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Boundedness of KKT Multipliers in Fractional Programming Problem Using Convexificators

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Here, using the idea of convexificators, we study boundedness and nonemptiness of lagrange multipliers satisfying the first order necessary conditions. We consider a class of nonsmooth fractional programming problems with equality and inequality constraints and an arbitrary set constraint. Within this context, we define a generalized Mangasarian-Fromovitz type constraint qualification and show that this constraint qualification is necessary and sufficient conditions for the Karush-Kuhn-Tucker (KKT) multipliers set to be nonempty and bounded.

***Keywords**: Fractional problem, Multiplier vector, Constraint qualification, Convexificator.*

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1. Introduction

We discuss bounded and nonemptiness of Lagrange multipliers satisfying the first order necessary conditions. Bounded and nonemptiness of the Karush-Kuhn-Tucker *(*KKT) multipliers set for optimization problems have been studied by several researchers in recent years. A necessary and sufficient regularity condition for nonempty and boundedness of KKT multipliers set for a differentiable scalar optimization problem with equality and inequality constraints were derived in Gauvin [9]. For a nonsmooth scalar optimization problem, a necessary and sufficient condition for the set of multiplier vectors to be nonempty and bounded were obtained ([19] and [20]). In the more general setting of Banach spaces, Jourani [13] introduced several constraint qualifications, and showed that the conditions guarantee the nonemptiness and the boundedness of the Lagrange multiplier sets for general nondifferentiable programming problems. Also, Li and Zhang [15] introduced constraint qualifications and studied existence and boundedness of the KKT multipliers set for a nonsmooth multiobjective optimization problem with inequality constraints and an arbitrary set constraint, where all functions were locally Lipschitz.

Convexificator is viewed as a generalization of the idea of subdifferential, in as much as many of the well-known subdifferentials, such as those by Clarke, Michel Penot and Treiman, are convexificators for locally Lipschitz functions. They are always closed sets, but not necessarily convex or compact, unlike the well-known subdifferentials which are convex and compact objects. The concept of convexificator was first introduced by Demyanov [4] in 1994 as a generalization of the notion of upper convex and lower concave approximation. Recently, the idea of convexificators has been employed to extend and strengthen various results in nonsmooth analysis and optimization (see [4, 5, 7, 11, 12, 16, 23]). For a locally Lipschitz function, most known subdifferentials are convexificators and these known subdifferentials may contain the convex hull of a convexificator; see, for instance, [5, 12, 24]. Optimality conditions were also discussed for vector minimization problems in terms of convexificators. For nonsmooth optimization problems, various results concerning Fritz-John type and KKT type necessary optimality conditions that use convexificators have been developed in [11, 16, 23, 24]. Later, Golestani et al. [10] obtained nonsmooth analogue of the generalized Mangasarian-Fromovitz constraint qualication by using the upper semi-regular

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convexificators and for efficient solutions, thay derived strong KKT necessary optimality conditions for a nonsmooth multiobjective optimization problem with inequality constraints and an arbitrary set constraint. Upper semi-regular convexificator is a strengthened version of an upper convexifictor. Babahadda and Gadhi [1] studied necessary optimality conditions with the help of an appropriate regularity condition using convexificators for bilevel programming problems. Recently, Gadhi [8] established necessary and sufficient optimality conditions for a multiobjective fractional programming problem in terms of convexificators.

Our aim here is to introduce generalized Mangasarian-Fromovitz type constraint qualification for a nonsmooth fractional programming problem with equality constraints, inequality constraints and an arbitrary set constraint via convexificator and show that are necessary and sufficint conditions for the KKT multipliers set to be nonempty and bounded. Since the Clarke and the Michel-Penot subdifferentials of a locally Lipschitz function are convexificators, the results in our work are valid with the convexificators being replaced respectively by the Clarke and the Michel-Penot subdifferentials.

The remainder of our work is organized as follows. In the Section1, we introduce notations and give the basic definitions of convexificators and derive some preliminary results to be used in the rest of the article. In the Section2, we introduce an extended version of the Mangasarian-Fromovitz type constraint qualification for a nonsmooth fractional programming problem with equality constraints, inequality constraints and an arbitrary set constraint via convexificator. Furthermore, for this problem with locally Lipschitz objective and inequality constraint functions and continuously differentiable equality constraint functions, a necessary and sufficient condition is presented for the set of KKT multipliers to be nonempty and bounded.

2. Preliminaries

Throughout our work, $\mathbb{R}^n$ is the usual $n-$dimensional Euclidean space. Let S be a nonempty subset of $\mathbb{R}^n$. The convex hull and closure of S are denoted by coS and $cl\,S$, respectively. The negative and strictly negative polar cones S^- and S^s are defined respectively by

$$S^- := \{u \in \mathbb{R}^n \mid \langle x, u\rangle \leq 0 \ \ \forall x \in S\}, \qquad S^S := \{u \in \mathbb{R}^n \mid \langle x, u\rangle < 0 \ \ \forall x \in S\}.$$

The Clarke tangent cone $T^C(S,x)$ and the Clarke normal cone $N^C(S,x)$ to S at $x \in cl\,S$ are defined respectively by

$$T^C(S,x) \coloneqq \{d \in \mathbb{R}^n : \forall\, x_k \in S,\ x_k \longrightarrow x, \forall\, t_k \downarrow 0, \exists\, d_k \longrightarrow d \text{ such that } x_k + t_k d_k \in S, \forall k\},$$

$$N^C(S,x) \coloneqq T^C(S,x)^- \coloneqq \{\xi \in \mathbb{R}^n : \langle \xi, d\rangle \leq 0 \ \forall\, T^C(S,x)\}.$$

It is well-known that $T^C(S,x)$ and $N^C(S,x)$ are always nonempty, closed and convex. Let $f\colon \mathbb{R}^n \to \overline{\mathbb{R}} \coloneqq \mathbb{R} \cup \{+\infty\}$ be an extended real valued function and,

$$f^-(x;v) \coloneqq \lim_{t\downarrow 0} inf\ \frac{f(x+tv)-f(x)}{t} \quad \text{and} \quad f^+(x;v) \coloneqq \lim_{t\downarrow 0} inf\ \frac{f(x+tv)-f(x)}{t}$$

denote, respectively, the lower and upper Dini directional derivatives of f at x in direction v. It is worth mentioning that if $f\colon \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz, then both the lower and upper Dini derivatives exist finitely. Now, we recall the definitions of upper and lower convexificators from [12]:

- f is said to have an upper convexificator (respectively, upper regular convexificator) at $x \in \mathbb{R}^n$ if there is a closed set $\partial^* f(x) \subset \mathbb{R}^n$ such that for each $v \in \mathbb{R}^n$,

$$f^-(x;v) \leq sup_{\xi\in\partial^* f(x)}\langle\xi,v\rangle \quad \left(respectively, \quad f^+(x;v) = sup_{\xi\in\partial^* f(x)}\langle\xi,v\rangle\right).$$

- *f* is said to have a lower convexificator (resp., lower regular convexificator) at *x* if there is a closed set $\partial^* f(x) \subset \mathbb{R}^n$ such that for each $v \in \mathbb{R}^n$,

$$f^+(x;v) \geq inf_{\xi\in\partial^* f(x)}\langle\xi,v\rangle \quad \left(respectively, \quad f^-(x;v) = inf_{\xi\in\partial^* f(x)}\langle\xi,v\rangle\right).$$

A closed set $\partial^* f(x) \subset \mathbb{R}^n$ is said to be a convexificator of *f* at *x* if and only if it is both upper and lower convexificator of *f* at *x*.

Convexificators are not necessarily convex or compact. These relaxations allow applications to a large class of nonsmooth functions. The upper convexificator is also known as the Jeyakumar-Luc subdifferential of *f* at *x* [24]. We point out that if a continuous function $f:\mathbb{R}^n \to \overline{\mathbb{R}}$ admits a locally bounded upper convexificator at *x*, then it is locally Lipschitz around the point (see [12]). In [11], the notion of convexificator was extended and used to unify and strengthen various results in nonsmooth analysis and optimization. Along the lines of [7], we now give the definition of upper semi-regular convexificators which will be useful later:

- The function $f:\mathbb{R}^n \to \overline{\mathbb{R}}$ is said to have an upper-semi regular convexficator at $x \in \mathbb{R}^n$ if there is a closed set $\partial^* f(x) \subset \mathbb{R}^n$ such that for each $v \in \mathbb{R}^n$, $f^+(x;v) \leq sup_{\xi\in\partial^* f(x)}\langle\xi,v\rangle$.

- *f* is said to have a lower semi-regular convexificator at $x \in \mathbb{R}^n$ if there is a closed set $\partial^* f(x) \subset \mathbb{R}^n$ such that for each $v \in \mathbb{R}^n$, $f^-(x;v) \geq inf_{\xi\in\partial^* f(x)}\langle\xi,v\rangle$.

Obviously, an upper (lower) regular convexificator of *f* is also an upper (lower) semi-regular convexificator of *f* and each upper (lower) semi-regular convexificator is an upper (lower) convexificator. Moreover, convex hull of an upper semi-regular convexificator of a locally Lipschitz function may be strictly contained in the Clarke and Michel-Penot subdifferential (see Example 2.1 of [12]).

Example 1.1. Let $\mathbb{Q}$ denote the set of rationals and consider $f:\mathbb{R} \to \mathbb{R}$ given as follows

$$f(x) = \begin{cases} \sin 2x, & if\ x \in \mathbb{Q}\cap[0,+\infty) \\ x^3 - 3x & if\ x \in \mathbb{Q}\cap(-\infty,0] \\ 0 & otherwise\,. \end{cases}$$

Observe that in this case for the point x = 0 we have the upper and lower Dini derivatives given as follows:

$$f(x) = \begin{cases} 2v & if\ v \geq 0 \\ -3v & if\ v < 0 \end{cases}, \quad f^-(0;v) = 0, \qquad (\forall\, v \in \mathbb{R}).$$

The sets $\{-3, 2\}$ and $[-3, 2]$ are upper semi-regular convexificators of *f* at x = 0.

Since for all $v \in \mathbb{R}$, $f^-(x; v) \leq f^+(x; v)$, an upper semi-regular convexificator is also an upper convexificator of f at x. The converse is not necessarily true.

Remark 1.2. It is clear that every differentiable function has an upper regular convexificator given by $\partial^* \mathrm{f}(\mathrm{x}) = \{\nabla \mathrm{f}(\mathrm{x})\}$. Since a locally Lipschitz function is differentiable almost everywhere, it admits upper regular convexificator over a dense set. If $\mathrm{f}: \mathbb{R}^{\mathrm{n}} \to \mathbb{R}$ is locally Lipschitz, then the Clarke subdifferential $\partial_{\mathrm{C}} \mathrm{f}(\mathrm{x})$ [3], the Michel-Penot subdifferential $\partial^{\circ} \mathrm{f}(\mathrm{x})$ [17], the Mordukhovich subdifferential $\partial_{\mathrm{M}} \mathrm{f}(\mathrm{x})$ [18] and the Treiman subdifferential $\partial_{\mathrm{T}} \mathrm{f}(\mathrm{x})$ [22] are examples of upper semi-regular convexificators for f.

Let us now examine some calculus rules for upper semi-regular convexificators under appropriate conditions. The proofs for the following two lemmas are based on the technique used in [12, Rules 4.1 and 4.2].

Lemma 1.3. Let $\partial^* \mathrm{f}(\mathrm{x})$ be an upper semi-regular convexificator and $\partial^* \mathrm{f}(\mathrm{x})$ be a lower semi-regular convexificator of f at x. Then, $\lambda\, \partial^* \mathrm{f}(\mathrm{x})$ is an upper semi-regular convexifi cator for λf at x for every $\lambda > 0$ and $\lambda\, \partial^* \mathrm{f}(\mathrm{x})$ is an upper semi-regular convexificator for λf at x for every $\lambda < 0$.

Lemma 1.4. Assume that the functions $\mathrm{f}, \mathrm{g}: \mathbb{R}^{\mathrm{n}} \longrightarrow \mathbb{R}$ admit upper semi-regular convexificators $\partial^* \mathrm{f}(\mathrm{x})$ and $\partial^* \mathrm{g}(\mathrm{x})$ at x, respectively. Then, $\partial^* \mathrm{f}(\mathrm{x}) + \partial^* \mathrm{g}(\mathrm{x})$ is an upper semi-regular convexificator of $\mathrm{f} + \mathrm{g}$ at x.

3 Main Results

The present section will be devoted to developing nonempty and boundedness of the KKT multipliers set for a fractional optimization problem with equality and inequality constraints and an arbitrary set constraint in terms of lower and upper semi-regular convexificators.

Consider the following nonsmooth fractional programming problem:

$$
\begin{array}{llll}
(\mathrm{P}) & \min & \dfrac{f(x)}{g(x)} & \\
 & \mathrm{s.t.} & k_i(x) \leq 0, & i \in I := \{1, \dots, m\}, \\
 & & h_j(x) = 0, & j \in J := \{1, \dots, q\}, \\
 & & x \in S, &
\end{array}
$$

where f, g, k_i and h_j (for all $i \in I$ and $j \in J$) are function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ with $f(x) \geq 0$ and $g(x) > 0$. The active constraint indices at the feasible point $\bar{x}$ is denoted by $I(\bar{x})$. Here, we assume that all the functions $f, g, k_i, i \in I(\bar{x})$ are locally Lipschitz around $\bar{x}$ and all the functions $k_i, i \notin I(\bar{x})$, are continuous at $\bar{x}$. Suppose, in addition, that all the functions $h_j, j \in J$ are continuously differentiable and S is an arbitrary subset of $\mathbb{R}^n$. Also, for the vector function $h := (h_1, \dots, h_q)$, we define

$$H := \{x \in \mathbb{R}^n : h(x) = 0\}.$$

Let $\bar{x}$ be a feasible point for problem (P). Denote the set of all KKT multiplier vectors associated with the inequality and equality constraints by $\Lambda(\bar{x})$, i.e., $(\mu, \lambda) \in \mathbb{R}^m \times \mathbb{R}^q$, belong to $\Lambda(\bar{x})$ if and only if

$$0 \in co\, \partial^* f(\bar{x}) - \varphi(\bar{x})co\, \partial_* g(\bar{x}) + \sum_{i=1}^{m} \mu_i\, co\partial^* k_i(\bar{x}) + \sum_{j=1}^{q} \lambda_i \nabla h_j(\bar{x}) + N^C(S, \bar{x}),$$
$$\mu_i \geq 0, \qquad \mu_i k_i(\bar{x}) = 0, \;\; \forall i \in I,$$

where $\varphi(\bar{x}) = \frac{f(\bar{x})}{g(\bar{x})}$.

In the rest of our work, using the idea of upper semi-regular convexificators, we introduce a constraint qualification of Mangasarian-Fromovitz type for the fractional optimization problem (P) and show that this constraint qualification is necessary and suffcint for the KKT multiplier set to be nonempty and bounded.

Let us first consider the following nonsmooth optimization problem:

$$(\widetilde{\mathrm{P}}) \qquad \min \quad F(x)$$
$$s.t. \quad G_i(x) \leq 0, \quad i \in I, \quad x \in C,$$

where $F : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ and $G_i : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ are real-valued functions (for $i \in I$) and C is a subset of $\mathbb{R}^n$. The active constraint indices at the feasible point $\bar{x}$ is denoted by $\widehat{I(\bar{x})}$.

Let $\bar{x}$ be a feasible point for problem $(\widetilde{\mathrm{P}})$. We say that the $(CQ1)$ is satisfied at $\bar{x}$ if $G^s \cap T^C(C, \bar{x})$ is nonempty, where

$$G := \bigcup_{i \in \widehat{I(\bar{x})}} co\partial^* G_i(\bar{x}).$$

In order to establish our main theorem, we present the following auxiliary result without proof, since it follows along the lines of the proof given for [10, Theorem 1].

Lemma 2.1. Let $\bar{x}$ be a local optimal solution for $(\widetilde{\mathrm{P}})$. Suppose that F and G_i^s are locally Lipschitz functions at $\bar{x}$ which admit bounded upper semi-regular convexificators $\partial^* F(\bar{x})$ and $\partial^* G_i(\bar{x})$ for all $i \in I$. If (CQ1) holds at $\bar{x}$, then there exists a vector $\mu \in \mathbb{R}^m$ such that

$$0 \in co\partial^* F(\bar{x}) + \sum_{i=1}^{m} \mu_i\; co\partial^* G_i(\bar{x}) + N^C(C, \bar{x}),$$
$$\mu_i \geq 0, \qquad \mu_i G_i(\bar{x}) = 0, \;\; \forall i \in I.$$

Using the idea of upper semi-regular convexificators, we introduce the following nonsmooth analogue of the generalized Mangasaria-Fromovitz constraint qualification which is called $(CQ2)$.

Definition 2.2. Let $\bar{x}$ be a feasible point of problem (P). We say that the generalized Mangasaria-Fromovitz constraint qualification (CQ2) is satisfied at $\bar{x}$ if $\{\nabla h_j(\bar{x})\}_{j \in J}$ is a linearly independent set and there exists $d \in \mathrm{int}T^C(S, \bar{x})$ satisfying

$$\langle \xi_i, d \rangle \leq -b_i, \qquad \forall \xi_i \in co\,\partial^* k_i(\bar{x}),$$

And

$$\langle \nabla h_j(\bar{x}), d\rangle = 0, \qquad \forall j \in J.$$

for all $i \in I(\bar{x})$ and some $b_i > 0$.

Using the following lemma we can find a necessary and suffcient condition for nonempty and boundedness of $\Lambda(\bar{x})$.

Lemma 2.3. [14]. Let $\bar{x}$ be a feasible point of fractional programming problem (P). Then $\bar{x}$ is a local solution of (P) iff $\bar{x}$ be a local solution of the following scalar optimization problem:

$$\begin{array}{lll} (SP) & \min & f(x) - \varphi(\bar{x})\, g(x) \\ \text{s.t} & & k_i(x) \leq 0, \quad i \in I, \quad h_j(x) = 0, \quad \forall j \in J, \quad x \in S. \end{array}$$

Now, we are ready to prove our main result which establish the equivalence of $(CQ2)$ with the nonempty and boundedness of the KKT multiplier set at a local optimal solution of (P).

Theorem 2.4. Let $\bar{x}$ be a local optimal solution for (P). Suppose that f, g and k_i s are locally Lipschitz functions at $\bar{x}$. Assume that g admit bounded lower semi-regular convexificator $\partial_* g(\bar{x})$ and f and k_i s admit bounded upper semi-regular convexificators $\partial^* f(\bar{x})$ and $\partial^* k_i(\bar{x})$ for all $i \in I$. Also, suppose that h_js are continuously differentiable and that $\mathrm{int}T^C(S,\bar{x}) \neq \emptyset$. Then the following conditions are equivalent.

(i) (CQ2) is satisfied at $\bar{x}$,
(ii) $\Lambda(\bar{x})$ is a nonempty bounded subset of $\mathbb{R}^{m+q}$.

Proof. (**i**) $\Rightarrow$ (**ii**). Let us first show that $(CQ2)$ ensures the nonemptiness of $\Lambda(\bar{x})$. Since all $h_j s$ is continuously differentiable and the set $\{\nabla h_j(\bar{x})\}_{j\in J}$ are linearly independent, it can be shown that

$$\begin{gathered} N^C(H,\bar{x}) = span\{\nabla h_j(\bar{x}) : \ j \in J\}, \\ T^C(H,\bar{x}) = \{v \in \mathbb{R}^n : \ \langle \nabla h_j(\bar{x}), v\rangle = 0, \ \ j \in J\} \end{gathered} \tag{1}$$

Since $(CQ2)$ holds, $intT^C(S,\bar{x}) \cap T^C(H,\bar{x})$ is nonempty, thus using of [21, Theorem 5] we get

$$\begin{gathered} T^C(H,\bar{x}) \cap T^C(S,\bar{x}) \subseteq T^C(H \cap S,\bar{x}),, \\ N^C(H \cap S,\bar{x}) \subseteq \ N^C(H,\bar{x}) + \ N^C(S,\bar{x}). \end{gathered} \tag{2}$$

Since $\partial^* f(\bar{x})$ is an upper semi-regular convexificator of $f(\cdot)$ at $\bar{x}$ and $\partial_* g(\bar{x})$ is a lower semi-regular convexificator of $g(\cdot)$ at $\bar{x}$, using Lemma 1.3 and Lemma 1.4, we have that $\partial^* f(\bar{x}) - \varphi(\bar{x})\partial_* g(\bar{x})$ is an upper semi-regular convexifactor of $f(\cdot) - \varphi(\bar{x})g(\cdot)$ at $\bar{x}$. On the other hand, $CQ2$ together with (2) implies that $CQ1$ is satisfied for ($\tilde{P}$). Thus, by Lemma 2.1 and Lemma2.3 there exist nonnegative numbers $\mu_1, \ldots, \mu_m$ such that

$$0 \in co\big(\partial^* f(\bar{x}) - \varphi(\bar{x})\partial_* g(\bar{x})\big) + \sum_{i=1}^{m} \mu_i \, co\partial^* k_i(\bar{x}) + N^C(H \cap S,\bar{x})$$

$$\subseteq co\partial^* f(\bar{x}) - \varphi(\bar{x}) co\partial_* g(\bar{x}) + \sum_{i=1}^{m} \mu_i \, co\partial^* k_i(\bar{x}) + N^C(H \cap S, \bar{x}),$$
$$\mu_i k_i(\bar{x}\ ) = 0, \ \ \forall i \in I.$$

Therefore, from (1) and (2) there exists a vector $\lambda \in \mathbb{R}^q$ such that

$$0 \in co\partial^* f(\bar{x}) - \varphi(\bar{x}) co\partial_* g(\bar{x}) + \sum_{i=1}^{m} \mu_i \, co\partial^* k_i(\bar{x}) + \sum_{j=1}^{q} \lambda_i \nabla h_j(\bar{x}) + N^C(S, \bar{x}),$$
$$\mu_i k_i(\bar{x}\ ) = 0, \ \ \forall i \in I.$$

Thus $\Lambda(\bar{x})$ is a nonempty set.

Now, we show that $(CQ2)$ ensures the boundedness of $\Lambda(\bar{x})$. Since $\{\nabla h_j(\bar{x}) : j \in J\}$ is a linearly independent set, for each subset $\hat{J} \subseteq J$, by Gordan's theorem, there exists $\hat{d} \in \mathbb{R}^n$ such that

$$\langle \nabla h_j(\bar{x}), \hat{d}\ \rangle < 0, \qquad \forall j \in \hat{J},$$
$$\langle \nabla h_j(\bar{x}), \hat{d}\ \rangle > 0, \qquad \forall j \in J \setminus \hat{J}. \tag{3}$$

Let $d_0 \in intT^C(S, \bar{x})$ be a vector which is satisfied in $(CQ2)$. Thus, for all $i \in I(\bar{x})$,

$$\langle \xi_i\ , d_0 \rangle \le -b_i, \qquad \forall \xi_i \in co\partial^* k_i(\bar{x}).$$

Then, there exists a $\hat{\delta} > 0$ such that $\hat{\epsilon} \in (0, 1)$ may be chosen so small that for every $i \in I(\bar{x})$,

$$(1 - \hat{\epsilon}) \langle \xi_i\ , d_0 \rangle + \hat{\epsilon} \langle \xi_i\ , \hat{d} \rangle \le -\hat{\delta} < 0, \qquad \forall \xi_i \in co\partial^* k_i(\bar{x}), \tag{4}$$

and such that $\bar{d} = (1 - \hat{\epsilon}) d_0 + \hat{\epsilon}\, \hat{d} \in T^C(S, \bar{x})$. Since by $(CQ2)$, we have

$$\langle \nabla h_j(\bar{x}), d_0\ \rangle = 0, \qquad \forall j \in J,$$

from (3), we have

$$\langle \nabla h_j(\bar{x}), \bar{d}\ \rangle = \hat{\epsilon} \langle \nabla h_j(\bar{x}), \hat{d}\ \rangle \le -\hat{\epsilon}\, \widehat{\rho}\ \le -\widehat{\beta}\,, \qquad \forall j \in \hat{J},$$
$$\langle \nabla h_j(\bar{x}), \bar{d}\ \rangle = \hat{\epsilon} \langle \nabla h_j(\bar{x}), \hat{d}\ \rangle \ge \hat{\epsilon}\, \widehat{\rho}\ \ge \widehat{\beta}\,, \qquad \forall j \in J \setminus \hat{J}. \tag{5}$$

and from (4), for all $i \in I(\bar{x})$,

$$\langle \xi_i\ , \bar{d} \rangle \le -\hat{\delta}\ \le -\widehat{\beta}\ < 0, \qquad \forall \xi_i \in co\partial^* k_i(\bar{x})\,, \tag{6}$$

where $\widehat{\rho} = \min_{j \in J} \left| \langle \nabla h_j(\bar{x}), \hat{d} \rangle \right|$ and $\widehat{\beta} = \min\{\hat{\delta}, \hat{\epsilon}\, \widehat{\rho}\ \} > 0$. Now, suppose that $(\mu, \lambda) \in \mathbb{R}^m \times \mathbb{R}^q$ is an arbitrary multiplier vector in $\Lambda(\bar{x})$ and $\hat{J} = \{j \in J : \lambda_j > 0\}$. Therefore, there exist $\xi \in co\partial^* f(\bar{x})$, $\rho \in \partial^* g(\bar{x})$, $\xi_i \in co\partial^* k_i(\bar{x})\,, i \in I$ and $\eta \in N^C(S, \bar{x})$ such that

$$0 = \xi - \varphi(\bar{x})\rho + \sum_{i=1}^{m} \mu_i\, \xi_i + \sum_{j=1}^{q} \lambda_j \nabla h_j(\bar{x}) + \eta,$$
$$\mu_i = 0, \qquad \forall I \notin I(\bar{x}).$$

Since $\bar{d} \in T^C(S,\bar{x})$, we have

$$\langle \xi - \varphi(\bar{x})\rho \,, \bar{d}\,\rangle + \sum_{i=1}^{m} \mu_i \,\langle \xi_i \,, \bar{d}\,\rangle + \sum_{j=1}^{q} \lambda_j \,\langle \nabla h_j(\bar{x}) \,, \bar{d}\,\rangle \geq 0,$$

which combined with (5), (6) and the definition of $\widehat{J}$ gives

$$\langle \xi - \varphi(\bar{x})\rho \,, \bar{d}\,\rangle \geq \widehat{\beta} \left(\sum_{i=1}^{m} \mu_i + \sum_{j=1}^{q} |\lambda_j| \right).$$

Since $co\partial^* f(\bar{x}) - \varphi(\bar{x}) co\partial^* g(\bar{x})$ is bounded and there are only a finite number of possible subsets $\widehat{J}$, there is a finite upper bound on $\frac{\langle \xi - \varphi(\bar{x})\rho \,, \bar{d}\,\rangle}{\widehat{\beta}}$ independent of $\widehat{J}$, which is also an upper bound for

$$\sum_{i=1}^{m} \mu_i + \sum_{j=1}^{q} |\lambda_j| \,.$$

Therefore, since (μ,λ) is arbitrary, $\Lambda(\bar{x})$ is bounded.

$(ii) \Rightarrow (i)$. We now establish that the nonempty and boundedness of the set $\Lambda(\bar{x})$ ensures that $(CQ2)$ holds at $\bar{x}$. First, we show that $\{\nabla h_j(\bar{x})\}_{j\in J}$ is linearly independent. Suppose, on the contrary, that this is not true. Then, there exist numbers $\bar{\lambda}_1, \ldots, \bar{\lambda}_q$ not all being zero such that

$$\sum_{j=1}^{q} \bar{\lambda}_j \nabla h_j(\bar{x}) = 0. \tag{7}$$

Now, since the KKT multiplier set $\Lambda(\bar{x})$ is nonempty, there exists a multiplier vector $(\mu,\lambda) \in \mathbb{R}^m \times \mathbb{R}^q$ such that

$$0 \in co\partial^* f(\bar{x}) - \varphi(\bar{x}) co\partial_* g(\bar{x}) + \sum_{i=1}^{m} \mu_i \, co\partial^* k_i(\bar{x}) + \sum_{j=1}^{q} \lambda_i \nabla h_j(\bar{x}) + N^C(S,\bar{x}), \tag{8}$$
$$\mu_i k_i(\bar{x}\;) = 0, \;\; \forall i \in I.$$

From relations (7) and (8), we know that, for any $\gamma > 0$, we have

$$0 \in co\partial^* f(\bar{x}) - \varphi(\bar{x}) co\partial_* g(\bar{x}) + \sum_{i=1}^{m} \mu_i \, co\partial^* k_i(\bar{x}) + \sum_{j=1}^{q} \lambda_i \nabla h_j(\bar{x}) + \sum_{j=1}^{q} \gamma \;\bar{\lambda}_j \nabla h_j(\bar{x}) + N^C(S,\bar{x}),$$
$$\mu_i k_i(\bar{x}\;) = 0, \;\; \forall i \in I.$$

Thus,

$$(\mu_1, \ldots \mu_m, \lambda_1 + \gamma\bar{\lambda}_1, \ldots, \lambda_q + \gamma\bar{\lambda}_q) \in \Lambda(\bar{x}) \,, \qquad \forall\, \gamma > 0.$$

which, by noting that $\overline{\lambda_j} \neq 0$, for at least one $j \in J$, contradicts the hypothesis that $\Lambda(\bar{x})$ is bounded.

Now, we assert that $intT^C(S,\bar{x}) \cap T^C(H,\bar{x})$ is nonempty. Indeed, if the assertion is not true, then by a strict separation theorem, there exists a $v \in \mathbb{R}^n$ such that

$$\langle v, d \rangle > 0, \qquad \forall d \in intT^C(S,\bar{x}),$$

and

$$\langle v, d \rangle \leq 0, \qquad \forall d \in T^C(H,\bar{x}).$$

Thus, $v \neq 0$ and $v \in N^C(H,\bar{x})$. So, by (1), there exist $\rho_1, \dots, \rho_q$ not all being zero such that

$$v = \sum_{j=1}^{q} \rho_j \nabla h_j(\bar{x}). \tag{9}$$

Let $\rho_k \neq 0$ for some $k \in J$. Nonemtiness of $\Lambda(\bar{x})$ allows us to select $(\mu,\lambda) \in \Lambda(\bar{x})$. Thus, there exist $\xi \in co\partial^* f(\bar{x}), \rho \in co\partial_* g(\bar{x})$ and $\zeta i \in co\partial^* k_i(\bar{x}), i \in I$, such that for all $d \in T^C(S,\bar{x})$,

$$\langle \xi - \varphi(\bar{x})\rho , d \rangle + \sum_{i=1}^{m} \mu_i \langle \xi_i , d \rangle + \sum_{j=1}^{q} \lambda_j \langle \nabla h_j(\bar{x}), d \rangle \geq 0. \tag{10}$$

On the other hand, by the boundedness of the set $\Lambda(\bar{x})$, there exists $M > 0$ such that

$$|\lambda_j| < M, \qquad \forall j \in J.$$

Let for all $j \in J, \gamma_j := A\rho_j + \lambda_j$, where $A = \frac{2M}{|\rho_k|}$. Since $|\gamma_k| > M$, the vector (μ,γ) does not belong to $\Lambda(\bar{x})$, and so

$$0 \notin co\partial^* f(\bar{x}) - \varphi(\bar{x}) co\partial_* g(\bar{x}) + \sum_{i=1}^{m} \mu_i \, co\partial^* k_i(\bar{x}) + \sum_{j=1}^{q} \lambda_i \nabla h_j(\bar{x}) + \sum_{j=1}^{q} \gamma \ \nabla h_j(\bar{x}) + N^C(S,\bar{x}),$$

and, by a separation theorem, there exists $\bar{d} \in T^C(S,\bar{x})$ such that for all $\xi \in co\partial^* f(\bar{x}), \rho \in co\partial_* g(\bar{x})$ and $\zeta i \in co\partial^* k_i(\bar{x}), i \in I$, we have

$$\langle \xi - \varphi(\bar{x})\rho , \bar{d} \rangle + \sum_{i=1}^{m} \mu_i \langle \xi_i , \bar{d} \rangle + \sum_{j=1}^{q} \gamma_j \langle \nabla h_j(\bar{x}), \bar{d} \rangle < 0. \tag{11}$$

Without loss of generality, we may choose $\bar{d} \in intT^C(S,\bar{x})$. From (10) and (11) we have

$$A \sum_{j=1}^{q} \rho_j \langle \nabla h_j(\bar{x}), \bar{d} \rangle < 0.$$

Therefore, since $A > 0$, by (9) we have $\langle v, \bar{d} \rangle < 0$, which is a contradiction. Then, $intT^C(S,\bar{x}) \cap T^C(H,\bar{x})$ is nonempty.

Finally, we show that there exist $d \in intT^C(S,\bar{x}) \cap T^C(H,\bar{x})$ and numbers $b_i > 0$, $i \in I(\bar{x})$, such that

$$\langle \xi_i \ , d\rangle \leq -b_i, \qquad \forall \xi_i \in co\partial^* k_i(\bar{x}),$$

For this, we assert that

$$0 \notin co \bigcup_{i \in I(\bar{x})} co\partial^* k_i(\bar{x}) + N^C(H,\bar{x}) + N^C(S,\bar{x}). \tag{12}$$

Indeed, if the assertion (12) is not true, then there exist $t_i \geq 0$ and $(\lambda_1, \dots, \lambda_q)$ with $\sum_{i \in I(\bar{x})} t_i = 1$ such that

$$0 \in \sum_{i \in I(\bar{x})} t_i \, co\partial^* k_i(\bar{x}) + \sum_{j=1}^{q} \lambda_j \nabla h_j(\bar{x}) + N^C(S,\bar{x}). \tag{13}$$

Since $\Lambda(\bar{x})$ is nonempty, by the hypothesis, there exists $(\mu_1, \dots \mu_m, \gamma_1, \dots, \gamma_q)$ with $\mu_i \geq 0$, for $i \in I(\bar{x})$ such that

$$0 \in co\partial^* f(\bar{x}) - \varphi(\bar{x}) co\partial_* g(\bar{x}) + \sum_{i \in I(\bar{x})} \mu_i \, co\partial^* k_i(\bar{x}) + \sum_{j=1}^{q} \lambda_i \nabla h_j(\bar{x}) + N^C(S,\bar{x}). \tag{14}$$

From relations (13) and (14) we know that, for any $\beta > 0$,

$$\begin{aligned} 0 \in co\partial^* f(\bar{x}) - \varphi(\bar{x}) co\partial_* g(\bar{x}) + \sum_{i \in I(\bar{x})} (\mu_i + \beta t_i) \, co\partial^* k_i(\bar{x}) + \sum_{j=1}^{q} (\gamma_j + \beta \, \lambda_j) \nabla h_j(\bar{x}) \\ + N^C(S,\bar{x}). \end{aligned} \tag{15}$$

From the relation (15), we obtain

$$(\mu_1 + \beta \, t_1, \dots, \mu_m + \beta \, t_m, \gamma_1 + \beta\lambda_1, \dots, \gamma_q + \beta\lambda_q) \in \Lambda(\bar{x}), \qquad \forall \beta > 0.$$

Since $t_k > 0$, for some $k \in I(\bar{x})$, $\mu_k + \beta \, t_k \to +\infty$ as $\beta \to +\infty$, we contradict the hypothesis that $\Lambda(\bar{x})$ is bounded. Then, the assertion (12) is true. Therefore, by a strict separation theorem, there exist $\hat{d} \in \mathbb{R}^n$ and $b > 0$ such that

$$\langle \theta \ , \hat{d} \rangle \leq -b, \qquad \forall \theta \in \bigcup_{i \in I(\bar{x})} co\partial^* k_i(\bar{x}), \tag{16}$$

$$\langle \eta, \hat{d} \rangle \leq 0, \qquad \forall \eta \in N^C(S,\bar{x}). \tag{17}$$

$$\langle \sigma, \hat{d} \rangle \leq 0, \qquad \forall \sigma \in N^C(H,\bar{x}). \tag{18}$$

From (17) and (18), we have $\hat{d} \in T^C(S,\bar{x}) \cap T^C(H,\bar{x})$. Furthermore, (16) implies that for each $i \in I(\bar{x})$,

$$\langle \theta_i \ , \hat{d} \ \rangle \leq -b, \qquad \forall \theta_i \in co\partial^* k_i(\bar{x}).$$

Now, suppose that $d_0 \in T^C(S, \bar{x}) \cap T^C(H, \bar{x})$. We put $b_i = b$, for all $i \in I(\bar{x})$, and $d = \varepsilon\, d_0 + (1-\varepsilon)\, \widehat{d}$ with $\varepsilon \in (0,1)$ and sufficiently small. We see that $(CQ2)$ is satisfied and the proof is complete.

It is worth noting that Theorem 2.4 is not valid if in the definition of $\Lambda(\bar{x})$ the convex hull is removed. Let us illustrate this with the following example.

Example 2.5. Consider the problem*:*

$$(\mathrm{P}_1) \qquad \min \qquad \frac{f(x,y,z)}{g(x,y,z)}$$

$$s.t. \qquad h_j\,(x,y,z) = 0, \qquad x \in (x,y,z),$$

where $\mathrm{f}, \mathrm{g}, \mathrm{h}_\mathrm{j} : \mathbb{R}^3 \to \mathbb{R}, \mathrm{j} = 1, 2$ are defined by

$$f(x,y,z) = |x+y| + 1, \qquad\qquad g(x,y,z) = 2 - |z|,$$

$$h_1(x,y,z) = x + 2y + z, \qquad\qquad h_1(x,y,z) = 2x + y - z,$$

$$S = [-1,1] \times [-1,1] \times [-1,1].$$

Then, $\bar{\mathrm{x}} = (0,0,0)$ is the global minimizer for (P_1). We have

$$f^+\big((0,0,0), V\big) = |v_1 + v_2|, \qquad\qquad g^+\big((0,0,0), V\big) = -|v_3|$$

$$\nabla h_1(0,0,0) = (1,2,1), \qquad\qquad \nabla h_2(0,0,0) = (2,1,-1).$$

Observe that $\partial^*\mathrm{f}(\bar{\mathrm{x}}) = \{(1,1,0), (-1,-1,0)\}$ is an upper semi-regular convexificator of f a $\bar{\mathrm{x}}$ and $\partial_*\mathrm{g}(\bar{\mathrm{x}}) = \{(0,0,2), (0,0,-2)\}$ is a lower semi-regular convexificator of f at $\bar{\mathrm{x}}$.

Obviously, $\mathrm{T}^\mathrm{C}(\mathrm{S}, \bar{\mathrm{x}}) = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. So,

$$Ker\nabla h(\bar{x}) \cap intT^C(S, \bar{x}) \neq \emptyset.$$

Thus, there is no $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ such that

$$(0,0,0) \in \partial^* f(\bar{x}) - \frac{1}{2}\, \partial_* g(\bar{x}) + \lambda_1\, \nabla h_1(\bar{x}) + \lambda_2\, \nabla h_1(\bar{x}) + N^C(S, \bar{x}).$$

Remark 2.6. Since the Clarke subdifferential and the Michel-Penot subdifferential of a locally Lipschitz function are upper semi-regular convexificators. Then, the Theorem 2.4 ture is valid with the convexificators being replaced respectively by the Clarke and the Michel-Penot subdifferential.

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