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# Semidefinite Relaxation for the Dominating Set Problem

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*It is a well-known fact that finding a minimum dominating set and consequently finding the domination number of a general graph is an NP-complete problem. Here, we first model this problem as nonlinear binary optimization problems and then extract two closely related semidefinite relaxations. For each of these relaxations, different rounding algorithms are exploited to produce near-optimal dominating sets. Feasibility of the generated solutions and efficiency of the algorithms are analyzed.*

***Keywords***: *Dominating set, Semidefinite programming, Rounding algorithm.*

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## 1. Introduction

Given an undirected, connected graph $G = (V, E)$, a set $S \subseteq V$ is called a dominating set of $G$ if every vertex $v \in V$ is in $S$ or adjacent to a vertex in $S$. For a dominating set $S$ and a set $S' \subseteq V$, the relation $S \subseteq S'$ implies that $S'$ is a dominating set as well. A minimal dominating set is the one with no dominating set as a proper subset, and a minimum dominating set has minimum cardinality among all minimal dominating sets. This cardinality is of important interest and is referred to as *domination number*, denoted by $\gamma(G)$.

Dominating set problem has important applications in many practical fields (e.g. [1,5,7,8,10,15,16,17,18,21,23]), and its identification is of significant importance, while it is known as an NP-complete problem for an arbitrary graph [25]. In the literature, lower or upper bound of the domination number has been calculated in terms of some graph's parameters such as size, order, diameter, and minimal degree. For some special graphs like grids, circuits and paths; it can be obtained parametrically in terms of its order [6,13]. For some others, such as trees, directed paths, block graphs, interval and trapezoidal ones, it can be calculated algorithmically in a polynomial time [9].

Despite these interesting results, finding a minimum dominating set and consequently the domination number of an arbitrary graph still remains a challenge, and finding its sensible approximation in polynomial time is an appealing problem both in theory and practice. Quality of an approximation algorithm is measured by the approximation ratio, which refers to the ratio between the objective value of the algorithm and the exact optimal value of the problem, usually denoted by $\alpha$. For example, in an NP-hard minimization problem, an $\alpha$-approximation algorithm means that the objective value provided by the algorithm is not greater than $\alpha$ times of the exact value.

One of the first attempts to acquire a minimum dominating set using optimization approaches has been carried out by proposing a binary linear integer programming model [27] as

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$$\min \left\{ \sum_{i=1}^{n} x_i \mid (A+I)x \geq e \,,\ x_i \in \{0,1\}, i = 1, \dots, n \right\},$$

where $A$ is the adjacency matrix, $I$ denotes the $n \times n$ identity matrix, and $e$ denotes the $n$-vector of all ones. It is well-known that solving an integer optimization problem is the hardest. A trivial approximation algorithm, and obviously the first and the easiest one coming to mind, is the linear relaxation of the model by considering $x_i \in [0,1]$. Recall that the optimal objective value of this linear optimization problem is referred to as the fractional domination number. It is also demonstrated that deriving a near-optimal solution of the original problem from an optimal solution of the relaxed problem is another hard challenge [24].

In [19] it has been shown that a pair of L-reductions exists between the minimum dominating set problem and the set covering problem. This means that an efficient algorithm for one of these problems provides an efficient algorithm for the other in polynomial time. For these two problems, it is also proved that the approximation ratio is preserved under this reduction [19]; any polynomial-time $\alpha$-approximation algorithm for each of them presents a polynomial-time $\alpha$-approximation algorithm for the other problem. The greedy algorithm provides a logarithmic approximation factor for the set covering problem and consequently for the minimum dominating set problem. It is also proved that no algorithm can attain an approximation factor better than the greedy algorithm unless P = NP [26].

In addition to linear relaxation, Semidefinite programming (SDP) relaxation proved itself as a powerful tool to approximate a number of graph parameters. To name some examples, one can mention the problems of maximum independent set [3], vertex cover [14,22], maximum cut [12], and vertex coloring [20]. The success of SDP relaxation on these problems tempted us to exploit it in finding a feasible dominating set with the associate near-optimal domination number for a general graph. It is noted that the first published result has no particular reported details [11].

Here, we establish our methodology on a binary nonlinear integer programming model of [11], and present two further SDP relaxations, with two rounding procedures, respectively, with the aim of constructing an approximately minimum dominating set. In one, a randomized rounding method is applied to the obtained optimal solution of the associated SDP, which probabilistically leads to a feasible solution of the basic binary nonlinear integer programming model. For the second, we utilize the known hyperplane rounding algorithm [12]. In this procedure, providing a tool for reliability analysis of the relaxed problem requires a slight change of the objective function in the binary nonlinear integer programming model, which instead, produces $n - \gamma(G)$. By adding a reference variable to this model, identification of $n - \gamma(G)$ is transformed into a binary quadratic integer programming problem, and consequently its SDP relaxation is considered. Using the hyperplane rounding method similar to [12] on the produced optimal solution of the SDP relaxation leads to a potential dominating set. For both methods, we prove that the maximum probability that the generated set is not a dominating set, is strictly less than 1.

The reminder of our work is organized as follows. Section 2 gives a short review of SDP. A nonlinear binary optimization model is developed to find the value of $\gamma(G)$ in Section 3, and an equivalent problem is provided which is prepared for the SDP relaxation. The relaxed SDP model is provided in Section 4. Two rounding algorithms are presented in Section 5, where their performances are analyzed. Concluding remarks and some future research directions are presented in Section 6.

## 2. Short Review of SDP

Let $\mathbb{S}^n$ denote the set of symmetric $n \times n$ real matrices. The cone of symmetric positive semidefinite (definite) matrices is denoted by $\mathbb{S}^n_+$ ($\mathbb{S}^n_{++}$). $B \succcurlyeq D$ ($B \succ D$) means that $B - D$ is positive semidefinite (definite). For $B \in \mathbb{S}^n_{++}$, there is a lower triangular matrix $U$ where $B = U^T U$ (Cholesky decomposition).

Suppose that $A_1, \dots, A_m$ are linearly independent matrices in $\mathbb{S}^n$, $C \in \mathbb{S}^n$ and $b \in \mathbb{R}^m$. An SDP problem can be expressed as

$$\begin{array}{ll} \min & \langle C, X \rangle \\ \text{s.t.} & \langle A_i, X \rangle = b_i \,, \quad i = 1,2, \dots, m \\ & X \succcurlyeq 0, \end{array}$$

where $\langle B, D \rangle = \operatorname{tr}(B^T D) = \sum_{i,j} b_{ij}\, d_{ij}$.

The SDP problem is a special case of convex programming problems and can be solved in a polynomial time with an interior point method [2]. The interested reader is referred to [4,29] for a thorough discussion and applications of SDP.

## 3. Nonlinear Formulation of the Dominating set Problem

Let $G$ be a graph with the vertex set $V = \{v_1, v_2, \dots, v_n\}$ and the edge set $E$. The open neighborhood of a vertex $v$ consists of the set of adjacent vertices to $v$, that is, $N(v) = \{w \in V : vw \in E\}$. The closed neighborhood of $v$ is defined as $N[v] = N(v) \cup \{v\}$. The following labeling can be defined on $V$ with respect to a subset $S \subseteq V$:

$$y(v_i) = \begin{cases} +1, & v \in S \\ -1, & v \notin S. \end{cases}$$

For the sake of simplicity, we denote $y(v_i)$ by $y_i$ and refer to a vertex with the label 1 as (+1)-vertex and as (-1)-vertex, otherwise. Further, $N(i)$ ($N[i]$) stands for the open (closed) neighborhood of the vertex $v_i$. It is important to mention that a vertex in a dominating set $S$ is a (+1)-vertex induced by $S$.

In [11], the following valid nonconvex quadratic inequalities are proposed for $S$ being a dominating set:

$$\begin{array}{ll} \sum_{v_j \in N(i)} (1 - y_i y_j) \geq 2, & i = 1,2, \dots, n, \\ y_i \in \{-1, +1\}, & i = 1,2, \dots, n, \end{array} \tag{1}$$

where $y = (y_i)$ is the labeling induced by $S$. To keep self-consistency, a brief argument is provided. Observe that when $S$ is a minimal dominating set and $v_i \in S$, there must be at least a vertex $v_j \in N(i)$, where $y_i y_j = -1$. Otherwise, one can drop $v_i$ from $S$ and reduce its cardinality, which contradicts the minimality assumption of $S$. The following lemma summarizes this fact.

**Lemma 3.1.** Suppose that $S \subseteq V$ is a dominating set of $G$. Then, for each $v_i \in V$, there exists $v_j \in N(i)$ such that $y_i y_j = -1$.

Observe that when $v_i$ and $v_j$ are of the same sign, $y_i y_j = 1$, and $y_i y_j = -1$, otherwise. Therefore, Lemma 3.1 justifies the validity of (1).

Let us denote the set of $n$-vectors $y$ satisfying (1) by $Y$. Observe that this set is not empty because every graph admits a dominating set. Moreover, it includes many finitely feasible vectors and consequently any optimization problem over this set is not unbounded. Since the number of (+1)-vertices in any $y \in Y$ is $\frac{1}{2}\sum_{i=1}^{n}(1 + y_i)$, minimizing this function over $Y$ provides a minimum dominating set, and consequently $\gamma(G)$ is the optimal objective value. On the other hand, note that $\frac{1}{2}\sum_{i=1}^{n}(1 - y_i)$ stands for the number of (-1)-vertices in any $y \in Y$, and the optimal value of maximizing $\frac{1}{2}\sum_{i=1}^{n}(1 - y_i)$ over $Y$ produces $n - \gamma(G)$. Further, the optimal solution of this problem naturally corresponds to a minimum dominating set. Therefore, there are two closely related binary problems as

$$\min \left\{\frac{1}{2}\sum_{i=1}^{n}(1 + y_i) \mid y \in Y\right\}, \tag{2}$$

and

$$\max \left\{\frac{1}{2}\sum_{i=1}^{n}(1 - y_i) \mid y \in Y\right\}. \tag{3}$$

The next lemma outlines the equivalency of these two problems.

**Lemma 3.2.** A vector $y^* \in Y$ is an optimal solution of (2) if and only if it is optimal for (3).

**Proof.** Let $y^*$ be an optimal solution of (2) and $\hat{y}$ be a feasible solution of (3) such that $\sum_{i=1}^{n}(1 - \hat{y}_i) > \sum_{i=1}^{n}(1 - y_i^*)$. This results in the number of (-1)-vertices in $\hat{y}$ being greater than those in $y^*$. Therefore, $\sum_{i=1}^{n}(1 + \hat{y}_i) < \sum_{i=1}^{n}(1 + y_i^*)$, which is in contradiction with the optimality of $y^*$. This means that $y^*$ is an optimal solution of (3) as well. An analogous argument implies the opposite direction. □

Observe that the objective functions (2) and (3) are linear, while for analyzing our algorithms, we need a quadratic objective function. To convert these linear functions to quadratic ones, a reference variable $y_0 \in \{-1,1\}$ is introduced and problems (2) and (3) are rephrased as follows:

$$\begin{array}{lll}
(IP1)\ \min & \frac{1}{2}\sum_{i=1}^{n}(1 + y_0 y_i) & \\
\text{s.t.} & \sum_{v_j \in N(i)}^{n}(y_0^2 - y_i y_j) \geq 2, & i = 1,2,\dots,n \\
& y_i \in \{-1,+1\}, & i = 0,1,2,\dots,n,
\end{array}$$

and

$$
\begin{array}{lll}
(IP2)\ \max & \frac{1}{2}\sum_{i=1}^{n}(1-y_0y_i) & \\
\text{s.t.} & \sum_{v_j\in N(i)}^{n}(y_0^2-y_iy_j)\geq 2, & i=1,2,\dots,n \\
 & y_i\in\{-1,+1\}, & i=0,1,2,\dots,n.
\end{array}
$$

The following theorem provides an informative straightforward connection between the optimal solutions of problems (2),(3) and $(IP1)$, $(IP2)$, respectively.

**Theorem 3.3.** A vector $y^*\in Y$ is an optimal solution of (2) and (3) if and only if $(1,y^*)^T$ (and obviously $(-1,-y^*)^T$) is an optimal solution of $(IP1)$ and $(IP2)$.

**Proof.** We establish the result for problems (2) and $(IP1)$. The proof for the other ones goes similarly. It can be immediately observed from the definition of these two problems that $y$ is a feasible solution of (2) if and only if $(y_0,y)^T$ and$(y_0,-y)^T$ are feasible for $(IP1)$, where $y_0\in\{-1,+1\}$. Now, let $y^*$ be an optimal solution of (2). Further, let $(y_0,\tilde{y})^T$ be an optimal solution of $(IP1)$ such that

$$\sum_{i=1}^{n}(1+y_0\tilde{y}_i)<\sum_{i=1}^{n}(1+y_i^*). \tag{4}$$

There are two possibilities for $y_0$. When $y_0=1$, from (4), we get $\sum_{i=1}^{n}(1+\tilde{y}_i)<\sum_{i=1}^{n}(1+y_i^*)$. This clearly means that the number of (+1)-vertices in $\tilde{y}$ is less than of $y^*$, which contradicts the optimality of $y^*$ for (2). On the other hand, $y_0=-1$ leads to $\sum_{i=1}^{n}(1-\tilde{y}_i)<\sum_{i=1}^{n}(1+y_i^*)$. This also means that the number of $(-1)$-vertices in $\tilde{y}$ (or (+1)-vertices in $-\tilde{y}$) is less than the (+1)-vertices in $y^*$. This is a contradiction again. This completes the proof. □

The following result can be deduced immediately.

**Corollary 3.4.** Let $(y_0,y^*)^T$ be an optimal solution of $(IP1)$ or $(IP2)$. Then, $S=\{v_i|y_i^*=y_0\}$ is a minimum dominating set.

## 4. Semidefinite Programming Relaxation

Here, we establish an SDP relaxation of $(IP1)$. First, for $i=0,\dots,n$, the variable $y_i$ is substituted by an $(n+1)$-dimensional vector $u_i\in\mathcal{U}$, where $\mathcal{U}=\{(-1,0,\dots,0),(+1,0,\dots,0)\}$. Accordingly, the restriction $y_i\in\{-1,+1\}$ is replaced by $u_i\in\mathcal{U}$ and then problem $(IP1)$ is adapted to be

$$
\begin{array}{lll}
\min & \frac{1}{2}\sum_{i=1}^{n}(1+u_0^Tu_i) & \\
\text{s.t.} & \sum_{v_j\in N(i)}(u_0^Tu_0-u_i^Tu_j)\geq 2, & i=1,2,\dots,n \\
 & u_i\in\mathcal{U}, & i=0,1,2,\dots,n.
\end{array}
$$

Recall that $||u_i||=1$, for $u_i\in\mathcal{U}$, and this motivates us to expand $U$ to the standard $(n+1)$-dimensional unit sphere $\mathbb{S}^{n+1}=\{u\in\mathbb{R}^{n+1}|\ ||u||=1\}$, at the second step of the relaxation procedure. Thus, the following problem is obtained:

$$\begin{array}{lll} \min & \frac{1}{2}\sum_{i=1}^{n}(1+u_0^T u_i) & \\ \text{s.t.} & \sum_{v_j\in N(i)}(u_0^T u_0 - u_i^T u_j) \geq 2, & i=1,2,\dots,n \\ & u_i^T u_i = 1,\ u_i \in \mathbb{S}^{n+1}, & i=0,1,2,\dots,n. \end{array} \tag{5}$$

Some further notations are needed to complete the SDP relaxation procedure. Define $X_{ij} = u_i^T u_j$ and $E_{ij} = e_i e_j^T$ with $e_i$ as the $i$th standard unit vector of $\mathbb{R}^{n+1}$, where $i,j \in \{0,1,\dots,n\}$. Further, let $A_i = \sum_{v_j\in N(i)} \frac{1}{2}(2E_{00} - E_{ij} - E_{ji})$, where $i = 1,2,\dots,n$. Now, (5) can be presented as

$$\begin{array}{lll} \min & \frac{n}{2} + \langle C, X\rangle & \\ \text{s.t.} & \langle A_i, X\rangle \geq 2, & i=1,2,\dots,n \\ & X_{ii} = 1, & i=0,1,2,\dots,n \\ & \operatorname{rank}(X) = 1 & \\ & X \succcurlyeq 0, & \end{array} \tag{6}$$

where $C = (c_{ij})$, with $c_{i0} = c_{0i} = \frac{1}{4}$ for $i \in \{1,2,\dots,n\}$, and $c_{ij} = 0$ otherwise.

At the final step of the relaxation procedure, the nonconvex constraint, $\operatorname{rank}(X) = 1$, is dropped and the relaxed SDP of $(IP1)$ is formulated as

$$\begin{array}{lll} (SDP1)\ \min & \frac{n}{2} + \langle C, X\rangle & \\ \text{s.t.} & \langle A_i, X\rangle \geq 2, & i=1,2,\dots,n \\ & X_{ii} = 1, & i=0,1,2,\dots,n \\ & X \succcurlyeq 0. & \end{array}$$

With an analogous procedure, an SDP relaxation model is derived for $(IP2)$ as follows:

$$\begin{array}{lll} (SDP2)\ \max & \frac{n}{2} - \langle C, X\rangle & \\ \text{s.t.} & \langle A_i, X\rangle \geq 2 & j=1,2,\dots,n \\ & X_{ii} = 1, & i=0,1,2,\dots,n \\ & X \succcurlyeq 0. & \end{array}$$

The optimal solution of these problems can be approximated by a feasible interior point algorithm (e.g., see [28]) using a solver such as CVX. Recall that the iterative sequences provided by these algorithms converge to maximally complementary optimal solutions of the problem. Since any such algorithm practically starts with a strictly positive definite solution, this property is kept during each iteration, and stops at a solution with duality gap less than an accuracy parameter $\epsilon > 0$; thefinal provided solution is a positive definite matrix, which may be referred to as an $\epsilon$-approximate optimal solution. This property of the provided solution from the algorithm is essential for our rounding methods (see Section 5).

## 5. Rounding Methods

Here, we explain two rounding methods to extract a potentially dominating set from an optimal solution of SDP relaxation model, $X^*$. We further prove that the extracted set would be dominating with nonzero probability. The terminology feasibility probability refers to this fact. Therefore, *infeasibility probability* is the probability that the set might not be dominating.

### 5.1. A New Randomized Rounding Method

It is worth mentioning that the objective value of $(SDP1)$ is a lower bound for $\gamma(G)$, and an optimal solution of this problem does not indicate any dominating set. Let $X^*$ be a positive definite optimal solution of $(SDP1)$. To potentially produce a dominating set from $X^*$, we propose a new randomized rounding algorithm. First, matrix $X^*$ is decomposed by the Cholesky decomposition procedure as $X^* = U^T U$, where $U = [u_0, u_1, \dots, u_n]$. The following procedure runs $l$ times ($l$ will be specified through analysis of the algorithm). In iteration $k$, with probability $(1 + u_0^T u_i)/2$, the vertex $v_i$ is selected as a member of a probable dominating set $S_k$. Therefore, $\Pr[y_i = 1] = (1 + u_0^T u_i)/2$ and $\Pr[y_i = -1] = (1 - u_0^T u_i)/2$. At the end, a randomly generated candidate set is constructed as $S = \bigcup_{k=1}^{l} S_k$, which is just a candidate for the dominating set. For the sake of brevity, this randomly generated dominating set is denoted by RGDS in the sequel. In the following, first we show that this procedure generates a dominating set with positive probability. Further, the integrality gap of the solution is investigated.

Let us first calculate the probability of RGDS feasibility to problem (2). The next lemma provides an elementary result.

**Lemma 5.1.** Let $\beta = \max_{1 \le i \le n} | u_0^T u_i|$. Then, $0 \le \beta < 1$.

**Proof.** Since $X^*$ is a positive definite matrix and $X_{ii}^* = 1$ for $i = 1, \dots, n$, $||u_i|| = 1$. On the other hand,

$$0 < \det(X^*) = \det(U^T)\det(U) = \det(U)^2).$$

Hence, $U$ is invertible and none of the $u_i$ is a scaler multiplication of $u_0$. Let $\theta_{i0}$ be the acute angel between $u_0$ and $u_i$. Thus, $u_0^T u_i = ||u_0||\, ||u_i|| \cos\, \theta_{i0} = \cos\, \theta_{i0} < 1$. It follows that $\beta < 1$. □

The following lemma states that a vertex $v_i$ is not dominated by an RGDS with at most a constant probability strictly less than 1.

**Lemma 5.2.** The probability that a vertex $v_i$ is not dominated in each iteration of the randomized rounding is at most $(\frac{1+\beta^2}{2})^\delta$, where $\delta$ is the smallest degree of the vertices.

**Proof.** A vertex $v_i$ is not dominated when none of the vertices in $N[i]$ is in RGDS, i.e., for each $v_j \in N(i)$, $y_i y_j = 1$. An upper bound of $\Pr[y_i y_j = 1]$ is calculated for a fix $v_j \in N(i)$ as follows:

$$\begin{aligned}
\Pr[y_i y_j = 1] &= \Pr[y_i = 1, y_j = 1] + \Pr[y_i = -1, y_j = -1] \\
&= \Pr[y_i = 1]\Pr[y_j = 1] + \Pr[y_i = -1]\Pr[y_j = -1] \\
&= \frac{1 + u_0^T u_i}{2}\frac{1 + u_0^T u_j}{2} + \frac{1 - u_0^T u_i}{2}\frac{1 - u_0^T u_j}{2} \\
&= \frac{1}{2}\left(1 + u_0^T u_i u_0^T u_j\right) \\
&\le \frac{1 + \beta^2}{2} < 1,
\end{aligned} \tag{7}$$

Where the last inequality follows from Lemma 5.1 Observe that all $y_i$ are independently chosen from $\{-1,1\}$, and for distinct vertices $v_j, v_k \in N(i)$, the values of $y_i y_j$ and $y_i y_k$ are independent. Therefore, $v_i$ is not dominated with probability $\prod_{v_j \in N(i)} \Pr[y_i y_j = 1]$. Now, an upper bound to infeasibility of the candidate set derived from the above-mentioned rounding algorithm follows immediately:

$$\begin{aligned}
\Pr[v_i \text{ is not dominated}] &= \prod_{\mathrm{v_j \in N(i)}} \Pr[\mathrm{y_i y_j} = 1] \\
&\le \left(\frac{1 + \beta^2}{2}\right)^{|\mathrm{N(i)}|} \le \left(\frac{1 + \beta^2}{2}\right)^{\delta} < 1.
\end{aligned} \tag{8}$$

□

When the previous randomized rounding algorithm is repeated ln $n$ times, the probability that a vertex $v_i$ is not dominated in any round is at most $(\frac{1+\beta^2}{2})^{\delta \ln\, n} < 1$, provided that the candidate set in each round is chosen independently from the others. This means that this algorithm generates a dominating set with positive probability.

Now, the performance of the algorithm is analyzed. The expected cost of our approach is derived as follows:

$$\begin{aligned}
\mathrm{E[RGDS]} &= \sum_{i=1}^{n} (\Pr[y_i = 1]\Pr[y_0 = 1] + \Pr[y_i = -1]\Pr[y_0 = -1]) \\
&= \sum_{i=1}^{n} \Pr[y_i = 1] = \sum_{i=1}^{n} \frac{1 + u_0^T u_i}{2} = \mathrm{opt}(SDP1).
\end{aligned} \tag{9}$$

When the algorithm is repeated ln $n$ times, an RGDS is obtained, the expected value of the cost for deriving this solution is (ln $n$)opt($SDP1$), and

$$\mathrm{E[RGDS]} = (\ln\, n)\ \mathrm{opt}(SDP1) \le (\ln\, n)\ \mathrm{opt}(IP1),$$

where $\mathrm{opt}(\cdot)$ denotes the optimal function value of the corresponding optimization problem. This relationship shows that the algorithm produces a dominating set with positive probability and its expected cost does not exceed (ln $n$) opt($IP1$). In other words, the algorithm is an (ln $n$)-approximation algorithm.

### 5.2. Hyperplane Rounding Method

The hyperplane rounding method has been applied to the SDP relaxation of MaxCut problem to get a binary solution for its nonlinear integer programming model [12]. The success in practice as well as its elegant analysis encourages us to utilize and implement the algorithm for the minimum dominating set problem. For this purpose, $n - \gamma(G)$ is approximated instead of $\gamma(G)$ itself and problem (3) is relaxed to an SDP model. Suppose that $X^*$ is an optimal solution of the problem $(SDP2)$. The following rounding strategy is implemented to develop this solution into a feasible solution for problem (3).

Let $X^* = U^T U$ be the Cholesky decomposition of $X^*$, where $U = [u_0, u_1, \dots, u_n]$. A vector $a \in \mathbb{S}^{n+1}$ is selected randomly, and one may fix $y_i = 1$ for each vertex $v_i$ when $a^T u_i \geq 0$, and $y_i = -1$, otherwise. Consequently, a potential minimum dominating set might be $S = \{v_i | y_i = y_0\}$. Recall that this solution might not be feasible for the problem (3). Based on the following observation [12], we prove that the probability of infeasibility of the generated set to be dominating is strictly less than 1. Let $y_i$ and $y_j$, corresponding to vectors $u_i$ and $u_j$, be produced from the hyperplane rounding algorithm. Then

$$\Pr[y_i y_j = -1] = \frac{arccos\ (u_i^T u_j)}{\pi}, \tag{10}$$

$$\Pr[y_i y_j = 1] = 1 - \frac{arccos\ (u_i^T u_j)}{\pi}. \tag{11}$$

Moreover, for $0 \leq \theta \leq \pi$, the following inequality holds:

$$\frac{\theta}{\pi} \geq \alpha . \frac{1 - cos\ \theta}{2}, \tag{12}$$

where $\alpha \approx 0.878567$. Suppose that $y$ is generated by the hyperplane rounding algorithm. To complete the result, the next lemma is useful.

**Lemma 5.3.** Let $\overline{\beta} = \max_{i<j} | u_i^T u_j |, 1 \leq i, j \leq n$. Then, $0 \leq \overline{\beta} < 1$, and $0 \leq 1 - \frac{arccos\ (\overline{\beta})}{\pi} < 1$.

**Proof.** In Lemma 5.1, it was proved that the matrix $U$ is invertible. Then none of the $u_i$ is a scalar multiplication of the others. Since for $i, j \in \{1,2, \cdots, n\}$, the acute angel between $u_i$ and $u_j$, shown by $\theta_{ij}$, satisfies the relation $u_i^T u_j = ||u_i||\ ||u_j|| \cos\ \theta_{ij} = \cos\ \theta_{ij}$, therefore

$$-1 < u_i^T u_j = \cos\ \theta_{ij} < 1.$$

The first part of the lemma is now proved. The second part of the proof is based on the fact that $\arccos\ \theta$ is a strict descending function. □

This lemma induces an upper bound to infeasibility probability of $y$. The solution is infeasible when at least one of the $v_i$ is not dominated. For such a $v_i$, its sign must be the same as the sign of all of its neighbours, i.e., $y_i y_j = 1$, for all $v_j \in N(i)$. Since the signs of $y_i$ and $y_j$ are based on selecting the random vector $a$, they are independently selected as $+1$ or $-1$. Consequently,

$$\begin{aligned}\Pr[v_i \text{ is not dominated}] &= \prod_{v_j \in N(i)} \Pr[y_i y_j = 1] \\ &= \prod_{v_j \in N(i)} \left(1 - \frac{\arccos(u_i^T u_j)}{\pi}\right) \\ &\leq \left(1 - \frac{\arccos(\bar{\beta})}{\pi}\right)^{|N(i)|} \\ &\leq \left(1 - \frac{\arccos(\bar{\beta})}{\pi}\right)^{\delta} < 1,\end{aligned} \tag{13}$$

where $\delta$ is the smallest degree of vertices, and $\bar{\beta}$ is as defined in Lemma 5.3. This means that the solution generated by the hyperplane rounding method has a positive feasibility probability.

Considering the well-known facts (11) and (12), the following performance ratio for the hyperplane rounding algorithm can be derived. Let us denote the random variable $W$ as the number of (-1)-vertices in a solution generated by the hyperplane rounding algorithm. The following lemma provides an elementary result.

**Lemma 5.4.** Suppose $y$ is generated by the hyperplane rounding algorithm. Then,

$$\mathrm{E}[W] \geq \alpha \times \mathrm{opt}(SDP2),$$

where $\alpha \approx 0.878567$.

**Proof.** For the generated $y = (y_0, y_1, \dots, y_n)$, it holds

$$\begin{aligned}\mathrm{E}[W] &= \mathrm{E}\left[\frac{1}{2}\sum_{i=1}^{n}(1 - y_0 y_i)\right] = \frac{1}{2}\sum_{i=1}^{n}\mathrm{E}\left[(1 - y_0 y_i)\right] \\ &= \frac{1}{\pi}\sum_{i=1}^{n}\arccos(u_0^T u_i) \geq \frac{\alpha}{2}\sum_{i=1}^{n}(1 - u_0^T u_i) \geq \alpha \times (n - \gamma(G)).\end{aligned}$$

□

This lemma says that the hyperplane rounding algorithm generates a solution with the expected value of $n - \gamma(G)$, at least 0.878 times of its exact value, and therefore, it is a sensible approximation of $n - \gamma(G)$. Let the random variable $\overline{W}$ denote the number of (+1)-vertices in a generated solution by the hyperplane rounding algorithm. The next theorem provides an upper bound for $\mathrm{E}[\overline{W}]$.

**Theorem 5.5.** Suppose that $y$ is generated by the hyperplane rounding algorithm. Then, for any graph $G$ we have

$$\mathrm{E}[\overline{W}] \leq (1 + (1 - \alpha)\Delta)\gamma(G).$$

**Proof.** It is clear that $W + \overline{W} = n$. Consequently, $\mathrm{E}[W] + \mathrm{E}[\overline{W}] = n$. First, recall that for a given graph $G$, we have that $n \leq (1 + \Delta)\gamma(G)$ [16], where $\Delta$ is the largest degree of the vertices. Thus,

$$
\begin{aligned}
\mathrm{E}[\overline{W}] &= n - \mathrm{E}[W] \leq n - \alpha \times z^* \\
&= n - \alpha(n - \gamma(G)) = n(1-\alpha) + \gamma(G)\alpha \\
&\leq (1+\Delta)(1-\alpha)\gamma(G) + \gamma(G)\alpha = (1 + (1-\alpha)\Delta)\gamma(G).
\end{aligned}
$$

□

It can be immediately concluded from Theorem 5.5 that the hyperplane rounding algorithm is a 2-approximation algorithm for sparse graphs.

**Corollary 5.6.** For a graph $G$ with $\Delta \leq 7$, we have $\mathrm{E}[\overline{W}] \leq \beta.\gamma(G)$, where $\beta < 2$.

## 6. Concluding remarks

We addressed a famous NP-complete problem in graph theory, the dominating set problem, and proposed an SDP relaxation model. This problem has a main difference with the well-investigated MaxCut problem. The MaxCut problem is a binary unconstrained problem, while the dominating set problem includes some additional constraints. We analyzed the reliability of the proposed method. Two rounding methods were presented to elicit a possible dominating set from the optimal solution of the SDP relaxation. The practical efficiency of the approach was also investigated. Finding sharper bounds for the expected objective function value could be very useful. Additionally, classification of graphs to find out a rounding method for a classes is also useful. Moreover, the method might be extended to other kinds of constrained combinatorial problems including some other variants of dominating set problems.

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