Iranian Journal of Operations Research
Vol. 6, No. 2, 2015, pp. 87-99

Weapon Scheduling in Naval Combat Systems for Maximization of Defense Capabilities

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Air defense is a crucial matter for all naval combat systems. In this study, we consider a warship equipped with an air-defense weapon that targets incoming threats using surface-to-air missiles. We define the weapon scheduling problem as an optimal scheduling of a set of surface-to-air missiles of a warship to a set of attacking air threats. Optimal scheduling of the weapon results in an increase in the probability of successful targeting of all incoming threats. We develop a heuristic method to obtain a very fast and acceptable solution for the problem. In addition, a branch and bound algorithm is developed to find the optimal solution. In order to increase the efficiency of the algorithm, a lower bound, an upper bound and a set of dominance rules are proposed. Using randomly generated test problems, the performances of the proposed solution approaches are analyzed. The results indicate that in all practical situations, the branch-and-bound algorithm is able to solve the problem in less than a second.

***Keywords**: Weapon scheduling problem, Naval combat systems, Branch and bound algorithm*

Manuscript was received on 15/04/2016, revised on 09/07/2016 and accepted for publication on 23/07/2016.

1. Introduction

The security of a country's borders is of strategic importance and thus countries are severely concerned with defending their frontiers. Accordingly, maritime boundaries are no exception, specially taking into account the exclusive circumstances that complicate naval warfare. For the countries with long seashores and multitudes of security elements within their regions the strengthening of their naval forces using scientific methodologies is necessary to effectively solve problems faced with in the field of naval combat.

Given the increasingly complex structure of missiles, being a major threat for any naval fleet, we investigate increasing the chances of survival for a frigate while confronted with airborne threats, particularly missiles. One major method of air defense used in engaging incoming missiles is multi-layer defense consisting of fighter jets, surface-to-air missiles (SAM) and ultimately artillery as the last layer of defense.

Surface-to-air missiles, as the middle defense layer, are of great significance. Since weapons used in this layer have limited firing capacity, careful planning is vital in order to achieve an optimal performance. This can be done using weapon scheduling methods that have been utilized since the First World War. Special circumstances of combats dictate optimal consumption of resources and thus scheduling problems have gained importance. The next section investigates scheduling problems when jobs may fail, corresponding to the failure of shots on threats. Then, the literature regarding the weapon target assignment (WTA) problem is reviewed. There are various forms of the WTA problem

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one of which can be applied to air defense scheduling problem. Ultimately, the background for air defense scheduling problems is discussed.

It is worthwhile mentioning that throughout our work here, the term “threat” refers to hostile rockets fired at the warship which is responded by the “missiles” launched from the frigate.

1.1. Scheduling Problems When Jobs May Fail

Stochastic scheduling problems have mainly addressed jobs with stochastic processing times or stochastic release dates. Consideration the possibility of failure for each job was first introduced in sequential testing by Ünlüyurt [13]. Subsequently, this was applied to scheduling of research and development projects. De Reyck and Leus [5] argued that the main feature of R&D projects is the possibility of completion at any time. This is because the success of jobs in the project is of a probabilistic nature. Coolen et al. [4] investigated scheduling of R&D projects by considering each project to be a combination of modules with each module consisting of several activities. This was an extension of Reyck and Leus' work where instead of an activity, a module is in place and the success of the project depends on the success of all modules while the success of a module is dependent only on the success of a single alternative within the module. Similarly, in this paper, each missile fired at a threat is successful if it hits the threat and fails otherwise. Therefore, by considering each firing of a missile as a job, the outcome can probabilistically be a success or a failure.

1.2. Weapon Target Assignment Problems

The WTA problem is the assignment of a set of weapons to a set of targets (or threats) in order to minimize the expected value of the targets’ survival (Ahuja et al. [1]). This problem was initially introduced in the 1950’s and a comprehensive overview of the initial studies on this problem can be found in Eckler and Burr [6]. Ahuja et al. [1] developed an integer programming model for the WTA problem and used a branch and bound (B&B) algorithm to solve the problem. For the first time, the authors presented the optimal solution for the WTA problem for 200 weapons and 200 threats. Naeem and Masood [11] developed an optimal dynamic multi-air threat evaluation and weapon allocation algorithm using a variant of Stable Marriage Algorithm. They use a new dynamic weapon scheduling algorithm, allowing multiple engagements using shoot-look-shoot strategy, to compute a near-optimal solution.

Planning for air defense consists of two components (Kwon et al. [9]). The first component is the allocation of weapons, i.e., the missile launchers to threats and the second is scheduling of the launching sequence of each weapon. Cha and Kim [3] studied the weapon scheduling problem by minimizing the number of survived threats. Sherali et al. [12] considered a scheduling problem in which a set of illuminators (homing devices) should strike a set of targets using surface-to-air missiles in a naval battle. This problem is a production floor shop scheduling problem of minimizing the total weighted flow e, subject to time-window job availability and machine downtime side constraints. They developed a simple algorithm based on solving assignment problems.

1.3. Missile Allocation Problems

The main distinction between WTA problem and missile allocation problem is the type of threats. In the previous section, the planning for weapon assignment was made regardless of the nature of the threats while in this section, targets are mobile and thus a more detailed planning is required to take into account the various characteristics of the threats. Missile allocation problems have long existed in the literature of scheduling and a thorough classification of such problems can be found in Matlin

[10]. Yet, Karasakal et al. [7] introduced a new approach to this problem. The authors interpreted the missile allocation problem as an optimal assignment of air defense weapons to a set of air threats. They defined the goal of this problem as maximizing the number of neutralized threats while minimizing the resources used in the warship. This problem considered a naval fleet consisting of a multitude of warships with various weapons. Ultimately, two greedy heuristic algorithms were developed to allocate the optimal combination of weapons and threats as well as assigning weapons to all existing threats.

The novelties of this paper are threefold: (1) introducing the weapon scheduling problem in a modular form where the number of missiles fired at each threat is probabilistic, (2) defining the objective function based on the idea of weighted number of tardy jobs, and (3) solving the problem using an exact method based on a B&B algorithm.

The remainder of our work is structured as follows. Section 2 provides a description of the problem and presents a nonlinear stochastic model. Section 3 describes the proposed solution method. In Section 4, computational results are discussed. Concluding remarks are given in Section 5.

2. Problem Description and Modeling

A warship may be equipped with different weapons in order to engage potential threats and missiles, due to their high velocities, require a more elaborate response. Anti-missile weapons normally consist of short-range missiles with a specific range that cannot eliminate targets outside the specified range.

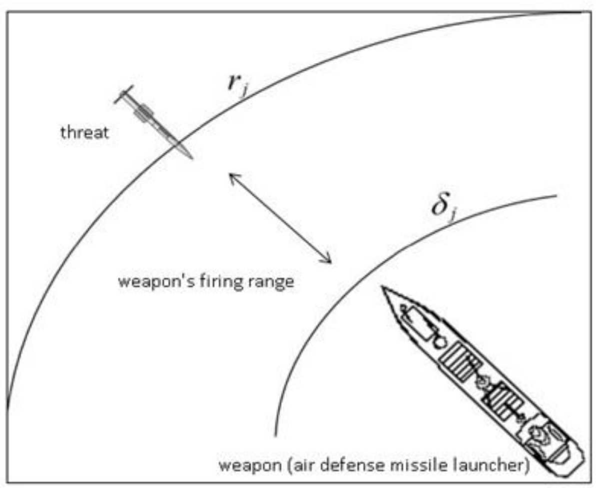

Fig. 1. Weapon's firing range and the threat's release date and due date

We consider a single weapon on a warship that should fire at incoming threats. This problem is similar to a single machine scheduling problem where the jobs indicate the threats and processing of jobs is equivalent to firing at threats. Jobs in this problem have release dates, i.e., when threats enter a weapon's specified firing range, denoted by r_j for threat j. Given the mobility of threats, they may escape a weapon's range which indicates a due date δ_j for threat j. After this due date, the job cannot be completed as it would be impossible to fire at a target outside the weapon's range. This condition is illustrated in Fig. 1. Depending on the velocity of a threat, its initial distance from the warship, and the velocity of the air defense missile, a number of missiles are fired at the threat. Since a warship can only contain a limited number of missiles, it is vital to optimally utilize such resources.

Accordingly, we assume a shoot-look-shoot (SLS) policy for firing the missiles. This means that a second shot will only be made if the first one misses its target. Under the SLS assumption, the maximum number of missiles fired at a threat can be calculated. This number is denoted by ν_j and is in form of a chain of jobs for any given threat. In other words, the jth chain consists of ν_j jobs (missiles fired) with precedence relations. If any of the missiles hits its target, the entire chain succeeds and processing of the remaining jobs in the chain would no longer be required.

Generally, there are three steps in each launching of a missile: (1) the setup time which is the time taken for target identification, rotation, angle deflection, etc. This is shown by s_{ij} and represents the amount of time it takes a weapon to switch from target i to target j; (2) the time taken for launching of a missile which is a constant for every launch and only depends on the type of the weapon or launcher; (3) the amount of time it takes a missile to reach its target which is a variable because the missile and the threat are in simultaneous movement. Consequently, the moment that the two missiles collide can be estimated as the initial distance between the two missiles divided by the sum of their velocities. It should be mentioned that we assume the course of both the missile and the threat to be linear, despite the fact that in reality, missiles move nonlinearly and in unpredictable directions. In scheduling problems, however, the nonlinear flight path of a missile only impacts the estimated traversed distance. Therefore, if the nonlinear course of a missile (or a threat) can be assumed as a multiple of the direct traversed distance, then the results of our work would still be valid.

The description of the problem is presented using a nonlinear stochastic mathematical model. In this model, $N = \{1, \dots, n\}$ is the set of threats and i and j would be the associated indices. As mentioned previously, for any given threat there exists a chain of jobs each being denoted by k.

Table 1. Model parameters

Parameter	Description
r_j	Release date of threat *j* into the weapon's firing range (earliest start time for processing of the first job in chain *j*)
δ_j	The moment when threat *j* escapes the weapon's firing range (due date for *j*th module)
d	Duration of the missile launching
p_j	Probability of a missile hitting threat *j*
s_{ij}	Setup time required for switching from threat *i* to threat *j*
Δ_j	Initial distance of the threat *j* from the warship
η_j	Precision of the threat *j* (probability of threat *j* hitting the warship)
V_j	Velocity of threat *j* (a constant)
ρ	Velocity of the missile fired at a threat (a constant)
M	A big number

Also, in order to schedule all module's jobs on a single machine, $A = \sum_{j=1}^{n} v_j$ positions (in this case, missile launchers) are required which are indexed as *r* in the model. In other words, each specific sequence of all shots at all threats includes *A* elements where each element indicates a position.

Table 1 explains the model parameters. The *precision parameter* indicates the probability of a threat hitting the warship and the subsequent damage if no action is taken against it. Using this parameter, the most hazardous threat can be identified. We assume that the total number of threats is known and targets have previously been identified by radars. We define the binary decision variable

X_{jkr} that gets the value of 1 if the kth missile to threat j is fired in the rth position and gets to be zero otherwise. Table 2 shows the stochastic variables of the model.

Table 2. Random parameters of the model

Variable	Description
$Y_{jkr} =$	Finish time of kth shot at threat j when the shot is made from the rth position.
$W_{jk} =$	A Bernoli stochastic variable with parameter p_j that gets to be 1 if the kth fire successfully hits threat j, and 0 otherwise.
L_{jkr}	The time it takes a fired missile to reach threat j when the kth shot is made from the rth position. This parameter can be approximated using the laws of physics.

It should be noticed that Y_{jkr} is a stochastic variable because the finishing time of all jobs within the jth chain, except for the first one, depends on the outcome of preceding jobs. Furthermore, the shots made at a threat are not independent of each other and to calculate the probability of a shots being within the feasible range, the outcome of preceding shots must be known.

A mathematical formulation of the model is proposed as follows.

$$\max \prod_{j=1}^{n}\left(1-\eta_j\left(\begin{array}{l}\sum_{r=1}^{A} X_{j1r}P(Y_{j1r}>\delta_j)+\\ \sum_{u=2}^{\nu_j}\left[\begin{array}{l}\sum_{s=u-1}^{A}\sum_{r=s+1}^{A} X_{j(u-1)s}\,X_{jur}\,P\left(Y_{jur}>\delta_j \mid (Y_{j(u-1)s}<\delta_j \text{ and } W_{j(u-1)}=0)\right)\\ \times P(Y_{j(u-1)s}<\delta_j \text{ and } W_{j(u-1)}=0)\end{array}\right]\\ +\sum_{s=\nu_j-1}^{A}\sum_{r=s+1}^{A} X_{j(\nu_j-1)s}\,X_{j\nu_j r}\,P\left(Y_{j\nu_j r}<\delta_j \text{ and } W_{j\nu_j}=0\right)\end{array}\right)\right) \quad (1)$$

s.t

$$\sum_{r=1}^{A} X_{jkr}=1, \qquad \forall j=1,\dots,n;\ k=1,\dots,\nu_j, \quad (2)$$

$$\sum_{j=1}^{n}\sum_{k=1}^{\nu_j} X_{jkr}=1, \qquad \forall r=1,\dots,A, \quad (3)$$

$$rX_{jkr}-sX_{jls}<0, \qquad \forall j=1,\dots,n; k,l=1,\dots,\nu_j \text{ and } k<l; \forall r,s=1,\dots,A, \quad (4)$$

$$Y_{jkr}-d-r_j\geq 0, \qquad \forall j=1,\dots,n; k=1,\dots,\nu_j, \quad (5)$$

$$Y_{jkr}+s_{ji}\leq Y_{il(r+1)}-d+M(2-X_{jkr}+X_{il(r+1)}), \qquad \forall i,j=1,\dots,n, i\neq j;\ k=1,\dots,\nu_j; l=1,\dots,\nu_i; r=1,\dots,A, \quad (6)$$

$$Y_{jkr}+L_{jkr}\leq Y_{j(k+1)s}-d+M(2-X_{jkr}+X_{j(k+1)s}), \qquad \forall i,j=1,\dots,n, i\neq j;\ k=1,\dots,\nu_j;\ r,s=1,\dots,A \,\&\, r<s, \quad (7)$$

$$L_{jkr}(\rho+V_j)=\Delta_j-V_j y_{jkr}, \qquad \forall j=1,\dots,n; k=1,\dots,\nu_j;\ r=1,\dots,A, \quad (8)$$

$$Y_{jkr}\sim \textit{General Discrete Distribution}, \qquad \forall j=1,\dots,n; k=1,\dots,\nu_j; r=1,\dots,A, \quad (9)$$

$$W_{jk} \sim Bernoli(p_j), \qquad \forall j = 1,...,n; k = 1,...,v_j, \tag{10}$$

$$X_{jkr} \in \{0,1\}, \qquad \forall j = 1,...,n; k = 1,...,v_j; r = 1,..., A. \tag{11}$$

The objective function indicates survival probability of a warship equipped with air defense weapons, i.e., the missiles. This is the probability of no threats hitting the warship. If U_j is defined as a binary variable with $U_j = 1$ denoting intact passing of the threat through the weapon's firing range, then $P(U_j = 1)$ indicates the corresponding probability. Therefore, if the intact passing of a threat through the weapon's firing range poses danger to the warship, then $\eta_j P(U_j = 1)$ would be the probability of the threat hitting the warship. Consequently, the probability of the warship surviving from all *n* threats would be $\prod_{j=1}^{n}\left(1 - \eta_j p(U_i = 1)\right) P(U_j = 1)$, meaning that in the weapon's range (distance between r_j to δ_j) either no missiles have been fired at the threat or the missiles fired have missed their target. This condition can be divided into three possible scenarios: (1) the first job from the jth module is scheduled to be carried out after its due date. Because of the precedence relations between jobs within a module, the subsequent jobs of the jth module will also be scheduled after δ_j. Thus, no missiles would be fired at threat *j* before δ_j, (2) this scenario considers the second to v_jth jobs of a module, where $u\ th$ job from module *j* finishes after its due date while $u - 1$ precedent jobs, although unsuccessfully, have been finished on time. Given the existence of precedence relations between the jobs, it suffices only to consider the assumption for the $(u - 1)$th job since $(u - 2)$th job and the preceding ones have been completed on time and without success. Because firing of the uth missile takes longer than δ_j, the jth threat will pass through the weapon's firing range, (3) this scenario investigates the condition where all the missiles assigned to the jth threat are fired on time and all such missiles miss their target. Consequently, the threat would safely pass through.

Constraints (2) indicates that firing of each missile should be carried out at a single position. Constraint (3) imply that in each position only one missile can be fired. Constraints (4) illustrate the sequence of missiles fired at a threat so that for example the position of the second missile fired at a threat should be larger than the position of the first missile fired at the same threat. Constraints (5) state the fact that the finish time for any job must be larger than sum of its release date and the time taken for launching. If firings of two consecutive missiles at two distinct threats are required, then the weapon must undergo a setup time in order to rotate, prepare, and aim at the second threat. Constraints (6) illustrate the condition. Furthermore, if two consecutive missiles should be fired at a threat, based on the SLS assumption, the second shot can only be made once the outcome of the first shot is known. This is shown by constraints (7). The time it takes a missile to reach its target can be estimated according to laws of physic. Constraints (10) indicate the amount of time it takes to know the outcome of a missile fired at a target. Ultimately, the types of the variables are defined by (9), (10) and (11).

3. Solution Method

Here we discusses the method used for solving the proposed problem. There are various solution methodologies for weapon assignment and scheduling in the literature. For example, Blodgett et al. [2] developed a metaheuristic approach based on Tabu search for resource management in naval warfare. Kwon et al. [8] used a Lagrangean relaxation approach to the targeting problem. Van Dongen and Joost [14] developed a simulation model for the analysis of ship air defence.

Given the nonlinear and stochastic characteristics of our model, it cannot be solved using optimization software and thus a B&B algorithm is developed here. A B&B algorithm explores the entire feasible space in order to find the optimal solution but as the dimension of the problem increases, the search space grows exponentially which considerably affects the computing time and hence bounding schemes are used.

It should be mentioned that our problem has a dynamic nature but the obtained solution of our developed B&B algorithm is static. Thus, this solution is considered as a baseline and rescheduling of the problem can be made using reimplementation of the B&B algorithm. In other words, we manage the dynamic feature of the problem using reimplementation of our fast B&B algorithm.

3.1. Branching Scheme

We use of a depth-first approach in developing the B&B algorithm. Starting from a root node, several nodes called *offsprings* are generated. The offsprings are then ranked using the $\frac{threat\ \ precision}{number\ of\ \ jobs\ within\ the\ module}$ ratio. The far left side node, that has the largest value based on the ratio, is then selected and the branching procedure will accordingly continue. The general structure derived from this procedure is called a *search tree*. The branching method should explore the entire solution space which means considering all possible sequences of firing the missiles. Therefore, in each stage, the branching is applied to all jobs that can be carried out, i.e., all the missiles that can be fired by the weapon. Since jobs form a chain, it is necessary to identify the jobs whose direct precedents are already scheduled. The branching is then applied to the first organized job of each chain. As an example, four sample threats are illustrated in Table 3. The order of threats based on the $\frac{threat\ precision}{number\ of\ \ jobs\ within\ the\ module}$ ratio is determined. The air defense missile has a velocity of 782 meters per second and the launching time is 4 seconds. The weapon's firing range is 700 meters to 15 kilometers. Number of shots fired is not an input and is determined by calculating the velocity of the threat and its initial distance.

Table 3. Characteristics of threats specified in the example

Threat No.	Velocity of threat (m/s)	Initial distance (m)	Possible no. of shots	Possibility of success	Threat precision
1	680	21000	2	0.75	0.99
2	510	14000	2	0.72	0.97
3	680	13000	2	0.75	0.96
4	510	15000	3	0.75	0.93

Fig. 2 shows part of the search tree and the branching procedure for this example. The $i-j$ notation inside each node represents the jth job from the ith module. Starting from the root node, if the $i-j$ node is placed before (above) the $i'-j'$ node, it implies that the jth job from the ith module must be carried out before the j'th job from the i'th module.

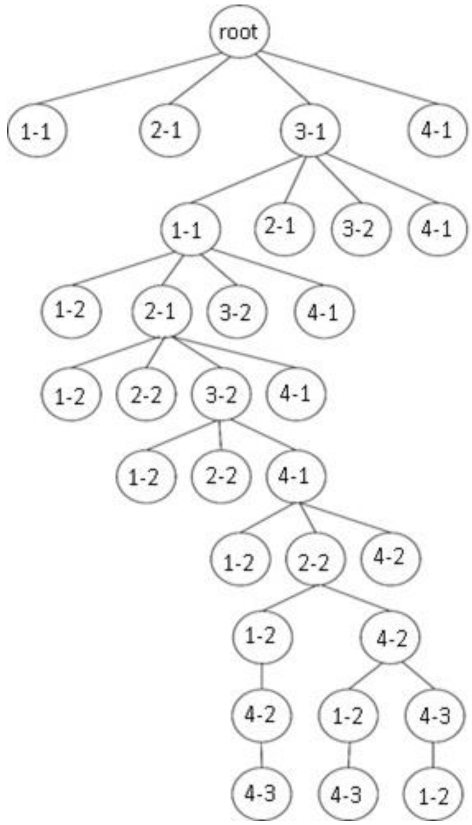

Fig. 2. Structure of the search tree

3.2. Elimination of Tardy Jobs

The problem discussed has several distinct characteristics one of which is the mobility of threats. Because of this characteristic, the threat, regardless of the action taken against it, will continue to change its location. For example, in some cases, there may be no missiles fired at a threat but the target, following its own flight path, may move out of the weapon's firing range. In such cases, the branching will not continue on the same node because branching is only applied to jobs the direct precedents of which are already finished. In other words, no missiles will be fired at a target that has escaped the weapon's firing range and such threat should be damaged by another weapon.

3.3. Lower Bound

A lower bound for the problem would be a sequence of jobs on the weapon. In order to improve the quality of the lower bound, job ordering rule is used. According to this rule, the remaining time for taking action against a threat is calculated using

$$rt_j = \frac{\Delta_j - V_j T}{V_j} \tag{12}$$

where T denotes the earliest time of weapon availability for firing at a target. Threats with shorter response times have higher priorities. Furthermore, when the outcome of a missile fired at a threat is not known, a second missile cannot be fired at the top-priority threat. In such cases, in order to preserve time, the next top-priority threat is selected.

It should be noted that the collision of the two missiles must take place within the weapon's firing range. After calculation of the lower bound in the proposed algorithm, if there exists a threat with no missiles scheduled to be fired at, a job from that threat's module is replaced by a job from another module. This is done by selecting and eliminating a job from the other module. This selection is based on the order of threats, number of missiles fired and their corresponding weights. This means that the priority (for elimination) is given to threats with most missiles fired at, and if there are the same number of missiles fired at two threats, the threat with the smaller weight (less potential damage) will be selected. Subsequently, the first job in the module of a threat with no missiles fired at is replaced by the last job in the module of the selected threat. Ultimately, it is necessary to validate the scheduling sequence in terms of setup times, firing range, and module due dates.

3.4. Upper Bound

In using a B&B algorithm, it is possible to calculate the objective function for the most recent partial scheduling. Moreover, using an upper bound, the value of the objective function from a node's offspring can be predicted for the best-case scenario. If this best possible value is lower than an existing solution (lower bound), the branching on that node can be terminated and consequently the node would be fathomed. An upper bound for this problem can be equivalent to the condition that all missiles are simultaneously fired at the threats. In other words, the entire set of jobs in a module is scheduled on the weapon in parallel. Parallel scheduling of jobs would require equal number of weapons while in reality only a single weapon is at disposal. The solution calculated under the assumption of parallel scheduling can be considered as an upper bound. If v_j' is the number of available jobs in a module at a given time, an upper bound can be calculated as

$$UB = \prod_{j=1}^{n} (1 - \eta_j (1 - p_j)^{v'_j}) \,. \tag{13}$$

In fact, this assumption considers a weapon firing at all threats simultaneously. This value can act as an upper bound for the obtained value of the objective function at a node because if it is assumed that all jobs are finished on time (before the due date δ_j), the first two terms of the objective function would then equal to zero which means the probability of tardiness of jobs becomes zero and the only non-zero term in the objective function would be the third term which represents the probability of all jobs finishing on time without success. In other words, it is assumed that all missiles fired at a threat j are within the weapon's firing range. $(1 - p_j)^{v'_j}$ is the probability of all v_j' missiles missing their target, i.e., the jth threat and $\eta_j(1 - p_j)^{v'_j}$ is the probability of the jth threat hitting the warship. Therefore, $\left(1 - \eta_j(1 - p_j)^{v'_j}\right)$ is the survival probability of the warship from threat j. In order to construct the upper bound, all possible finishing times for partial scheduling of a node are calculated and then the number of remaining jobs for each module is determined and used in the equation for calculating the upper bound.

3.5. Bounding Scheme

The search tree explores all possible solutions to the problem while many nodes ultimately lead to poor solutions. Furthermore, considering all possible combinations significantly increases the computing time. Thus, two bounding schemes are used to limit the search process.

The first bounding scheme limits the search process for nodes when improving the solution is not possible. All solutions from the B&B algorithm are feasible whether the weapon is scheduled to fire at a single threat or a multitude of threats. But, when no further shots can be scheduled, i.e., all threats are outside the weapon's firing range, no more nodes will be generated and thus the node can be fathomed.

The second bounding scheme is used when the upper bound of the value of the objective function at a node is less than an already existing feasible solution. In this case, the node is fathomed to prevent generation and consideration of poor solutions.

4. Computational Results

Here we discusses the computational results obtained from executing an implementation of the algorithm on 200 test problems. Then, in order to evaluate the proposed lower bound, solutions in the existence and absence of proposed bounds are compared. The proposed B&B algorithm is implemented in the Visual C++ 2010 environment and all the test problems have been saved on a notebook with CPU Core i7, 6 GB RAM, and Windows 7 operating system.

4.1. Setting of Test Problems

In order to generate the test problems, two factors were taken into consideration: the number of threats (n) and initial distance of the threats from the warship (Δ). We considered five values n: $5, 10, 15, 20$ and 25. The initial distance of the threats from the warship was defined as a random variable in the specified intervals. The intervals were selected based on the radar range (25 kilometers) and characteristics of medium-range air defense weapons (a range of 700 meters to 15 kilometers). Hence, the initial distance of the threats from the warship was generated uniformly as a random number in the intervals [1, 25], [5, 25], [10, 25] and [15, 25] kilometers. For each combination of distance interval and number of threats, 10 test problems were randomly generated where in each category the problems varied in terms of velocity of threats, setup times and weapon precision. In all of the 200 randomly generated test problems, precision of threats were randomly generated in the interval [0.9, 1] with uniform distribution, the velocity of the threats was selected randomly with equal chance from the three values 680, 510 and 306 as meters per second and the velocity of the air defense missile was set to 782 meter per second. The setup time for launching a missile is randomly determined from a [0,1] minute interval. Furthermore, the probability of the missile hitting any target (or threat) was determined as a random value from the [0.7, 0.8] interval. As mentioned before, the launching time for all missiles is a constant and depends on the type of the weapon. Here, the launching time was set to be 4 seconds.

4.2. Evaluation of B&B Performance

In this section, the results of the proposed B&B algorithm in terms of CPU run time are discussed. Initially, the algorithm was run with a sufficiently large time limit (approximately four hours) and the optimal solutions of all test problems were obtained. Since the four-hour time limit (TL) is impractical for the applications of this type of a problem, the algorithm was run for six different *TL*: 0.5, 1, 5, 10, 30, and 60 seconds (the corresponding results are shown in Table 4). For each time limit, the average percent deviation of the best found solution from the optimal solution was calculated and the number of optimal solutions found (*#opt*) in each time limit was noted. The average percent deviation (*APD*) was estimated using

$$APD = \frac{optimal\ solution - best\ found\ solution}{optimal\ solution} \times 100. \quad (14)$$

Table 4 presents the results obtained by the B&B algorithm on 200 test problems.

Table 4. Performance of the proposed B&B algorithm

TL	0.5	1	5	10	30	60
APD	7.98	7.28	6.04	5.04	3.97	3.51
#opt	123	126	140	148	161	162

Table 4 illustrates the efficiency of the proposed B&B algorithm. In 0.5 seconds the algorithm reached a solution having value 62% of the optimal solution and this increased to 81% for 60 seconds. It should be mentioned that for practical problems number of threats are not usually more than five and our developed B&B algorithm was able to find all optimal solutions of the 40 test problems with $n = 5$ in a CPU run time less than a second.

Table 5 shows the average Δ for the objective function, i.e., the survival probability of the warship for 200 test problems in 20 sets.

Table 5. Average value Δ of the objective function for test problems with 60 seconds time limit

n \ *Δ*	[1, 25]	[5, 25]	[10, 25]	[15, 25]
5	0.5861	0.8436	0.9047	0.9348
10	0.5864	0.7132	0.8325	0.9153
15	0.2298	0.5471	0.7382	0.7912
20	0.1927	0.326	0.616	0.6759
25	0.1667	0.343	0.3748	0.4376

Quite expectedly, as the lower bound of the initial distance of threats increases, the survival probability of the warship is improved because an increased lower bound for the initial distance of the threat leads to an increase in the number of missiles fired at the threat and consequently the chance of the warship for survival improves. Also, it is evident that an augmented number of threats will reduce the chance of survival.

4.3. Evaluation of the Lower Bound's performance

Here, we compare the results obtained from the solution procedure developed as the lower bound with the optimal results. Fig. 3 illustrates the *APD* based on the initial distance intervals. In test problems with $\Delta \in [1, 25]$, *APD* is about 16% and deviations for problems with $\Delta \in [5, 25]$ and $\Delta \in [10, 25]$ increased but ultimately plunged for the last interval. This sharp decrease can be attributed to the design of the lower bound because the method used for calculating the lower bound is proportional to large distances. This is why the number of initial solutions equal to the optimal solution in this interval is more than the other intervals. It should be noted that the solution found using lower bounds was equal to the optimal solution in 29 of the test problems.

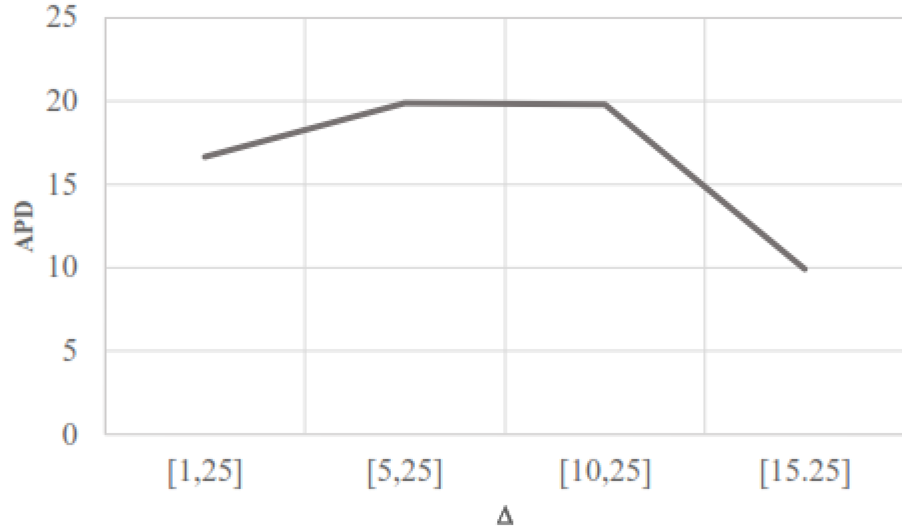

Fig. 3. *APD* diagram based on the intervals for initial distance of threats

4.4. Impact of the Upper Bound

In order to evaluate the impact of the developed upper bound, we ran the B&B algorithm without the upper bound and its corresponding dominance rule. Similar to Table 4, we report the new results in Table 6. It is concluded that with shorter CPU run times, the upper bound has more impact on the efficiency of the B&B algorithm.

Table 6. Impact of the upper bound on B&B performance

TL	0.5	1	5	10	30	60
APD	21.83	18.36	12.47	8.14	5.48	4.83
#opt	72	78	91	121	143	158

4.5. Operationalization of the Proposed Solution Method

As mentioned previously, our problem involves stochastic variables and the proposed solution method maximizes the expected value of a warship's survival. In reality, however, different circumstances could occur. For example, while the optimal solution may consist of three missiles fired at threat *j*, the first missile might hit the target and thus the launching of the other two would not be required. The solution we presented here is a baseline scheme for scheduling air defense weapon which is only optimal before the launching of the missiles. In practice, the scheduling of a weapon must be updated after the missiles are fired. As long as the missiles fired miss their targets or no new threat is introduced, the scheduling of the weapon using the B&B algorithm works fine. But, if a missile hits a target or new threats are posed, a new problem is at hand and thus the algorithm should run again.

5. Conclusions

We investigated weapon scheduling for a warship's air defense system. The problem was formulated using the literature of R&D projects scheduling and the concept of possible failure of jobs in projects. A mathematical model with stochastic variables and nonlinear objective function was constructed for calculating the survival probability of the warship. The model being nonlinear and stochastic, a B&B algorithm was developed to solve the model. Computational results indicated that the proposed algorithm could reach 81% of the optimal solution in 60 seconds.

Given the novelty of this problem in the field of scheduling and its immediate applications in naval combats, further developments can be considered for the model. Developing dynamic solution approaches for this problem is an interesting research topic. Also, the same problem can be considered with several air defense weapons of the same kind. In this case, the problem takes the form of a parallel machine scheduling problem. Developments of more efficient exact and heuristic algorithms for the proposed problem are always useful.

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