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On Optimality Conditions via Weak Subdifferential and Augmented Normal Cone

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In this paper, we investigate relation between weak subdifferential and augmented normal cone. We define augmented normal cone via weak subdifferential and vice versa. The necessary conditions for the global maximum are also stated. We produce preliminary properties of augmented normal cones and discuss them via the distance function. Then we obtain the augmented normal cone for the indicator function. Relation between weak subifferential and augmented normal cone and epigraph is also explored. We also obtain optimality conditions via weak subdifferential and augmented normal cone. Finally, we define the Stampacchia and Minty solution via weak subdifferential and investigate the relation between Stampacchia and Minty solution and the minimal point.

Keywords: *Subgradient, Weak subgradient, Normal cone, Augmented normal cone, Frechet differentiable function, Locally Lipchitz function.*

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1. Introduction

A convex set having a supporting hyperplane at each boundary point turns to central notion in convex analysis, namely subgradient of a possible nonsmooth even extended real valued function [4, 5]. Subgradient plays a key role in the derivation of optimality conditions and duality results. Since a nonconvex set has no supporting hyperplane at each boundary point, the notion of subgradient has been generalized by most researchers to optimality conditions for nonconvex problems. For more study, see [3, 4]. The various of different subdifferentials can be divided into 2 large groups:

- “simple” subdifferentials, and
- “strict” subdifferentials.

A simple subdifferential is defined at a given point and it does not take into account “differential” properties of a function in its neighborhood. Simple subdifferential are not widely used directly because of rather poor calculus. Contrary to simple subdifferentials, strict subdifferentials incorporate differential properties of a function near a given point.

The notion of weak subdifferential, as a generalization of the classical subdifferential, was introduced by Azimov and Gasimov [1, 2]. It uses explicitly defined supporting conic surfaces instead of supporting hyperplanes. The main reason for difficulties arising in passing from the convex

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analysis to the nonconvex one is that the nonconvex cases may arise in many different forms, each of which may require a special approach. The main ingredient is the method of supporting the given nonconvex set. Subgradients play important roles in the derivation of optimality conditions and duality results. The first canonical generalized gradient was introduced by Clarke [4, 5]. He applied the generalized gradient systematically to nonsmooth problems in various problems. Since a nonconvex set has no supporting hyperline at each boundary point, the notion of subgradient has been generalized by most researchers to optimality conditions for nonconvex problems [4, 5, 14]. By using the notion of subgradients, a collection of zero duality gap conditions for a wide class of nonconvex optimization problems was derived [1, 2]. Augmented normal cone via weak subdifferential was defined by Kasimbeyli and Mammadov [13]. Here we give some important properties of augmented normal cones via weak subdifferentials. By using the definition and properties of the weak subdifferential, as described in [9, 10, 11, 12], we stablish results on connection with augmented normal cones and weak subdifferential for nonsmooth and nonconvex problems.

The remainder of our work is organized as follows. The definitions and preliminaries of weak subdifferential and augmented normal cone are provided in Section 2. In Section 3, we state some useful properties of augmented normal cones, and then prove some results connecting augmented normal cones and weak subdifferentials for nonsmooth and nonconvex problems in Section 4. Section 5 provides relations between the Stampacchia and Minty solution and the minimal point.

2. Preliminaries

Let X be a real normed space and let X^* be the topological dual of X. By $\|\cdot\|$, we denote the norm of X and by $\langle x^*, x\rangle$, the value of the linear functional $x^* \in X^*$ at the point $x \in X$. Let $\emptyset \neq S \subset X$.

Definition 2.1. [11, 12] Let $f: X \to \mathbb{R}$ be a function and $\bar{x} \in X$ be a given point. The set

$$\partial f(\bar{x}) = \{x^* \in X^* : (\forall x \in X)\ f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle\}$$

is called the subdifferential of f at $\bar{x} \in X$.

The next definition generalizes the notion of subdifferential.

Definition 2.2. [11, 12] Let $f: X \to \mathbb{R}$ be a function and $\bar{x} \in X$ be a given point. A pair $(x^*, c) \in X^* \times \mathbb{R}^+$, where $\mathbb{R}^+$ is the set of nonnegative real numbers, is called the weak subgradient of f at $\bar{x} \in X$ if the following inequality holds:

$$(\forall x \in X)\quad f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\|.$$

The set

$$\partial^w f(\bar{x}) = \{(x^*, c) \in X^* \times \mathbb{R}^+ \ : (\forall x \in X)\quad f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\|\}$$

of all weak subgradients of f at $\bar{x} \in X$ is called the weak subdifferential of f at $\bar{x} \in X$. If $\partial^w f(\bar{x}) \neq \emptyset$ then f is said to be weakly subdifferentiable at $\bar{x}$.

Remark 2.1. [3] It is clear that when f is subdifferentiable at $\bar{x}$, then f is also weakly subdifferentiable at $\bar{x}$; that is, if $x^* \in \partial f(\bar{x})$ then by the definition of weak subgradient we get $(x^*, c) \in \partial^w f(\bar{x})$, for every $c \geq 0$. But the converse may fail (consider $f(x) = -|x|,\ X = \mathbb{R}$).

The next definition is used in the sequel.

Definition 2.3. [8] Let $f: X \to \mathbb{R}$. If there is a continuous linear map $f'(\bar{x}): X \to \mathbb{R}$ with the property

$$\lim_{\|h\|\to 0} \frac{|f\,(\bar{x}+h) - f\,(\bar{x}) - \langle f'(\bar{x}), h\rangle|}{\|h\|} = 0,$$

then $f'(\bar{x}): X \to \mathbb{R}$ is said to be Fréchet derivative of f at $\bar{x} \in X$ and f is called the Fréchet differentiable at $\bar{x}$.

Remark 2.2. [3] It follows from Definition 2.2 that the pair $(x^*, c) \in X^* \times \mathbb{R}^+$ is a weak subdifferential of f at $\bar{x} \in X$ if and only if a continuous (super linear) concave function $g: X \to \mathbb{R}$, defind by $g(x) = f(\bar{x}) + \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\|,\ \ x \in X$, satisfies

$$(\forall x \in X)\quad g(x) \leq f(x) \quad \text{and} \quad g(\bar{x}) = f(\bar{x}).$$

This condition means that g supports f from below. Hence, it follows that if f is weakly subdifferentiable at $\bar{x}$ and $(x^*, c) \in \partial^w f(\bar{x})$, then the graph of function g becomes a supporting surface to the epigraph of f on X at the point $\big(\bar{x}, f(\bar{x})\big)$.

For the gradient of g at $\bar{x}$, that is $\nabla g(\bar{x})$, we can obtain

$$\nabla g(\bar{x}) = x^* - \frac{c(x-\bar{x})}{\|x - \bar{x}\|},$$

and for the norm of $\nabla g(\bar{x})$, we get

$$\|\nabla g(\bar{x})\| \leq \|x^*\| + c.$$

In fact, we get the bounded of the norm of $\nabla g(\bar{x})$, which will be useful in estimating the subgradients for finding the extremal points of a nonsmooth function.

Theorem 2.1. [11] Let the weak subdifferential of $f: X \to \mathbb{R}$ at $\bar{x}$ be nonempty. Then, the set $\partial^w f(\bar{x})$ is closed and convex.

3. Main Results

Here we first recall the definition of augmented normal cone as given in [13] and then state our main results.

Definition 3.1. The set

$$N_S(\bar{x}) = \{x^* \in X^*\,; \langle x^*, x - \bar{x}\rangle \leq 0, \quad \forall x \in S\}$$

is called a normal cone to S at $\bar{x}$.

Definition 3.2. The set

$$N_S^a(\bar{x}) = \{(x^*, c) \in X^* \times \mathbb{R}^+ ; \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\| \leq 0, \quad \forall x \in S\},$$

is called an augmented normal cone to S at $\bar{x}$. Note that if there exists $x^* \in X^*$ such that $(x^*, 0) \in N_S^a(\bar{x})$, then $x^* \in N_S(\bar{x})$.

Remark 3.1. From the definitions of normal and augmented normal cones, we have

$$x^* \in N_S(\bar{x}) \Longrightarrow (x^*, c) \in N_S^a(\bar{x}), \qquad \forall c \geq 0.$$

Remark 3.2. For $(x^*, c) \in N_S^a(\bar{x})$ with $\|x^*\| \leq c$, the inequality $\langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\| \leq 0$ is obviously satisfied for all $x \in X$. An augmented normal cone consisting of only such elements is called trivial and denoted by $N_S^{triv}(\bar{x})$. Obviousely,

$$N_S^{triv}(\bar{x}) \subset N_S^a(\bar{x}).$$

Example 3.1. Let $X = S$, $\bar{x} \in X$. Then, we have

$$\begin{aligned} N_S^a(\bar{x}) = N_X^a(\bar{x}) &= \{(x^*, c) \in X^* \times \mathbb{R}^+ ; \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\| \leq 0, \ (\forall x \in S)\} \\ &= \{(x^*, c) \in X^* \times \mathbb{R}^+ ; \ \|x^*\| \leq c\} = N_X^{triv}(\bar{x}). \end{aligned}$$

Proposition 3.1. Let $c_1 \leq c_2$. Then, we have

$$(x^*, c_1) \in N_S^a(\bar{x}) \Longrightarrow (x^*, c_2) \in N_S^a(\bar{x}).$$

Proof. Let $(x^*, c_1) \in N_S^a(\bar{x})$. Then, by definition of augmented normal cone, we have

$$\langle x^*, x - \bar{x}\rangle - c_1\|x - \bar{x}\| \leq 0, \qquad \forall x \in S.$$

So, by assumption $c_1 \leq c_2$, we obtain

$$\langle x^*, x - \bar{x}\rangle - c_2\|x - \bar{x}\| \leq 0, \qquad \forall x \in S.$$

Therefore, $(x^*, c_1) \in N_S^a(\bar{x})$ and the result is at hand. Since for any $\bar{x} \in S$, we have $(0,0) \in N_S^a(\bar{x})$, the augmented normal cone is a nonempty and uncountable set. ■

Proposition 3.2. The set $N_S^a(\bar{x})$ is a closed convex cone

Proof. The proof directly follows from the definition of $N_S^a(\bar{x})$. ■

Proposition 3.3. $(x^*, c) \in N_S^a(\bar{x})$ if and only if the function $g: X \to \mathbb{R}$, defined by

$$g(x) = \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\|,$$

satisfies

$$g(x) \leq 0 \quad (\forall x \in S), \qquad g(\bar{x}) = 0.$$

In this case, we get

$$\nabla g(\bar{x}) = x^* - \frac{c(x-\bar{x})}{\|x-\bar{x}\|}.$$

Proof. Consider $g(x) = \langle x^*, x-\bar{x}\rangle - c\|x - \bar{x}\|$ to investigate the result. ■

The next proposition states the necessary condition for a global maximum.

Proposition 3.4. Let $f: X \to \mathbb{R}$ be a function that attains a global maximum at $\bar{x}$. Then, we have

$$\partial^w f(\bar{x}) \subset N_X^{triv}(\bar{x}) \subset N_X^a(\bar{x}).$$

Proof. If $\partial^w f(\bar{x}) \neq \emptyset$, then there exists a pair (x^*, c) such that

$$f(x) - f(\bar{x}) \geq \langle x^*, x-\bar{x}\rangle - c\|x-\bar{x}\|, \quad \forall x \in X.$$

With the assumption that f attains a global maximum at $\bar{x}$, we have

$$\langle x^*, x-\bar{x}\rangle - c\|x - \bar{x}\| \leq 0, \qquad \forall x \in X.$$

So,

$$\|x^*\| \leq c,$$

and we have $(x^*, c) \in N_X^{triv}(\bar{x})$ and the proof is complete by noting $N_S^{triv}(\bar{x}) \subset N_S^a(\bar{x})$. ■

Corollary 3.1. Let $f: X \to \mathbb{R}$ be a function that attains a global minimum at $\bar{x}$. Then, we have

$$\partial^w\big(-f(\bar{x})\big) \subset N_X^{triv}(\bar{x}).$$

The following example shows that the inclusion in Proposition 3.1 can be a proper inclusion.

Example 3.2. Let $X = \mathbb{R}, f(x) = -|x|$. Then, we have

$$\partial^w f(0) = \{(\alpha, c)\,;\ |\alpha| \leq c-1\}$$

and

$$N_{\mathbb{R}}^{triv}(0) = \{(\alpha, c)\,;\ |\alpha| \leq c\}.$$

Therefore, $\partial^w f(\bar{x}) \neq N_X^{triv}(\bar{x})$, and we note that f has a global maximum at $\bar{x} = 0$. The converse of Proposition 3.1 may not be true. Consider the next example.

Example 3.3. Let

$$f(x) = \begin{cases} 0, & x \in \mathbb{Q}, \\ 1, & x \in \mathbb{Q}^c. \end{cases}$$

Then,

$$\partial^w f(0) = N_X^{triv}(0) = \{(\alpha, c)\ ;\ |\alpha| \le c\},$$

while f attains a global minimum at $\bar{x} = 0$.

Proposition 3.5. Let $f: X \to \mathbb{R}$ be a function that attains a global minimum at $\bar{x}$. Then, we have

$$N_X^a(\bar{x}) \subset \partial^w f(\bar{x}).$$

Proof. Let $(x^*, c) \in N_X^a(\bar{x})$. Then, we have

$$\langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\| \le 0, \quad \forall\, x \in X.$$

Since f attains a global minimum at $\bar{x}$, then we obtain

$$f(x) - f(\bar{x}) \ge 0, \quad \forall x \in X,$$

from which, we get

$$f(x) - f(\bar{x}) \ge \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\| \le 0, \qquad \forall\, x \in X,$$

so that $(x^*, c) \in \partial^w f(\bar{x})$, and the proof is complete. ■

The next proposition states a link between weak subdifferential of f, $-f$ and augmented normal cone at $\bar{x}$ for the function attaining a global minimum at $\bar{x}$. This is a necessary condition for optimality.

Proposition 3.6. Let $f: X \to \mathbb{R}$ be a function that attains a global minimum at $\bar{x}$. Then, we have

$$\partial^w\big(-f(\bar{x})\big) \subset N_X^a(\bar{x}) \subset \partial^w f(\bar{x}).$$

Proof. The proof directly follows from Corollary 3.2 and Proposition 3.3. ■

Corollary 3.2. Let f be a constant function. Then, we have

$$N_X^c(\bar{x}) = \partial^w f(\bar{x}) = \partial^w\big(-f(\bar{x})\big).$$

Proof. The proof follows from Propositions 3.4. ■

As a particular case, consider the weak subdifferentiability of an indicator function. Let δ_S be an indicator function of a set $S \subset X$ such that

$$\delta_S(x) = \begin{cases} 0, & x \in S \\ \infty, & \text{o.w.} \end{cases}$$

Kasimbeily [10] generalized the well-known theorem in convex analysis that states a relationship between the subdifferential of the indicator function and the supporting hyperplane of a convex set. Here, generalize that result by presenting a relationship between the weak subdifferential of the indicator function of any set and the augmented normal cone.

Proposition 3.7. Let δ_S be an indicator function of a set $S \subset X$. Then, we have

$$N_S^a(\bar{x}) = \partial^w \delta_S(\bar{x}).$$

Proof. Assume $(x^*, c) \in N_S^a(\bar{x})$. Therefore, we have

$$\langle x^*, x - \bar{x} \rangle - c\|x - \bar{x}\| \leq 0, \quad \forall\, x \in S.$$

We know that

$$\delta_S(x) - \delta_S(\bar{x}) = 0, \quad \forall x \in S,$$
$$\delta_S(x) - \delta_S(\bar{x}) = \infty, \quad \forall x \notin S,$$

so that we obtain

$$\delta_S(x) - \delta_S(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle - c\|x - \bar{x}\|, \qquad \forall x \in X,$$

that is, $(x^*, c) \in \partial^w \delta_S(\bar{x})$. Conversely, if $(x^*, c) \in \partial^w \delta_S(\bar{x})$, then we have

$$\delta_S(x) - \delta_S(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle - c\|x - \bar{x}\|, \qquad \forall x \in X.$$

If $x \in S$, then we obtain

$$\delta_S(x) - \delta_S(\bar{x}) = 0,$$

and consequently,

$$\langle x^*, x - \bar{x} \rangle - c\|x - \bar{x}\| \leq 0, \quad \forall x \in S.$$

This means $(x^*, c) \in N_S^a(\bar{x})$, and the proof is complete. ■

In the sequel, we state some properties of the augmented normal cone.

Proposition 3.8. Let $S_1 \subset S_2$. Then, we have

$$N_{S_2}^a(\bar{x}) \subset N_{S_1}^a(\bar{x}).$$

Proof. Assume that $(x^*, c) \in N_{S_2}^a(\bar{x})$. Then,

$$\langle x^*, x - \bar{x} \rangle - c\|x - \bar{x}\| \leq 0, \quad \forall x \in S_2.$$

Since $S_1 \subset S_2$, we obtain

$$\langle x^*, x - \bar{x} \rangle - c\|x - \bar{x}\| \leq 0, \quad \forall x \in S_1.$$

that is, $(x^*, c) \in N_{S_1}^a(\bar{x})$ and the proof is complete. ■

Remark 3.3. With the last proposition, if $S_1 = S_2$, then we have $N_{S_2}^c(\bar{x}) = N_{S_1}^c(\bar{x})$. But the converse may not be true. Consider the next example.

Example 3.4. Let $S_1 = [0,1]$, $S_2 = [0,2]$. Then

$$N_{S_1}^a(0) = N_{S_2}^a(0) = \{(\alpha, c) : \alpha \leq c\},$$

while $S_1 \neq S_2$.

Proposition 3.9. $N_S^a(\bar{x}) = N_{cl\ S}^a(\bar{x})$.

Proof. Since $S \subset cl\ S$, $N_{cl\ S}^a(\bar{x}) \subset N_S^a(\bar{x})$. Conversely, for any $x \in cl\ S$, there exists $\{x_n\} \subset S$ such that $x_n \to x$. Now, assume $(x^*, c) \in N_S^a(\bar{x})$, so that

$$\langle x^*, x_n - \bar{x} \rangle - c\ \|x_n - \bar{x}\| \leq 0, \qquad \forall x_n \in S.$$

By taking the limit inferior of both sides of the last inequality when $n \to \infty$, we have

$$\langle x^*, x - \bar{x} \rangle - c\ \|x - \bar{x}\| \leq 0, \qquad \forall x \in cl\ S,$$

which means that $(x^*, c) \in N_{cl\ S}^a(\bar{x})$, and the proof is complete. ■

Proposition 3.10. Let S be a cone. Then,

$$N_S^a(\lambda \bar{x}) = N_S^a(\bar{x}), \qquad \forall\ \lambda > 0.$$

Proof. It follows from the hypothesis that

$$\begin{aligned}(x^*, c) \in N_S^a(\lambda\, \bar{x}) &\Leftrightarrow \langle x^*, \lambda x - \lambda \bar{x} \rangle - c\ \|\lambda x\ - \lambda \bar{x}\| \leq 0 \ \ (\forall x \in S) \\ &\Leftrightarrow \lambda(\langle x^*, x - \bar{x} \rangle - c\ \|x\ - \bar{x}\|) \leq 0 \ \ (\forall x \in S) \\ &\Leftrightarrow (x^*, c) \in N_S^a(\bar{x}).\end{aligned}$$

This completes the proof. ■

Proposition 3.11. Let $S_1, S_2 \subset X$, $S_1 \cap S_2 \neq \emptyset$. Then,

$$N_{S_1 \cup S_2}^a(\bar{x}) = N_{S_1}^a(\bar{x}) \cap N_{S_2}^a(\bar{x}) \subset N_{S_1 \cap S_2}^a(\bar{x}).$$

Proof. Suppose that $(x^*, c) \in N_{S_1 \cup S_2}^a(\bar{x})$. Then

$$\langle x^*, x - \bar{x} \rangle - c\|x -\ \bar{x}\| \leq 0, \qquad \forall x \in S_1 \cup S_2,$$

and so we have

$$\langle x^*, x - \bar{x} \rangle - c\|x -\ \bar{x}\| \leq 0, \qquad \forall x \in S_1,$$

and

$$\langle x^*, x-\bar{x}\rangle - c\|x - \bar{x}\| \leq 0, \qquad \forall x \in S_2,$$

which means $(x^*, c) \in N^a_{S_1}(\bar{x}) \cap N^a_{S_2}(\bar{x})$. Also, we obtain

$$\langle x^*, x-\bar{x}\rangle - c\|x - \bar{x}\| \leq 0, \qquad \forall x \in S_1 \cap S_2,$$

and the last inclusion is at hand. Conversely, assume that $(x^*, c) \in N^a_{S_1}(\bar{x}) \cap N^a_{S_2}(\bar{x})$. Then, we have

$$\langle x^*, x-\bar{x}\rangle - c\|x - \bar{x}\| \leq 0, \qquad \forall x \in S_1,$$

and

$$\langle x^*, x-\bar{x}\rangle - c\|x - \bar{x}\| \leq 0, \qquad \forall x \in S_2,$$

to get

$$\langle x^*, x-\bar{x}\rangle - c\|x - \bar{x}\| \leq 0, \qquad \forall x \in S_1 \cup S_2,$$

so that $(x^*, c) \in N^a_{S_1 \cup S_2}(\bar{x})$, to complete the proof. ∎

The following example shows that the converse of the last inclusion may not be true.

Example 3.5. Let $X = \mathbb{R}$, $S_1 = \{0, 1\}$, $S_2 = \{0, 2\}$, $\bar{x} = 0$. Then we have

$$N^a_{S_1}(\bar{x}) = N^a_{S_2}(\bar{x}) = \{(\alpha, c) \in \mathbb{R} \times \mathbb{R} : \alpha \leq c\},$$

while $N^a_S(\bar{x}) = \mathbb{R}^2$.

Remark 3.4. Since $S_1 \cap S_2 \subset S_1,\ S_2$, by Proposition 3.11, we have

$$N^a_{S_1}(\bar{x}), N^a_{S_2}(\bar{x}) \subset N^a_{S_1}(\bar{x}) \cap N^a_{S_2}(\bar{x}),$$

so that

$$N^a_{S_1}(\bar{x}) \cap N^a_{S_2}(\bar{x}) \subset N^a_{S_1 \cap S_2}(\bar{x}),$$

and similarly,

$$N^a_{S_1}(\bar{x}) \cup N^a_{S_2}(\bar{x}) \subset N^a_{S_1 \cap S_2}(\bar{x}).$$

Proposition 3.12. Let $S = S_1 \cap S_2 \neq \emptyset$. Then,

$$N^a_{S_1}(\bar{x}) + N^a_{S_2}(\bar{x}) \subset N^a_S(\bar{x}).$$

Proof. Assume that $(x_1^*, c_1) \in N^a_{S_1}(\bar{x})$ and $(x_2^*, c_2) \in N^a_{S_2}(\bar{x})$. Therefore,

$$\langle x_1^*, x-\bar{x}\rangle - c_1\|x - \bar{x}\| \leq 0, \qquad \forall x \in S_1,$$

$$\langle x_2^*, x-\bar{x}\rangle - c_2\|x-\bar{x}\| \le 0, \qquad \forall x\in S_2.$$

Now, for any $x\in S=S_1\cap S_2$, we obtain

$$\langle x_1^*+x_2^*, x-\bar{x}\rangle - (c_1+c_2)\|x-\bar{x}\| \le 0, \qquad \forall x\in S,$$

that is, $(x_1^*+x_2^*, c_1+c_2)\in N_S^a(\bar{x})$ and the proof is complete. ■

The next example shows that the converse of the last inclusion may fail.

Example 3.6. Let $X=\mathbb{R}$, $S_1=\{0,1\}$, $S_2=\{0,2\}$, $\bar{x}=0$. Then, we have

$$N_{S_1}^a(\bar{x})=N_{S_2}^a(\bar{x})=\{(\alpha,c)\in\mathbb{R}\times\mathbb{R}:\alpha\le c\},$$

while $N_S^a(\bar{x})=\mathbb{R}^2$.

Proposition 3.13. Let $S=S_1+S_2$, $\bar{x}=\bar{x}_1+\bar{x}_2$, $\bar{x}_i\in S_i$, $i=1,2$. Then,

$$N_S^a(\bar{x})=N_{S_1}^a(\bar{x})\cap N_{S_2}^a(\bar{x}).$$

Proof. Assume $(x^*,c)\in N_S^a(\bar{x})$. Then, we have

$$\langle x^*, x-\bar{x}\rangle - c\|x-\bar{x}\|\le 0 \ \ (\forall x\in S),$$

and we get

$$\langle x^*, (x_1+x_2)-(\bar{x}_1+\bar{x}_2)\rangle - c\|(x_1+x_2)-(\bar{x}_1+\bar{x}_2)\|\le 0, \qquad \forall x\in S=S_1+S_2.$$

From the last inequality, with $x_2=\bar{x}_2$ and $x_1=\bar{x}_1$, respectively, we obtain

$$\begin{aligned}\langle x^*, x_1-\bar{x}_1\rangle - c\|x_1-\bar{x}_1\|\le 0, &\qquad \forall x\in S_1 \Rightarrow (x^*,c)\in N_{S_1}^a(\bar{x}_1),\\ \langle x^*, x_2-\bar{x}_2\rangle - c\|x_2-\bar{x}_2\|\le 0, &\qquad \forall x\in S_2 \Rightarrow (x^*,c)\in N_{S_2}^a(\bar{x}_2).\end{aligned}$$

And if we follow the stages of the above argument conversely, the proof is complete. ■

Proposition 3.14. Let $\hat{S}=S\times S$, $\hat{x}=(\bar{x},\bar{x})$. Then,

$$N_{\hat{S}}^a(\hat{x})=\left\{\big((x^*,y^*),c\big)\in X^*\times X^*\times\mathbb{R}^+ : (\,(x^*+y^*),2c)\in N_S^a(\bar{x})\right\}.$$

Note that $\|(x,y)\|=\|x\|+\|y\|$, for all $x,y\in X$.

Proof. It follows from the hypothesis that

$$\begin{aligned}\big((x^*,y^*),c\big)\in N_{\hat{S}}^a(\hat{x}) &\Leftrightarrow \langle (x^*,y^*),(x,x)-\hat{x}\rangle - c\|(x,x)-\hat{x}\|\le 0 \quad \forall (x,x)\in\hat{S}\\ &\Leftrightarrow \langle x^*+y^*, x-\bar{x}\rangle - 2c\|x-\bar{x}\|\le 0 \quad \forall x\in S\\ &\Leftrightarrow (x^*+y^*,2c)\in N_S^a(\bar{x}),\end{aligned}$$

to complete the proof. ■

Proposition 3.15. Let $X = X_1 \times X_2$, $S = S_1 \times S_2$, $\bar{x} = (\bar{x}_1, \bar{x}_2)$, $\bar{x}_i \in S_i \subset X_i$, $i = 1, 2$. Then,

$$\pi\big(N_S^a(\bar{x})\big) = \pi\left(N_{S_1}^a(\bar{x}_1)\right) \times \pi\left(N_{S_2}^a(\bar{x}_2)\right).$$

Proof. We have the following inequalities:

$$\begin{aligned}\big((x^*, y^*), c\big) \in N_S^a(\bar{x}) &\Leftrightarrow \langle (x^*, y^*), (x_1, x_2) - (\bar{x}_1, \bar{x}_2)\rangle - c\|(x_1, x_2) - (\bar{x}_1, \bar{x}_2)\| \le 0 \quad \forall (x_1, x_2) \in S \\ &\Leftrightarrow \langle x^*, x_1 - \bar{x}_1\rangle - c\|x_1 - \bar{x}_1\| \le 0 \quad \forall x_1 \in S_1 \\ &\quad\; \langle y^*, x_2 - \bar{x}_2\rangle - c\|x_2 - \bar{x}_2\| \le 0 \quad \forall x_2 \in S_2 \\ &\Leftrightarrow \big((x^*, c), (y^*, c)\big) \in N_{S_1}^a(\bar{x}_1) \times N_{S_2}^a(\bar{x}_2).\end{aligned}$$

■

4. Augmented Normal Cones and Weak Subdifferentials

Kruger [14] used another approach to define the normal cone based on first considering the Fréchet subdifferential of the distance function. Recall that the distance function to S is defined by

$$d_S(x) = \inf_{y \in S} \|x - y\|.$$

We generalize this approach for augmented normal cones related by weak subdifferential in the sequel. Contrary to the indicator function whose weak subdifferential can be used for defining the augmented normal cone, the distance function is Lipschitz continuous. This makes it more convenient in some situations.

Proposition 4.1. $\partial^w d_S(\bar{x}) \subset \{(x^*, c) \in N_S^c(\bar{x}) : \|x\| \le c + 1\}$.

Proof. Suppose that $(x^*, c) \in \partial^w d_S(\bar{x})$. Then, we have

$$d_S(x) - d_S(\bar{x}) \ge \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\| \quad \forall x \in X.$$

For $x \in S$, we obtain

$$\langle x^*, x - \bar{x}\rangle - c\,\|x - \bar{x}\| \le 0, \qquad \forall x \in S,$$

and therefore, $(x^*, c) \in N_S^c(\bar{x})$. For $x \notin S$, we have

$$\|x - \bar{x}\| \ge \inf_{y \in S}\|x - y\| = d_S(x) \ge \langle x^*, x - \bar{x}\rangle - c\|x - \bar{x}\| \quad \forall x \notin S,$$

and thus,

$$\langle x^*, x - \bar{x}\rangle \le (c + 1)\|x - \bar{x}\|, \qquad \forall x \notin S.$$

From the above inequalities, for any $x \in X$ we obtain

$$\langle x^*, x - \bar{x}\rangle \le (c + 1)\|x - \bar{x}\|, \qquad \forall x \in X,$$

and consequently,

$$\|x^*\| \leq c + 1.$$

■

Remark 4.1. If we investigate the normal cone related by subdifferential, then we obtain

$$\partial d_S(\bar{x}) \subset \{x^* \in N_S(\bar{x}) : \ \|x\| \leq 1\}.$$

This is a similar result found by Kruger [14] for the Fréchet subdifferential. The following example shows that the converse of the inclusion may fail.

Example 4.1. Consider $S = [0, 1]$, $\bar{x} = 0$. Then, we have

$$\partial^w d_S(0) = \emptyset, \qquad \{(x^*, c) \in N_S^a(0) : \ \|x\| \leq c + 1\} \neq \emptyset.$$

It follows from Proposition 3.8 that an augmented normal cone is a particular case of a weak subdifferential. The converse is also true: the weak subdifferential of an arbitrary function can be equivalently defined through the augmented normal cone to its epigraph. Recall that the epigraph of f is the set

$$epi\ f = \{(u, \mu) \in X \times \mathbb{R} : \ f(u) \leq \mu\}.$$

The following result shows the relationship between weak subdifferential of f and the augmented normal cone related by $epi\ f$.

Proposition 4.2. The followings hold

- If $(x^*, c) \in \partial^w f(\bar{x})$, then $\big((x^*, -1), c\big) \in N_{epi\ f}^c\ \big(\bar{x}, f(\bar{x})\big)$.
- If $\mu \geq f(\bar{x})$ and $\big((x^*, \lambda), c\big) \in N_{epi\ f}^c\ (\bar{x}, \mu)$, then $|\lambda| \leq c$.

Proof. Suppose that $(x^*, c) \in \partial^w f(\bar{x})$. Then, we have

$$f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle - c\ \|x - \bar{x}\|, \qquad \forall x \in X,$$

so that

$$\langle (x^*, -1), (x - \bar{x}, f(x) - f(\bar{x}))\rangle \leq c\|x - \bar{x}\|,$$

and

$$c\|x - \bar{x}\| \leq c\|x - \bar{x}\| + c|f(x) - f(\bar{x})|.$$

Now, we know that

$$c\|x - \bar{x}\| + c\ |f(x) - f(\bar{x})| = c\big\|\big(x - \bar{x}, f(x) - f(\bar{x})\big)\big\|, \qquad \forall x \in X.$$

Therefore, with the above inequalities, we get

$$\left\langle (x^*,-1)\,,\left(x-\bar{x},f(x)-f(\bar{x})\right)\right\rangle \le c\left\|\left(x-\bar{x},f(x)-f(\bar{x})\right)\right\|, \qquad \forall x\in X,$$

and so $\left((x^*,-1),c\right)\in N^c_{epi\,f}\left(\bar{x},f(\bar{x})\right)$.

Next, suppose that $\left((x^*,\lambda),c\right)\in N^c_{epi\,f}(\bar{x},\mu)$. Then, we have

$$\langle (x^*,\lambda),(x-\bar{x},u-\mu)\rangle \le c\|x-\bar{x},u-\mu\|, \qquad \forall (x,u)\in epi\,f.$$

Set $x=\bar{x}$, $u=f(\bar{x})$. Then,

$$\lambda(f(\bar{x})-\mu)\le c|f(\bar{x})-\mu|,$$

and therefore,

$$(\lambda+c)(f(\bar{x})-\mu)\le 0,$$

so that with $\mu\ge f(\bar{x})$, we have

$$\lambda\ge -c.$$

Similarly, from

$$\langle (x^*,\lambda),(x-\bar{x},u-\mu)\rangle \le c\|(x-\bar{x},u-\mu\,)\|, \qquad \forall (x,u)\in epi\,f,$$

and $\mu=f(\bar{x})$, we have

$$\left\langle (x^*,\lambda),\left(x-\bar{x},u\,-\,f(\bar{x})\right)\right\rangle \le c\,\left\|\left(x-\bar{x},u-\,f(\bar{x})\right)\right\|, \qquad \forall (x,u)\in epi\,f.$$

For arbitrary $\epsilon>0$, we set $x=\bar{x}$, $u=f(\bar{x})+\epsilon$, and therefore

$$\lambda\epsilon\le c|\epsilon|,$$

so that $\lambda\le c$, and the desired result is at hand. ■

5. Stampacchia and Minty Solution via Weak Subdifferential

Here, we first consider the variational inequalities of the Stampacchia type in terms of the weak subdifferentials as follows:
(Stampacchia-∂^w): Find $\bar{x}\in K$ such that for any $x\in K$, there exists $(\bar{x}^*,c)\in\partial^w f(\bar{x})$ such that

$$\langle \bar{x}^*,x-\bar{x}\,\rangle - c\,\|x-\bar{x}\|\ge 0.$$

Proposition 5.1. Let K be a nonempty convex subset of a linear normed space X and let $\bar{x}$ be a Stampacchia-∂^w solution. Then, $\bar{x}$ is also a minimal point of the variational optimality problem.

Proof. To the contrary, suppose that $\bar{x}$ is not a minimal point of the variational optimality problem. Then, there exists $\tilde{x}\in K$ such that

$$f(\tilde{x}) - f(\bar{x}) < 0.$$

Now, for every $(\bar{x}^*, c) \in \partial^w f(\bar{x})$, we have

$$f(x) - f(\bar{x}) \geq \langle \bar{x}^*, x - \bar{x} \rangle - c\|x - \bar{x}\|, \qquad \forall x \in X.$$

Specially for $x = \tilde{x}$, we obtain

$$0 > f(\tilde{x}) - f(\bar{x}) \geq \langle \bar{x}^*, \tilde{x} - \bar{x} \rangle - c\|\tilde{x} - \bar{x}\|,$$

and therefore,

$$\langle \bar{x}^*, \tilde{x} - \bar{x} \rangle - c\, \|\tilde{x} - \bar{x}\| < 0,$$

which is a contradiction to the fact that $\bar{x}$ is a solution of the Stampacchia-∂^w. ■

Next, consider the variational inequalities of the Minty type in terms of the weak subdifferentials as follows:
(Minty-∂^w): Find $\bar{x} \in K$ such that for any $x \in K$ and $(x^*, c) \in \partial^w f(x)$,

$$\langle x^*, x - \bar{x} \rangle + c\, \|x - \bar{x}\| \geq 0.$$

Proposition 5.2. Let K be a nonempty convex subset of a linear normed space X and let $\bar{x}$ be a minimal point of the variational optimality problem. Then, $\bar{x}$ is also a Minty-∂^w solution.

Proof. Suppose that $\bar{x}$ is a minimal point of the variational optimality problem. Then, for every $x \in K$ and $(x^*, c) \in \partial^w f(x)$, by the definition of weak subdifferential, we have

$$f(\tilde{x}) - f(x) \geq \langle x^*, \tilde{x} - x \rangle - c\|\tilde{x} - x\|, \qquad \forall \tilde{x} \in X.$$

Specially for $\tilde{x} = \bar{x}$, we have

$$f(\bar{x}) - f(x) \geq \langle \bar{x}^*, \bar{x} - x \rangle - c\, \|\bar{x} - x\|, \quad \forall \bar{x} \in X.$$

Using the assumption, we obtain

$$0 \geq \langle \bar{x}^*, \bar{x} - x \rangle - c\|\bar{x} - x\|.$$

Therefore, we have

$$\langle \bar{x}^*, x - \bar{x} \rangle + c\|\bar{x} - x\| \geq 0,$$

which means that $\bar{x}$ is a Minty-∂^w solution. ■

Remark 5.1. The converse of Proposition 5.2 may not be true. Consider the following example.

Example 5.1. Let $X = K = \mathbb{R}$, and

$$f(x) = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0. \end{cases}$$

Obviously, $\bar{x} = 0$ is not a minimal point of $f(x)$. But, $\bar{x}$ is a Minty-∂^w solution. Indeed, for $x \neq 0$, we have

$$\partial^w f(x) = \{(x^*, c) : |x^*| \leq c\}.$$

Therefore,

$$\langle \bar{x}^*, \bar{x} - x\rangle + c\|\bar{x} - x\| = \begin{cases}(x^* + c)x, & x > 0,\\ (x^* - c)x, & x < 0,\end{cases}$$

and which is the nonnegative. Therefore, $\bar{x} = 0$ is a Minty-∂^w solution.

Proposition 5.3. Let K be a nonempty convex subset of a linear normed space X and let $\bar{x}$ be a Stampacchia-∂^w solution. Then, $\bar{x}$ is also a Minty-∂^w solution.

Proof. Combine the results of Theorem 5.1 and 5.2.

Remark 5.2. Example 5.1 also shows that the converse of Theorem 5.3 may not hold.

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