutting Planes

Recent research has been carried out on ways of finding cutting planes. Amaldi [3] introduced generation of coordinated cutting planes and the method reduces the number of branch-and-bound nodes. Generation of valid inequalities that simulteneously maximizes the cut violation and measure of diversity between the new and previously generated cutting planes have been suggested and the method was named to be bi-criteria method (Amaldi [2]). The cuts generated by this method were strong as compared to those obtained by maximizing the cut violation.

Choosing good separation methods may assist in handling large-size problems and effectively reducing the computational time (Ben-Ameur [7]). Neumaier and Shcherbina [19] introduced the idea of safe cuts which were further developed in the work of Cook et al. [9]. The floating-point arithmetic and mixed-integer rounding procedure were introduced. A research of Margot [18] attempted to compare cut generators using a method referred to as random diving towards a feasible solution. The method depends on the cut generator and precision of the LP. Integer linear combinations of rows of optimal simplex tableu were used to generate split cuts in a study of Cornuéjols and Nannicini [11].

3. Generating New Cutting Plane

3.1 Generation by taking binary variables only

We first construct a quartic valid inequality that is an inequality of up to four x variables without a y value. Let x_i, x_j x_k, and x_l be binary variables. We can generate the following five quartic inequalities:

$$x_i x_j x_k x_l \geq 0, \tag{10}$$
$$x_i x_j x_k (1 - x_l) \geq 0, \tag{11}$$
$$x_i x_j (1 - x_k)(1 - x_l) \geq 0, \tag{12}$$
$$x_i (1 - x_j)(1 - x_k)(1 - x_l) \geq 0, \tag{13}$$
$$(1 - x_i)(1 - x_j)(1 - x_k)(1 - x_l) \geq 0. \tag{14}$$

Similarly, one can formulate inequalities for other x combinations. Our work here considers the above inequalities (equations (10) to (14)), for brevity.

3.2 Generation by using inequalities

Given inequalities of the form

$$\mu^T x \leq \theta, \tag{15}$$
$$\nu^T x \leq \delta, \tag{16}$$
$$\psi^T x \leq \tau, \tag{17}$$
$$\alpha^T x \leq \zeta, \tag{18}$$

and binary variables x_m and x_n, we can generate quartic valid inequalities as

$$(\theta - \mu^T x)(\delta - \nu^T x)(\tau - \psi^T x) x_m \geq 0, \tag{19}$$
$$(\theta - \mu^T x)(\delta - \nu^T x)(\tau - \psi^T x)(1 - x_m) \geq 0, \tag{20}$$
$$(\theta - \mu^T x)(\delta - \nu^T x) x_m x_n \geq 0, \tag{21}$$
$$(\theta - \mu^T x)(\delta - \nu^T x) x_m (1 - x_n) \geq 0, \tag{22}$$
$$(\theta - \mu^T x)(\delta - \nu^T x)(1 - x_m)(1 - x_n) \geq 0. \tag{23}$$

It is also possible to take one linear inequality, say (15), and three binary variables, say x_m , x_n and x_r, to generate

$$(\theta - \mu^T x) x_m x_n x_r \geq 0, \tag{24}$$
$$(\theta - \mu^T x) x_m x_n (1 - x_r) \geq 0, \tag{25}$$
$$(\theta - \mu^T x) x_m (1 - x_n)(1 - x_r) \geq 0. \tag{26}$$
$$(\theta - \mu^T x)(1 - x_m)(1 - x_n)(1 - x_r) \geq 0. \tag{27}$$

We can take four linear inequalities, (15) to (18), of the system $Ax \leq b$ and form the quartic equality as

$$(\theta - \mu^T x)(\delta - \nu^T x)(\tau - \psi^T x)(\zeta - \alpha^T x). \tag{28}$$

We can deal with quartic terms of the form $x_i x_j x_k x_l$ using Lemma 3.1 below.

Lemma 3.1. Let x_i, x_j, x_k and x_l be variables all constrained to belong to the interval $[0,1]$. Let $y_{ij} = x_i x_j = (x_i x_j)^2$, $y_{ik} = x_i x_k = (x_i x_k)^2$, $y_{il} = x_i x_l = (x_i x_l)^2$, $y_{jk} = x_j x_k = (x_j x_k)^2$, $y_{jl} = x_j x_l = (x_j x_l)^2$, and $y_{kl} = x_k x_l = (x_k x_l)^2$ Then, the following lower and upper bounds on $x_i x_j x_k x_l$ hold:

$$x_i x_j x_k x_l \geq \max\{0;\ y_{ij} + y_{ik} + y_{il} - x_i;\ y_{ij} + y_{jk} + y_{jl} - x_j;\ y_{ik} + y_{jk} + y_{kl} - x_k;\ y_{il} + y_{jl} + y_{kl} - x_l\}, \tag{29}$$

$$x_i x_j x_k x_l \leq \min\{y_{ij},\ y_{ik},\ y_{il},\ y_{jk} + y_{jl} + y_{kl},\ 1 - x_i - x_j - x_k - x_l + y_{ij} + y_{ik} + y_{il} + y_{jk} + y_{jl} + y_{kl}\}. \tag{30}$$

Proof. The inequality $x_i x_j x_k x_l \geq 0$ is trivial. The fact that $(1 - x_i)(1 - x_j)(1 - x_k)(1 - x_l)$ must be non-negative proves the inequalities $x_i x_j x_k x_l \geq y_{ij} + y_{ik} + y_{il} - x_i,\ x_i x_j x_k x_l \geq y_{jk} + y_{jl} - x_j,$ $x_i x_j x_k x_l \geq y_{ik} + y_{jk} - x_k$ and $x_i x_j x_k x_l \geq y_{il} + y_{kl} - x_k$. Since $x_i x_j x_k(1 - x_l)$ is non-negative, it follows that $x_i x_j x_k x_l \leq y_{ij}$.

Similarly, inequalities $x_i x_j x_k x_l \leq y_{ik},\ x_i x_j x_k x_l \leq y_{il},\ x_i x_j x_k x_l \leq y_{jk},\ x_i x_j x_k x_l \leq y_{jl},$ $x_i x_j x_k x_l \leq y_{kl}$ hold. The fact that $(1 - x_i)(1 - x_j)(1 - x_k)(1 - x_l)$ is non-negative proves the inequality $x_i x_j x_k x_l \leq (1 - x_i - x_j - x_k - x_l + y_{ij} + y_{ik} + y_{il} + y_{jk} + y_{jl} + y_{kl})$. □

4. Weakening inequalities

Cutting planes can be produced by weakening the inequalities (24) to (27). Since $x_m,\ x_n$ and x_r are binary variables, at least two of them are equal. Suppose in this case that $x_n = x_r$. We have

$$x_m x_n x_r = x_m x_n^2 = y_{mn}. \tag{31}$$

The quartic inequality (24) can be rewritten as

$$\begin{aligned}
(\theta - \mu^T x) x_m x_n x_r &\geq 0,\\
\mu_i x_i x_m x_n x_r &\leq (\theta - \mu_m - \mu_n - \mu_r) x_m x_n x_r,\\
&\leq (\theta - \mu_m - \mu_n - \mu_r) x_m x_n^2,\\
&\leq (\theta - \mu_m - \mu_n - \mu_r) y_{mn}.
\end{aligned}$$

We can generalise the inequality as

$$\textstyle\sum_{i \in Q\setminus\{m,n,r\}} \mu_i x_i x_m x_n x_r \leq (\theta - \mu_m - \mu_n - \mu_r)(y_{mn} + y_{mr} + y_{nr}). \tag{32}$$

If $x_m = x_n$, then quartic inequality (25) can also be rewritten as

$$\begin{aligned}
(\theta - \mu^T x) x_m x_n (1 - x_r) &\geq 0,\\
\theta x_m x_n - \theta x_m x_n x_r &\geq (\mu^T x) x_m x_n - (\mu^T x) x_m x_n x_r,\\
\theta y_{mn} - \theta x_m x_n x_r &\geq \mu_i x_i x_m x_n - \mu_i x_i x_m x_n x_r - \mu_m x_m x_n x_r - \mu_n x_n x_r\\
&\quad - \mu_r x_n x_r,\\
\mu_i (x_i x_m x_n - x_i x_m x_n x_r) &\leq (\theta - \mu_m) y_{mn} - x_m x_n x_r (\theta - \mu_m) + \mu_n y_{nr} + \mu_r y_{nr},\\
\mu_i (y_{im} - x_i x_m x_n x_r) &\leq (\theta - \mu_m) y_{mn} - y_{mr} (\theta - \mu_m) + \mu_n y_{nr} + \mu_r y_{nr}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\textstyle\sum_{i \in Q\setminus\{m,n,r\}} \mu_i (y_{im} + y_{in} - x_i x_m x_n x_r) &\leq (\theta - \mu_m)(y_{mn} - y_{mr} - y_{nr})\\
&\quad + (\mu_n + \mu_r) y_{nr}.
\end{aligned} \tag{33}$$

Inequality (26) is rewritten as

$$\begin{aligned}
&(\theta - \mu^T x) x_m (1 - x_n)(1 - x_r) \geq 0,\\
&(\theta - \mu^T x)(x_m - 2x_m x_r + x_m x_n x_r) \geq 0,\\
&\theta x_m - 2\theta y_{mr} + \theta x_m x_n x_r - \mu_i [y_{im} - 2x_i x_m x_r + x_i x_m x_n x_r] - \mu_m x_m\\
&+ 2\mu_m y_{mr} - \mu_m x_m x_n x_r - \mu_n y_{nr} \geq 0,\\
&\mu_i [y_{im} - 2x_i x_m x_r + x_i x_m x_n x_r] \leq \theta x_m - 2\theta y_{mr} + \theta x_m x_n x_r\\
&- \mu_m x_m + 2\mu_m y_{mr} - \mu_m x_m x_n x_r - \mu_n y_{nr},\\
&\mu_i [y_{im} - 2x_i x_m^2 + x_i x_m x_n x_r] \leq \theta x_m - 2\theta y_{mr} + \theta x_m^2 x_r\\
&- \mu_m x_m + 2\mu_m y_{mr} - \mu_m x_m^2 x_r - \mu_n y_{nr}.
\end{aligned}$$

It follows that

$$\textstyle\sum_{i \in Q\setminus\{m,n,r\}} \mu_i [x_i x_m x_n x_r - y_{im}] \leq (\theta - \mu_m)(x_m - y_{mr} - y_{nr}) - \mu_n y_{nr}. \tag{34}$$

Similarly, (27) is rewritten as

$$\begin{aligned}
&(\theta - \mu^T x)(1 - x_m)(1 - x_n)(1 - x_r) \geq 0,\\
&\theta(1 - x_m - x_n - x_r + y_m + y_n + y_r) + \theta x_m x_n x_r - \mu_i (x_i - x_m - x_n - x_r\\
&- y_{im} - y_{in} - y_{ir} - x_i x_m x_n x_r) + \mu_m (x_m - y_{mr}) + \mu_n (x_n - y_{nr})
\end{aligned}$$

$$-\mu_r x_r \geq 0,$$
$$\mu_i(x_i - x_m - x_n - x_r - y_{im} - y_{in} - y_{ir} - x_i x_m x_n x_r) \leq \theta(1 - x_m - x_n - x_r)$$
$$+\theta(y_m + y_n + y_r) + \mu_m(x_m - y_{mr}) + \mu_n(x_n - y_{nr}) - \mu_r x_r + \theta x_m x_n x_r.$$

If $x_m = x_n = x_r$, then $x_m x_n x_r = x_r^2 x_r = x_r$. Therefore,

$$\sum_{i \in Q\setminus\{m,n,r\}} \mu_i(x_i - x_m - x_n - x_r - y_{im} - y_{in} - y_{ir} - x_i x_m x_n x_r)$$
$$\leq \theta(1 - x_m - x_n - x_r) + \theta(y_m + y_n + y_r) + \mu_m(x_m - y_{mr}) + \mu_n(x_n - y_{nr})$$
$$+x_r(\theta - \mu_r). \tag{35}$$

Using Lemma 3.1, inequalities (32) to (35) can be weakened by substituting $x_i x_m x_n x_r$ with the right hand side of (29) and (30) for lower and upper bounds respectively. Consider the mixed integer 0-1 program given by

$$\text{Min } c^T x$$
$$\text{subject to } \sum_{j=1}^{n} a_{kj} x_j \geq b_k, \text{ for } k = 1, \cdots, m, \tag{36}$$
$$0 \leq x \leq e, \; x_i \text{ binary, for } i \in B, \; x_i \text{ continuous, for } i \in C, \tag{37}$$

where $B = \{1, \cdots, n\}$, $C = \{n_1 + 1, \cdots, n\}$, and $e = (1, \cdots, 1)^T$. Multiplying the MIP by the bound-factors x_i and $(1 - x_i)$, $\forall i \in B$, and linearizing the problem for $x_i^2 = x_i$, $\forall i \in B$, and substituting $y_{ij} = x_i x_j$, $\forall i < j$, we obtain the first level RLT problem as follows:

$$\text{Min } c^T x$$
$$\text{subject to } (a_{ki} - b_k)x_i + \sum_{j \neq i} a_{kj} y_{(ij)} \geq 0, \; \forall \; k = 1, \cdots, m, \; \forall i \in B, \tag{38}$$
$$b_i x_i + \sum_{j \neq i} a_{kj}\left(x_j - y_{(ij)}\right) \geq b_k, \; \forall \; k = 1, \cdots, m, \; \forall i \in B, \tag{39}$$
$$0 \leq y_{ij} \leq x_i \text{ and } 0 \leq (x_j - y_{ij}) \leq (1 - x_i), \; \forall i \in B, \; j \in N, \; i < j, \tag{40}$$

where $N = \{1, \cdots, n\} \equiv B \cup C$.

Theorem 4.1. Let RLT be feasible, and define $\widehat{\text{R}}$LT as a formulation of RLT to which the implied original constraints (36) have been added. Then, there exists a dual optimal solution of $\widehat{\text{R}}$LT such that for each $k = \{1, \cdots, m\}$ and $i \in B$, the dual variable associated with at least one of the equalities (38) and (39) is zero. Hence, deleting such constraints from the $\widehat{\text{R}}$LT with associated dual multipliers being zero would yield a reduced first level RLT relaxation that preserves the lower bounding objective value of RLT.

Proof. See Sherali et al. [24]. □

By Theorem 4.1 we can safely discard (32) to (35) and concentrate on (32).

Theorem 4.2. For any set $\{m, n, r\} \subset A$, let M, N, R and K be disjoint subsets of $Q\setminus\{m, n, r\}$. Let $P = Q\setminus[\{m, n, r\} \cup M \cup N \cup R \cup K]$, and let $\mu^T x \leq \theta$ be one of the inequalities of the system $Ax \leq b$. Then, the following exponential large inequalities hold:

$$\sum_{i \in M \cup K} \mu_i y_{im} + \sum_{i \in N \cup K} \mu_i y_{in} + \sum_{i \in R \cup K} \mu_i y_{ir} - \sum_{i \in K} \mu_i x_i \leq -\mu(K^{min})$$
$$+\mu(M^{max} \cup K^{min})x_m + \mu(N^{max} \cup K^{min})x_n + \mu(R^{max} \cup K^{min})x_r \tag{41}$$
$$+\left(\theta - \mu\left[\{m, n, r\} \cup M^{max} \cup N^{max} \cup R^{max} \cup K^{min} \cup P^{min}\right]\right)$$
$$(y_{mn} + y_{mr} + y_{nr}).$$

Proof. From Lemma 3.1, the inequality (32) can be weakened by replacing $x_i x_m x_n x_r$ with $y_{im} + y_{in} + y_{ir} - x_i$, when $i \in K^{max}$, $y_{im} + y_{mn} + y_{mr} - x_m$, when $i \in M^{max}$ $y_{in} + y_{mn} + y_{nr} -$

x_n, when $i \in N^{max}$, $y_{ir} + y_{mr} + y_{nr} - x_r$, when $i \in R^{max}$, 0, when $i \in P^{max}$, y_{im}, when $i \in M^{min}$, y_{in}, when $i \in N^{min}$, y_{ir}, when $i \in R^{min}$, $1 - x_i - x_m - x_n - x_r + y_{im} + y_{in} + y_{ir} + y_{mn} + y_{mr} + y_{nr}$, when $i \in K^{min}$ and $y_{mn} + y_{mr} + y_{nr}$, when $i \in P^{min}$.

Rearranging proves (41). □

The produced inequalities are exponential in number. The separation can be achieved in a polynomial time as (10) to (28) are polynomial in number. There are z choices for the inequality $\mu^T x \leq \theta$ and $\binom{|A|}{3}$ choices for $\{m, n, r\}$. If we assume that $\mu,\ \theta,\ m,\ n,$ and r are fixed. Then (41) is rewritten as

$$\begin{aligned}&\textstyle\sum_{i\in M^{max}} \mu_i(y_{im} + y_{mn} + y_{mr} - x_m) + \sum_{i\in N^{max}} \mu_i(y_{in} + y_{mn} + y_{nr} - x_n)\\ &\quad + \textstyle\sum_{i\in R^{max}} \mu_i(y_{ir} + y_{mr} + y_{nr} - x_r) + \sum_{i\in K^{max}} \mu_i(y_{im} + y_{in} + y_{ir} - x_i)\\ &\qquad + \textstyle\sum_{i\in M^{min}} \mu_i y_{im} + \sum_{i\in N^{min}} \mu_i y_{in} + \sum_{i\in R^{min}} \mu_i y_{ir} + \sum_{i\in K^{min}} \mu_i(1 - x_i -\\ &x_m - x_n - x_r + y_{im} + y_{in} + y_{ir} + y_{mn} + y_{mr} + y_{nr}) + \textstyle\sum_{i\in P^{min}} \mu_i(y_{mn} + y_{mr} + y_{nr})\\ &\quad \leq (\theta - \mu_m - \mu_n - \mu_r)(y_{mn} + y_{mr} + y_{nr}).\end{aligned} \tag{42}$$

We examine the left hand side of the inequality to find the most violated $\{m, n, r\}$ inequality. Below, an algorithm is suggested to carry-out this task:

START Place i in either set $M,\ N,\ R,\ K, or\ P$
if $\mu_i > 0$ **then**
  Find max$\{y^*_{im} + y^*_{mn} + y^*_{mr} - x^*_m;\ \ y^*_{in} + y^*_{mn} + y^*_{nr} - x^*_n;\ \ y^*_{ir} + y^*_{mr} + y^*_{nr} - x^*_r;\ \ y^*_{im} + y^*_{in} + y^*_{ir} - x^*_i;\ \ 0\}$
  (Break ties arbitrarily)
else if $\mu_i < 0$ **then**
  Find min$\{1 - x^*_i - x^*_m - x^*_n - x^*_r + y^*_{im} + y^*_{in} + y^*_{ir} + y^*_{mn} + y^*_{mr} + y^*_{nr};\ \ y^*_{mn} + y^*_{mr} + y^*_{nr}\}$
  (Break ties arbitrarily)
else $\mu_i = 0$
  Place i in any set
STOP

5. Strengthening Inequalities

We strengthen the inequalities using the disjunctive argument.

Lemma 5.1. For any pair $\{m, n, r\} \subset A$, for every binary vector $x \in \{0,1\}^a$, the following disjunctive terms are satisfied:

$$\begin{aligned}&(x_m = 0 \wedge x_n = x_r = 1) \vee (x_n = 0 \wedge x_m = x_r = 1) \vee (x_r = 0 \wedge x_m = x_n = 1)\\ &\vee (x_m = 1 \wedge x_n = x_r = 0) \vee (x_n = 1 \wedge x_m = x_r = 0) \vee (x_r = 1 \wedge x_m = x_n = 0)\\ &\vee (x_m = x_n = 0 \wedge x_r = 1) \vee (x_m = x_n = 1 \wedge x_r = 0) \vee (x_m = x_r = 0 \wedge x_n = 1)\\ &\vee (x_m = x_r = 1 \wedge x_n = 0) \vee (x_n = x_r = 0 \wedge x_m = 1) \vee (x_n = x_r = 1 \wedge x_m = 0)\\ &\vee (x_m = x_n = x_r = 1) \vee (x_m = x_n = x_r = 0).\end{aligned} \tag{43}$$

Using Lemma 5.1, we can strengthen the inequality, if we let m, n, r, M, N, R, K, μ and θ be defined as in Theorem 4.2.

Theorem 5.2. Let

$$H_m = \min\{\mu M^+; (\theta - \mu(\{m\} \cup N^- \cup R^- \cup K^- \cup P^-)\},$$

$$H_n = \min\{\mu N^+; (\theta - \mu(\{n\} \cup M^- \cup R^- \cup K^- \cup P^-)\}$$

and

$$H_r = \min\{\mu R^+; (\theta - \mu(\{r\} \cup M^- \cup N^- \cup K^- \cup P^-)\}.$$

Then, equation (44) is valid for Q:

$$\sum_{i\in M\cup K} \mu_i y_{im} + \sum_{i\in N\cup K} \mu_i y_{in} + \sum_{i\in R\cup K} \mu_i y_{ir} - \sum_{i\in K} \mu_i x_i \leq (H_m + \mu K^-)x_m \\ (H_n + \mu K^-)x_n + (H_r + \mu K^-)x_r + (\theta - H_m - H_n - Hr - \mu(\{m,n,r\} \\ \cup P^-))y_{mn}y_{mr}y_{nr} - \mu K^-. \tag{44}$$

Proof. Using the disjunctive terms in (43), taking $x_m = x_n = x_r = 0$ and substituting into the left hand side of (44), we get $-\sum_{i\in K} \mu_i x_i$ which should not exceed $-\mu K^-$. Substituting $x_m = 0$ and $x_n = x_r = 1$ reduces the left hand side of (44) to $\sum_{i\in M} \mu_i x_i$ which cannot exceed H_m. We can do the same for the other disjunctive terms. If $x_m = x_n = x_r = 1$, then the left hand side reduces to $\sum_{i\in M\cup N\cup R\cup K} \mu_i x_i$ which also cannot exceed $\theta - \mu(\{n,m,r\} \cup P^-)$. Therefore, the left hand side does not exceed the right hand side of (44) in any of the disjunctive terms. □

6. Computational Analysis

To explore efficiency of the proposed method for generating cutting planes, separation and strengthening, 26 instances were taken from the second DIMACS challenge on max-clique, coloring, and satisfability; see Johnson and Trick [15]. The instances were subjected to the proposed method, coordinated cutting plane (COORD) method of Amaldi et al. [3], local branching (LB) method (Rodriguez-Martin and JoseSalazar-Gonzalez[22]) and the Gomory method (Gomory [12]). The instances were solved in MATLAB 7.0.4.365 environment using the Cplex solver provided by TOMLAB 7.9, an optimization environment in MATLAB, recording the number of cuts, time and the number iterations required to reach the bound; see Holmstrom [13] for a full description of TOMLAB and how it can be used.

In order to ensure that the proposed inequalities are not repeated, each linear inequality was also multiplied by the complement of each variable, to obtain n linear inequalities. The complexity of the proposed algorithm in ensuring that there were no repeatitions is a challenge needing further investigation. This has led to a limitation on the number of variables for the instances with a maximum of 15 variables. Verticies in nonincreasing order were sorted and added to the subset of vertices one at a time until a maximal clique was formed. Those that violeted a constraint were discarded. This technique was used to ensure that the inequalities already generated were not repeated in consecutive iterations. The method was used by Amaldi et al. [3].

Table 1 presents the comparison of the results recorded for the Gomory, COORD, local branching (LB) algorithms and our proposed method. The results show that the Gomory, COORD and LB algorithms are slower than our proposed method to achieve the bound. The proposed method required fewer number of iterations to achieve the bound, while the Gomory, COORD and LB required the on average, 5 more iterations to achieve the bound. The proposed method produced fewer cuts as iterations increased, thereby needing less CPU time as compared to the other methods. Considering all the instances, LB produced between 1.81 and 2.21, COORD produced between 1.21 and 1.34 and Gomory produced between 2.0 and 2.9 times more cuts than the proposed method. The computational results has shown the proposed method to be promising.

Table 1. Comparison of proposed method with the Gomory, COORD and LB methods

Instances		**COORD**			**Local Branching**			**Gomory**			**Proposed**		
	Bound	Iter	Cuts	Time	Iter	Cuts	Time	Iter	Cuts	Time	Iter	Cuts	Time
1-FullIns_ 3	3.3	13	13	0.0	11	11	0.0	14	14	0.0	10	8	0.0
1-Insertions_4	2.8	218	218	24.3	188	188	22.5	218	218	24.3	165	158	19.0
3-FullIns_3	5.2	36	36	0.2	19	19	0.1	35	35	0.2	17	17	0.1
3-Insertions_3	2.3	202	202	8.6	181	181	8.7	205	205	9.6	167	153	8.7
c-fat200-1	12.0	186	186	1.4	166	166	1.4	188	188	1.6	142	142	1.0
c-fat200-5	66.7	254	254	6.6	204	204	5.2	255	255	7.0	188	185	5.0
david	11.0	31	31	0.2	24	24	0.2	31	31	0.2	23	21	0.2
huck	11.0	26	26	0.1	23	23	0.1	25	25	0.1	20	20	0.1
jean	10.0	25	25	0.1	20	20	0.1	26	26	0.1	18	14	0.1
johnson16-2-4	8.0	20	20	3.4	20	20	5.7	20	20	5.7	20	20	2.9
johnson8-2-4	4.0	10	10	0.0	10	10	0.0	11	11	0.1	8	8	0.0
mug88_1	3.0	364	364	4.0	270	270	3.1	366	366	4.1	243	236	2.7
myciel3	2.9	14	14	0.0	12	12	0.0	14	14	0.0	10	9	0.0
petersen	2.5	13	13	0.0	7	7	0.0	14	14	0.0	6	6	0.0
queen5_5	5.0	5	5	0.0	5	5	0.0	5	5	0.0	5	5	0.0
queen8_8	8.4	123	123	3.4	112	112	3.1	124	124	3.8	102	102	2.4
queen9_9	9.0	252	252	12.1	256	256	12.8	251	251	12.1	251	251	12.0
r125.1c	46.0	54	54	26.3	54	54	93.6	52	52	27.0	48	48	23.1
sudokuc	9.0	20	20	1.3	22	22	2.1	24	24	1.3	20	20	1.3
ship-shipc	19.0	29	29	0.8	30	30	1.2	32	32	2.0	29	25	0.7
knights8_8c	32.0	110	110	3.9	67	67	3.3	115	115	4.0	68	68	3.3
kneser8-3c	28.0	96	96	4.9	59	59	5.4	96	96	5.0	59	59	5.0
barleyc	20.0	27	27	0.6	26	26	0.8	27	27	0.6	25	24	0.8
alarmc	18.0	19	19	0.2	19	19	0.3	19	19	0.2	18	18	0.2
1ubqc	30.5	57	57	3.7	49	49	4.2	59	59	3.7	40	40	2.9
hamming6-2	32.0	105	105	1.9	82	82	1.5	111	111	2.9	71	71	1.4

Fig. 1 and fig. 2 present graphical illustrations of the performance of the Gomory, COORD, LB and the proposed method for the cases queen8_8 and huck, respectively, as a function of the iterations. The graphical charts illustrate notable improvement provided by the proposed method, notably in the earlier generation of the cutting planes. The proposed method also provides tighter bounds as compared to the Gomory, COORD and LB algorithms, thereby converging faster than the other considered algorithms.

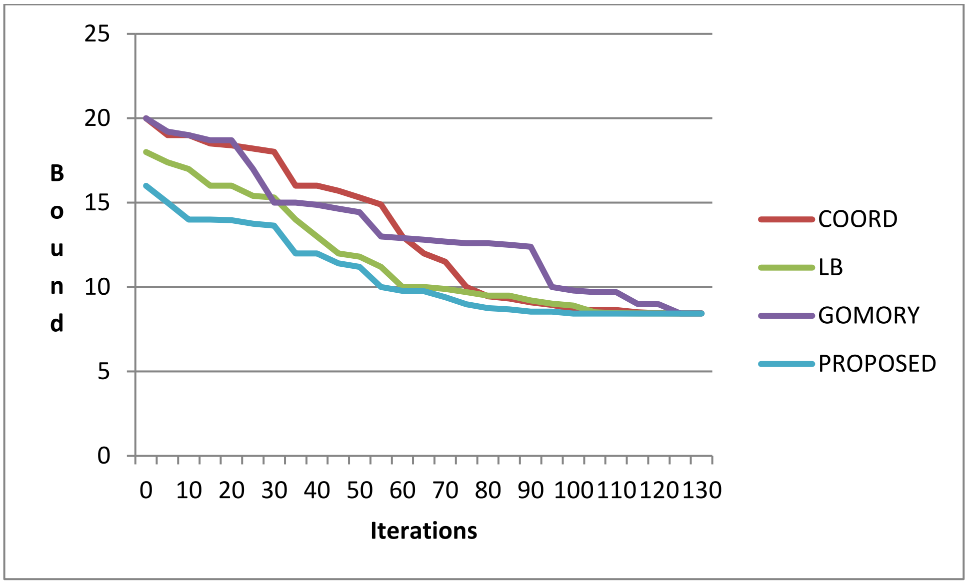

Figure 1. Comparison of the proposed method , COORD, LB and Gomory using queen8_8 as the instance.

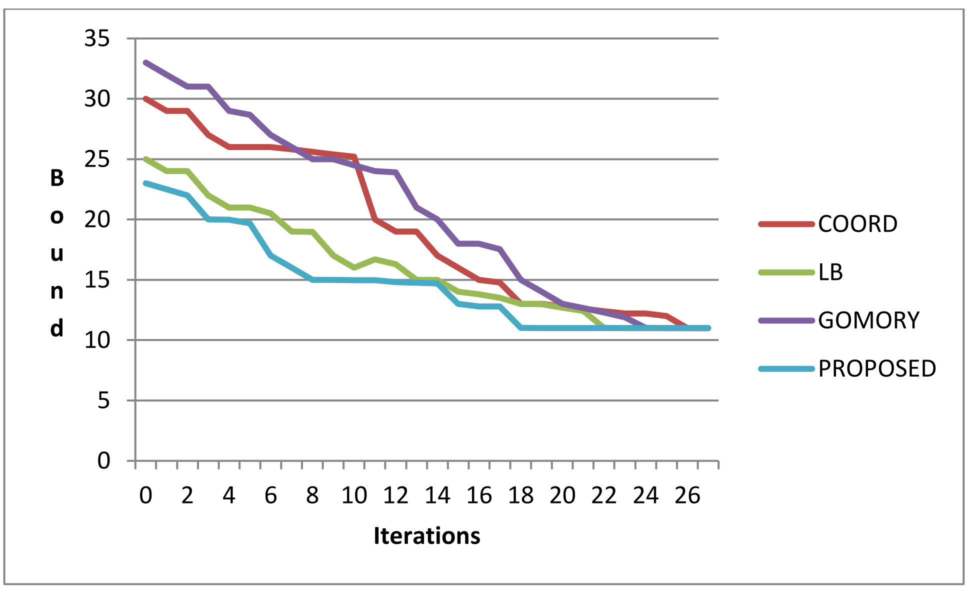

Figure 2. Comparison of the proposed method , COORD, LB and Gomory using huck as the instance.

5.1 Statistical analysis

An analysis of variance (ANOVA) parametric statistical test was performed using the SPSS version 20.0.0 software package to compare the time value. This was done to compare the obtained results. The results of the LB algorithm and the proposed method are compared, since it is reported in the literature that LB algorithm presents best results (Rodriguez-Martin and JoseSalazar-Gonzalez[22]). The significance level is considered at 5%. The null hypothesis is taken to be that the results of the proposed method and that of the LB algorithm are not significantly different while the alternative hypothesis is that the difference is significant. Table 2 presents the ANOVA results from SPSS. The results in Table 2 show that the p-value is 0.0 ($0.0 < 0.05$) and we may conclude that there is a significant difference between the local branching algorithm and proposed method interms of the execution time and number of iterations in achieving optimality. The ANOVA table shows that the proposed method performs better as compared to the local branching method.

Table 2. SPSS output table for the ANOVA test of local branching and the proposed method

Source of variation	Sum of squares	DF	Mean square	F	P- Value
Between groups	76.0	1	76.0	5.7	0.0
Within groups	992.0	25	10.0		
Total	1068.0	26			

7. Conclusion

There are several ways of generating cutting planes of a MIP. Using binary values to develop quartic inequalities seems to be promising. Generated inequalities can be weakened easily to generate the cutting planes of a MIP. Separation is achieved in a polynomial time and the proposed algorithm to carry-out this task seems to be promising. It is therefore recommended that further research be focused on development of prototypes or solvers that include several generations of first-level reduced MIPs, which are easier to solve. The proposed algorithm for cutting planes generation becomes complicated as the number of variables increases and may lead to repeats of inequalities already generated. It is therefore recommended that the proposed algorithm be modified to overcome this difficulty.

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