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Comparison of Selected Advanced Numerical Methods for Greeks Calculation of Vanilla Options

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Option valuation has been a challenging issue of financial engineering and optimization for a long time. The increasing complexity of market conditions requires utilization of advanced models that, commonly, do not lead to closed-form solutions. Development of novel numerical procedures, which prove to be efficient within various option valuation problems, is therefore worthwhile. Notwithstanding, such novel approaches should be tested as well, the most natural way being to assume simple plain vanilla options under the Black and Scholes model first; because of its simplicity the analytical solution is available and the convergence of novel numerical approaches can be analyzed easily. Here, we present the methodological concepts of two relatively modern numerical techniques, i.e., discontinuous Galerkin and fuzzy transform approaches, and compare their performance with the standard finite difference scheme in the case of sensitivity calculation (a so-called Greeks) of plain vanilla option price under Black and Scholes model conditions. The results show some interesting properties of the proposed methods.

Keywords*: Black and Scholes model, Numerical methods, Option valuation, The Greeks.*

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1. Introduction

Valuation of options, a specific nonlinear type of financial derivatives playing an important role in economics, has been a challenging issue of financial engineering and optimization for a long time. The standard ways to option valuation, as well as replication and hedging, date back to the 70's with the seminal papers of Black and Scholes [1] and Merton [19], Cox et al. [5] and Boyle [2]. While Black and Scholes [1] and Merton [19] derived their respective models within continuous time by solving partial differential equations (and thereafter called Black–Scholes–Merton partial differential equations) for riskless portfolio consisting of option itself and its underlying asset, Cox et al. [5] provided an approximate solution in a two-stage discrete time setting via recursive backward procedure. Alternatively, Boyle [2] suggested that in order to obtain the (discounted) expectation of the option payoff function the Monte Carlo simulation technique can be useful, i.e., instead of riskless portfolio construction and utilization of no-arbitrage principles the risk neutral behavior of all agents

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is assumed. It is a well-known result of quantitative finance (see, e.g., Duffie [6]) that these approaches are equivalent under the assumption of complete markets, or, at least, when an equivalent martingale measure exists. Although the aforementioned approaches slightly differ in details, for example, only the model of Cox et al. [5] can be used for valuation of American options, they all should lead to identical prices under equivalent circumstances.

Notwithstanding, the increasing complexity of market conditions requires utilization of advanced models that, commonly, do not lead to closed-form solutions. Thus, in real applications we often need to work with advanced processes or complex payoff functions and simple valuation procedures are no longer efficient and development of novel methods or numerical procedures, which prove to be efficient within various option valuation problems, is therefore worthwhile. Notwithstanding, such novel approaches should be tested as well, the most natural way being to assume simple plain vanilla options under the Black and Scholes model first; because of its simplicity the analytical solution is available and the convergence of novel numerical approaches can be analyzed easily.

Here, we focus on two novel numerical techniques (discontinuous Galerkin method and fuzzy transform approach) and propose their usage in numerical solution of complex option valuation problems. Obviously, in order to evaluate their performance, only a simple problem based on numerical approximation of the Black–Scholes–Merton partial differential equations accompanied by boundary and terminal conditions of plain vanilla options is assumed.

After reviewing some basic concepts about option pricing in Section 2, we present the methodological concepts of two relatively modern numerical techniques (Section 3), i.e., discontinuous Galerkin and fuzzy transform approaches, and compare their performance with the standard finite difference scheme in the case of sensitivity calculation (a so-called Greeks) of plain vanilla option price under Black and Scholes model conditions (Section 4). The results show some interesting properties of the proposed methods. We conclude in Section 5.

2. Option Valuation and Sensitivity Calculation

Options are nonlinear types of financial derivatives, which give the holder the right (but not the obligation) to buy the underlying asset in future (at maturity time) at a prespecified exercise price. Simultaneously, the writer of the option has to deliver the underlying asset if the holder asks. Therefore, the valuation is quite challenging.

The standard market model proposed independently by Black and Scholes [1] and Merton [19] is valid, in its basic form, only under idealized market conditions, including perfect market with the underlying asset price following log-normal distribution without any dividends and its returns having constant volatility, i.e., assuming fixed maturity T, the underlying asset price $S = S(t)$ follows geometric Brownian motion

$$\mathrm{d}S(t) = \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}Z(t), \tag{1}$$

where $Z = Z(t)$ is the standard Brownian motion (i.e., the Wiener process), μ is constant drift (long term return) and σ represents the volatility (standard deviation) of the underlying asset price returns.

Since the option value function $V(S,t)$ depends solely on time and the underlying asset price, the increments $\mathrm{d}V(S,t)$ can be specified as follows:

$$\mathrm{d}V(S,t) = \left(\frac{\partial V(S,t)}{\partial t} + \mu S \frac{\partial V(S,t)}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2}\right)\mathrm{d}t + \sigma S \frac{\partial V(S,t)}{\partial S}\mathrm{d}Z. \tag{2}$$

Clearly, the sensitivity of the option price V to the change of the underlying asset price S consists of the time increment $\mathrm{d}t$ and the standard normal variable change $\mathrm{d}Z$.

Since both V and S depend on the same source of uncertainty $\mathrm{d}Z$, it might be feasible to construct a riskless portfolio of option V and $-\Delta = \frac{\partial V(S,t)}{\partial S}$ shares of the underlying asset S. From that, we can derive a Black–Scholes–Merton partial differential equation (BSM PDE) for pricing the European option contracts on a single asset:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{3}$$

for $S > 0$ and $t > 0$.

Such an equation is valid for any option under given constraints. However, a particular solution depends on exact features of the option, i.e., on the payoff function which determines the option value at maturity T. This terminal condition in the case of vanilla options has the form

$$V(S,T) = \begin{cases} \max(S - \mathcal{K}, 0), & \text{(call)} \\ \max(\mathcal{K} - S, 0), & \text{(put)} \end{cases} \tag{4}$$

where $\mathcal{K}$ denotes the strike price, i.e., the specified price at which an option contract can be exercised.

The pricing equation (3) equipped with one of the terminal conditions (4) constitutes the Cauchy problem in (S,t)-domain whose analytical solutions are given by the Black–Scholes formula as

$$V(S,t;\sigma,r,\mathcal{K},T) = \begin{cases} S\Phi(d_1) - \mathcal{K}e^{-r(T-t)}\Phi(d_2), & \text{for a call,} \\ \mathcal{K}e^{-r(T-t)}\Phi(-d_2) - S\Phi(-d_1), & \text{for a put,} \end{cases} \tag{5}$$

where

$$d_1 = \frac{\ln(S/\mathcal{K}) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = \frac{\ln(S/\mathcal{K}) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

and Φ stands for the cumulative distribution function of the standard normal distribution.

2.1. The Greeks

Recall that the option value (5) depends on several underlying parameters and it is obvious that any parameter change should consequently influence it. The sensitivity analysis and measurement show how significant these changes will be; since Greek letters are commonly used to denote such sensitivity measures, we often call them *the Greeks* of an option. The availability of analytical solutions in the closed form, such as those for the European call and put options as presented in (5), implies the ability of deriving corresponding closed form representations for the sensitivity measures as well; see, for example, Hull [16].

In the rest of the paper we focus only on the first-order Greeks that are represented by the first derivatives with respect to the underlying parameters. The simplest sensitivity measure is Delta given by:

$$\Delta_V = \frac{\partial V}{\partial S} = \eta \Phi(\eta d_1), \tag{6}$$

where η indicates the option type ($\eta = 1$ for call and $\eta = -1$ for put option, respectively). Since Delta measures the sensitivity of the theoretical option value with respect to the changes in the underlying asset's price, its level is particularly important in hedging portfolios consisting of options.

Furthermore, the rate of change of an option value due to the passage of time is measured by Theta, given by:

$$\Theta_V = \frac{\partial V}{\partial t} = -\eta r \mathcal{K} e^{-r(T-t)} \Phi(\eta d_2) - \frac{\sigma S \phi(d_1)}{2\sqrt{T-t}}, \tag{7}$$

where ϕ is a density function of the standard normal distribution. This sensitivity measure is also referred to as an option's time decay, because the option loses its value as actual time approaches maturity (i.e., $t \to T$), ceteris paribus.

In volatile markets the value of some option positions can be particularly sensitive to changes in the volatility of the underlying asset price returns. In such cases, the derivative of the option value with respect to such volatility, Vega, should be taken into account:

$$\mathcal{V}_V = \frac{\partial V}{\partial \sigma} = S\sqrt{T-t}\,\phi(d_1), \tag{8}$$

The last of the first-order Greeks considered here is Rho as given below:

$$\rho_V = \frac{\partial V}{\partial r} = \eta (T-t) \mathcal{K} e^{-r(T-t)} \Phi(\eta d_2), \tag{9}$$

which measures the sensitivity to the riskless interest rate. Since the value of an option is less sensitive to changes in the riskless interest rate than to the changes in other parameters, Rho is the least used of the first-order Greeks. For further research, we refer to, e.g., Hull [16].

2.2. Finite Difference Methods

We now present a standard numerical technique, the *finite difference method*. This technique is very closely related to the Black and Scholes model because it approximates relevant partial differential equations by finite differences. Thus, finite difference method (FDM), as one of the simplest approximations of partial differential equations, replaces partial differentials of BSM PDE given by (3) using suitable (finite) differences.

Rewrite BSM PDE, cf. (3) with riskless rate and option value on the right as follows:

$$\frac{\partial V(S,t)}{\partial t} + rS \frac{\partial V(S,t)}{\partial S} + \sigma^2 S^2 \frac{1}{2} \frac{\partial^2 V(S,t)}{\partial S^2} = rV. \tag{10}$$

Here, we can see three terms with partial derivatives, showing subsequently the first order sensitivities of the option price to time increments ($\mathrm{d}t$) and the underlying asset price increments ($\mathrm{d}S$), respectively, and the second order sensitivity to the underlying asset price increments. These partial derivatives can be approximated by discrete increments since first order derivative of any function $f(x)$ can be depicted using its first central difference approximation (CDA):

$$f'(x) = \lim_{h\to 0} \frac{f(x+h/2) - f(x-h/2)}{h}. \quad (11)$$

Besides the above mentioned CDA, we can obviously use forward difference approximation (FDA) as well as backward difference approximation (BDA).

As an example, we will show an implicit approach to the definition of FDM according to which we replace $\mathrm{d}S$ in $\frac{\partial V(S,t)}{\partial S}$ by its discrete version ΔS. For this purpose, consider the central difference approximation:

$$\frac{\partial V(S,t)}{\partial S} = \frac{V(S+\Delta S,t) - V(S-\Delta S,t)}{2\Delta S}, \quad (12)$$

which is generally the most precise among the three available cases (i.e., forward, backward, and central approximations).

We can proceed in a similar way in the case of $\partial\, V(S,t)/\partial t$, but since t shows, by definition, positive increments only, FDA should be preferred:

$$\frac{\partial V(S,t)}{\partial t} = \frac{V(S,t+\Delta \mathrm{t}) - V(S,t)}{\Delta \mathrm{t}}, \quad (13)$$

The second order partial derivative $\partial^2 V(S,t)/\partial S^2$ can be expressed as a~difference between forward and backward approximations with respect to ΔS, and thus we have:

$$\frac{\partial^2 V(S,t)}{\partial S^2} = \left(\frac{V(S+\Delta \mathrm{S},t) - V(S,t)}{\Delta \mathrm{S}} - \frac{V(S,t) - V(S-\Delta \mathrm{S},t)}{\Delta \mathrm{S}}\right) / \,\Delta S$$
$$= \frac{V(S+\Delta \mathrm{S},t) - 2V(S,t) + V(S-\Delta \mathrm{S},t)}{(\Delta S)^2}, \quad (14)$$

Now, after selecting suitable finite differences to replace particular partial derivatives in BSM PDE of (3), we also replace $V(S,t)$ by the $f_{i,j}$, which states for option value at time i and (price) state j that (note that we switch the positions of state and time to make it):

$$\frac{f_{i+1,j} - f_{i,j}}{\Delta t} + rj\Delta S \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta S} - \sigma^2 j^2 \Delta S^2 \frac{f_{i,j+1} + f_{i,j-1} - 2f_{i,j}}{2\Delta S^2} = rf_{i,j}, \quad (15)$$

for $i = 0, \dots, M$ and $j = 1, \dots, N-1$. This way, we can depict the complete structure of option prices for all the points in time, including the initial time $i = 0$ ($t = 0$) and maturity time $i = M$ ($t = T$), and selected underlying asset prices S.

Subsequently, we can organize particular option values as follows:

$$a_j f_{i,j-1} + b_j f_{i,j} + c_j f_{i,j+1} = f_{i+1,j}. \tag{16}$$

This can be read as that the weighted average of option values in a given time moment i for three states of the underlying asset price $(j-1, j, j+1)$ is equal to option value in the next time moment $(i+1)$, given the central state j. The weights are specified by coefficients a_j, b_j, and c_j, each being dependent on final state j, so that

$$a_j = \frac{1}{2}\Delta t(rj - \sigma^2 j^2), \qquad b_j = 1 + \sigma^2 j^2 \Delta t + r\Delta t, \qquad c_j = -\frac{1}{2}\Delta t(rj + \sigma^2 j^2).$$

3. Alternative Numerical Approaches

In this section we focus on two relatively novel numerical techniques based on the discontinuous Galerkin method (DGM) and the fuzzy transform technique (FT), both falling into the class of variational methods. These methods significantly extend the standard numerical tools used in option valuation. These new approaches represent a very powerful tool for the numerical simulation of option valuation, since they allow for better capturing of some features of different options under various market conditions with respect to the discretization of the computational domain as well as the order of the polynomial approximation.

3.1. Discontinuous Galerkin Approach

The DGM combines the ideas and techniques of the finite volume method (FVM) and the finite element method (FEM) to take advantage of their strengths while eliminating their shortcomings. The FEM is a high-order method primarily designed for problems in which the exact solution is sufficiently regular and no steep derivatives or discontinuities in the data or solutions are present. The starting point is a variational formulation of the solved PDE and a concept of a weak solution as an element of the suitable infinite-dimensional function space (usually called the space of trial functions). Then, we can compute a discrete solution using the Ritz–Galerkin method as soon as a finite-dimensional subspace of the space of trial functions is specified. There are various ways to define these spaces. However, they are typically constructed as spaces of continuous piecewise polynomial functions with respect to the decomposition of the computational domain into finite elements. The basis of such a space is finite and is formed by the basis functions that generate this space. Therefore, the FEM in its simplest form can be regarded as a special way of constructing these spaces, which are called finite element spaces; see Ciarlet [4].

On the other hand, the FVM based on discontinuous, piecewise constant approximations allows us to capture discontinuities in the solution but has a low order of accuracy. The FVM was originally developed for the discretization of conservation laws. Similar to the FDM, the values are calculated at discrete places in a meshed geometry. The essential idea is to divide the domain into many discretization cells, called finite volumes, and approximate the integral conservation law for each of these volumes. More precisely, the volume integrals in the solved PDE that contain a divergence term are converted into surface ones using the divergence theorem. Then, these terms are evaluated using the numerical fluxes that are conserved from one finite volume to its neighbor; that is, the flux entering a given volume is identical to that leaving the adjacent one. This feature is called local conservativity. To construct the discrete solution, we assume that the solution in each finite volume is constant; thus, the finite volume approach produces the piecewise constant approximations corresponding to the discrete unknowns (see Eymard et al. [8]). Taking all of the above into account, the DGM can be viewed as a generalization of the finite volume techniques into higher-order schemes or as an imaginary bridge between the finite element and the finite volume.

The DGM provides the numerical solution of the PDEs composed of piecewise polynomial functions on a finite element mesh without any requirement for the continuity of the solution across particular elements. Therefore, this approach is suitable for problems on which other techniques fail or have difficulties. Although the DGM was developed in the early 1970s (see Reed and Hill [25]), its potency in option valuation problems has not been fully exploited yet. From this point of view, this method is perceived to be a very promising numerical tool.

In what follows, we introduce the *discrete problem* to option valuation within the DG framework. The proposed pricing methodology related to numerical solving of the BSE PDE requires truncation of the spatial domain to a bounded interval $\Omega = (0, S_{\max})$, where $S_{\max} > 0$ stands for the sufficiently large asset price. Therefore, we need to impose the option values at both endpoints of the domain Ω. These values are set in accordance with the theoretical European option prices (5) as $S \to 0+$ and $S \to +\infty$, i.e.,

$$V(0,t) = \begin{cases} 0, \\ \mathcal{K}e^{-r(T-t)}, \end{cases} \qquad V(S_{\max},t) = \begin{cases} S_{\max} - \mathcal{K}e^{-r(T-t)}, & \text{(call)} \\ 0, & \text{(put)} \end{cases} \tag{17}$$

Let $0 = t_0 < t_1 < \cdots < t_M = T$ be a partition of the interval $[0,T]$ with the constant time step $\tau = T/M$ (for simplicity) and denote $U_h^m \in S_h^p$ to be the approximation of the solution $V(\cdot, t_m)$. The set S_h^p is the finite-dimensional space of piecewise polynomial functions of order p, constructed over the partition of Ω with mesh size h.

The discrete solution is computed by the θ-scheme that reads: Find $U_h^{m+1} \in S_h^p$, $m = 0, \ldots, M-1$, such that the following conditions are satisfied:

$$\begin{aligned} (U_h^m, v_h) + \theta\tau\mathcal{A}_h(U_h^m, v_h) = \left(U_h^{m+1}, v_h\right) - (1-\theta)\tau\mathcal{A}_h\left(U_h^{m+1}, v_h\right) \\ -\theta\tau\ell_h(v_h)(t_m) - (1-\theta)\tau\ell_h(v_h)(t_{m+1}), \ \ \forall v_h \in S_h^p, \end{aligned} \tag{18}$$

$$\left(U_h^M, v_h\right) = (V(\cdot,T), v_h), \qquad \forall\, v_h \in S_h^p, \tag{19}$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$, the bilinear form $\mathcal{A}_h(\cdot,\cdot)$ stands for the DG semi-discrete variant of the degenerate parabolic partial differential operator from (3), accompanied with penalties and stabilizations, and the form $\ell_h(\cdot)(t)$ balances the Dirichlet boundary conditions (17); for more details, see Hozman et al. [15].

Note that the value of the parameter θ lies in interval $[0,1]$ and the equation (18) results into a sequence of linear algebraic problems. at each time level. The existence and uniqueness of the discrete solution are guaranteed under the ellipticity of the form $(\cdot,\cdot) + \theta\tau\mathcal{A}_h(\cdot,\cdot)$ on the left-hand side of (18) (cf. Hozman and Tichý [12]). The starting data (19) in the recursive formulation (18) are defined as L^2-projection of payoff function (4) onto the space S_h^p.

The properties of the θ-scheme depend on the value θ. The stability property holds for $\frac{1}{2} \leq \theta \leq 1$ and for $0 \leq \theta < \frac{1}{2}$ we have a stability bound for the step size τ. The case $\theta = \frac{1}{2}$ is well known as the

Crank--Nicolson method, which is practically unconditionally stable and gives the second order convergence in time. In other cases, $\theta \neq \frac{1}{2}$, we obtain the first order schemes only; see Trefethen [29]. Therefore, we consider the Crank--Nicolson method in our empirical study.

The discrete problem (18) is equivalent to a system of linear algebraic equations at each time level and can always be expressed in a matrix form; see Hozman and Tichý [13]. Let $B = \{w_i\}_{i=1}^{\mathrm{DOF}}$ denote the basis of the space S_h^p with degrees of freedom (DOF). Then, the discrete solution at each time level $t_m \in [0, T]$ can be written as a linear combination of basis functions in the form

$$U_h^m(S) = \sum_{i=1}^{\mathrm{DOF}} \xi_i^m w_i(S), \qquad \xi_i^m \in \mathbb{R}, \;\; S \in \overline{\Omega}. \tag{20}$$

More precisely, this discrete solution U_h^m is identified with the coefficient vector $\mathbf{u}^m = \{\xi_i^m\}_{i=1}^{\mathrm{DOF}} \in \mathbb{R}^{\mathrm{DOF}}$ with respect to the basis B. Then, (18) reads as

$$(\mathbf{M} + \theta\tau\mathbf{A})\mathbf{u}^m = (\mathbf{M} - (1-\theta)\tau\mathbf{A})\mathbf{u}^{m+1} - \tau(\theta\mathbf{f}^m + (1-\theta)\mathbf{f}^{m+1}), \tag{21}$$

Where the terminal vector $\mathbf{u}^M$ is given by U_h^M arising from (19).

The system matrix in (21) is a composition of the mass matrix **M** and the matrix **A** arising from the bilinear form $\mathcal{A}_h$, defined as

$$\mathbf{M} = \left\{(w_j, w_i)\right\}_{i,j=1}^{\mathrm{DOF}}, \qquad \mathbf{A} = \left\{\mathcal{A}_h(w_j, w_i)\right\}_{i,j=1}^{\mathrm{DOF}}. \tag{22}$$

The right-hand side of (21) contains a weighted average of the following two vectors

$$\mathbf{f}^m = \{\ell_h(w_i)(t_m)\}_{i=1}^{\mathrm{DOF}}, \qquad \mathbf{f}^{m+1} = \{\ell_h(w_i)(t_{m+1})\}_{i=1}^{\mathrm{DOF}}. \tag{23}$$

Approximate evaluation of Greeks. In order to illustrate the robustness of the presented approach, the numerical scheme (21) is used not only for the evaluation of option prices but also for their sensitivity measures. Considering the polynomial approximation at least of the first order (linear), Delta can be directly computed from the derivatives of the basis functions $\{w_i'\}_{i=1}^{\mathrm{DOF}}$, using the relation (20) as

$$\Delta_V(t_m) \approx \frac{\partial U_h^m}{\partial S} = \sum_{i=1}^{\mathrm{DOF}} \xi_i^m w_i'(S). \tag{24}$$

On the other hand, the remaining Greeks are numerically computed using the central finite difference

$$\Theta_V(t_m) \approx \frac{U_h^{m+1} - U_h^{m-1}}{2\tau}, \tag{25}$$

$$\mathcal{V}_V(t_m, \sigma) \approx \frac{U_h^m(\sigma+\delta) - U_h^m(\sigma-\delta)}{2\delta}, \tag{26}$$

$$\rho_V(t_m, r) \approx \frac{U_h^m(r+\delta) - U_h^m(r-\delta)}{2\delta}, \tag{27}$$

where $0 < \delta \ll 1$ and τ are sufficiently small values. This approach involves solving the option valuation problem twice to obtain the solution in two different stages, $t \pm \tau$, $\sigma \pm \delta$ and $r \pm \delta$, respectively. Note that this approach provides only the pointwise approximation with respect to the underlying parameters t, σ, and r.

3.2. F-transform Technique

The second numerical method considered here is based on the fuzzy transform (F-transform, for short) technique. The F-transform technique was introduced by Perfilieva [20] (see also Perfilieva [21]) to approximate real valued functions usually from the L^2 space and has two phases: direct and inverse. The direct F-transform transforms a continuous (or integrable) function defined in a bounded interval into a finite vector of real numbers, which are called the components of the F-transform. The inverse F-transform returns the vector of the F-transform components to a continuous function that approximates the original function. The key parameter of the F-transform is a fuzzy partition of the domain of the considered functions by means of fuzzy sets that form the basis function. Setting fuzzy partitions affects the quality of the approximation of functions using the F-transform.

The first application of the F-transform in the numerical solution of ordinary differential equations, in particular the Cauchy problem, was described by Perfilieva [22] (see also Perfilieva [20]) and partial differential equations of special types for multivariable functions by Štěpnička and Valášek [27, 28]. In Chen and Schen [3], a novel algorithm based on the F-transform has been proposed to obtain an approximate solution for a class of second-order ordinary differential equations with classical initial conditions. In Khastan et al. [17], new numerical methods based on the higher degree F-transform for solving the Cauchy problem have been presented. A further development of the F-transform based numerical method introduced by Perfilieva [22] for solving a boundary value problem (BVP) for a second-order ODE with Dirichlet boundary conditions can be found in Perfilieva et al. [23]. The proposed methods outperformed the second order Runge-Kutta method. An extension of the shooting method for nonlinear boundary value problems with the help of F-transform was proposed in [24]. A generalization of the Štěpnička and Valášek approach to the numerical solution of partial differential equations was then proposed by Holčapek and Valášek [9, 10].

The principal of the numerical solution of ordinary or partial differential equations lies in the substitution of the respective F-transform components for all the functions and their (partial) derivatives in the differential equation. The F-transform components of the derivatives of functions are then expressed by the method of finite differences (cf. Duffy [7]). The result of the substitution of the F-transform components and the expression of derivatives is a system of linear algebraic equations with unknown F-transform components of a function, which is a solution of the differential equation. The approximate solution of the differential equation is obtained by the inverse F-transform. The contribution of the F-transform to the numerical solution of differential equations consists mainly of the reduction of the number of linear algebraic equations, the solution of which becomes very complicated for an increasing dimension of function spaces.

Approximate evaluation of Greeks. In contrast to the computation of the sensitivity measures (Greeks) in the case of the DGM, here we compute also the Delta values by finite differences. The reason is that the derivatives of basic functions of a fuzzy partition (if they exist) at all nodes are equal to zero, which makes the approximation of Delta extremely imprecise (biased). Precisely, as specified in [15], we can consider the following approximation formulas:

$$\Delta_V(S,t_m) \approx \frac{U_1^m - U_0^m}{h}\varphi_0(S) + \sum_{k=1}^{\mathrm{DOF}-1} \frac{U_{k+1}^m - U_{k-1}^m}{2h}\varphi_k(S) + \frac{U_{\mathrm{DOF}}^m - U_{\mathrm{DOF}-1}^m}{h}\varphi_{\mathrm{DOF}}(S), \qquad (28)$$

where φ is the uniform generating function of the fuzzy partition, h is the bandwidth, U_k is the k-th component of the direct F-transform with respect to the restricted fuzzy partition to Ω, i.e., the vector $\mathbf{U} = \{\, U_k \mid k = 0, \dots, \mathrm{DOF}\,\}$ of real numbers given by

$$U_k = \frac{\int_\Omega u(x)\varphi_k(x)\mathrm{d}x}{\int_\Omega \varphi_k(x)\mathrm{d}x}, \qquad k = 0, \dots, \mathrm{DOF}. \qquad (29)$$

It is easy to see that the first order forward and backward differences in (28) are used to approximate the derivatives at the boundaries and the central differences to improve the accuracy of the approximation. The remaining Greeks, namely, Theta, Vega and Rho, are numerically computed by the formulas (25), (26) and (27), respectively.

4. A Comparative Example

The experimental analysis provided in this section shows extensive comparison of the FDM, DG and FT methods in connection with the plain vanilla option price sensitivity measurement of the first order within the Black and Scholes setting. In particular, we consider data modified from Kopa et al. [18], in line with Hozman and Tichý [11] and Hozman et al. [14], who have already analyzed the valuation problem of vanilla put option at the same market data (DAX options) using the DG approach. We consider here only intermediate (maturity of 193 days) close to the ATM options with the current underlying (German stock market index) value being $S_{ref} = 4715.879$. The fixed parameters of the model are the riskless interest rate r (0.039) and the volatility σ (0.4422), which is derived from true option prices observed at the market (implied volatility approach).

Numerical approximation is crucially related to the discretization of the computational domain Ω, its length being deliberately chosen as eight times the strike price to suppress the influence of the inaccurate Dirichlet boundary condition (17). Together with this, we choose the time step $\tau = 1/3600$ so that the effect of time discretization on numerical results is negligible.

For a more detailed comparison, each of the methods is considered in the form of linear as well as nonlinear (quadratic or cosine) approximations. The quality of the approximation can be easily observed by comparing the numerical results with the theoretical prices according to the Black and Scholes model. Therefore, at $t = 0$, we compute the relative error e_{L^2} measured in the L^2-norm over the whole computational domain and pointwise relative error e_{pw} evaluated in the reference point S_{ref}, i.e.,

$$e_{L^2} = \frac{\left\| U_h^0(S) - V(S,0) \right\|}{\| V(S,0) \|}, \qquad e_{pw} = \frac{\left| U_h^0(S_{ref}) - V(S_{ref},0) \right|}{\left| V(S_{ref},0) \right|} \qquad (30)$$

where U_h^0 denotes the approximate solution obtained by one of the two numerical approaches and $V(S,t)$ is the analytical solution given by the BS formula (5), respectively. The formulas (30) can be subsequently extended to the calculation of the relative errors e_{L^2} and e_{pw} for selected sensitivity measures using (6) – (9).

According to the theoretical results from Rivière [26], it is known that the DG technique produces (in general) optimal convergence of spatial derivatives of approximate solutions, all measured in the L^2-norm for sufficiently regular problems. In case of the remaining first order derivatives (i.e., except for the one with respect to the underlying asset price), similar convergence results in the L^2-norm could be expected. On the other hand, the general results of pointwise error estimates are not available.

The calculations are performed on a sequence of consecutive uniformly refined meshes with linear as well as nonlinear basis functions, for the particular scenario of vanilla options with $\mathcal{K} = 4700$ and $T = 193/360$ days.

We use linear approximations and analyze the first-order Greeks: Delta, Theta, Vega and Rho. These sensitivity measures are computed by (24) using the central finite differences (25) – (27) with steps $\tau = 1/3600$ and $\delta = 0.002$. Corresponding relative errors are apparent from Figure 1.

It is easy to see that relative errors in the L^2-norm decrease with the mesh refinement, i.e., proportionally to DOF. More precisely, the order of convergence for the first-order Greeks corresponds at least to the order of a polynomial approximation, though the pointwise errors exhibit oscillatory behavior in some cases. One can easily conclude that all considered methods are quite comparable with respect to both relative errors and it is not easy to identify the best method.

Concerning the nonlinear basis functions, the relative errors exhibit similar behavior as the linear basis functions. More precisely, in line with the theory, the convergence of Delta values in the L^2-norm is optimal for DGM, i.e., quadratic for the approximation with quadratic basis functions. Compared to this behavior, the raised cosine basis functions in FT also exhibit quadratic trend in the L^2-errors for the Delta values. As expected, the results for the remaining first-order Greeks are of the same quality and their orders of accuracy are consistent with the Delta ones. On the other hand, from the results for pointwise errors, nothing more rigorous can be inferred. Thus, for now, one should be satisfied with the claim that errors have more or less downward trends.

5. Conclusions

We proposed two novel numerical approaches newly applied for option valuation problem with special attention being paid to the evaluation of the sensitivity parameters, the so-called Greek letters. The first technique is derived from the discontinuous Galerkin method, which is based on discontinuous piecewise polynomial approximations. The second technique is based on the F-transform, the application of which to the original continuous problem leads to a new one for the unknown components of this transform. The resulting problem is then discretized using the finite difference method. In the case of linear approximation, the results are very similar amongst all the methods, but, for nonlinear basis functions, the differences in these approaches appear to be significant, especially due to the different types of basis functions (parabola vs. raised cosine). On the other hand, the benefits of the F-transform could mainly be reflected in the possibility of reducing the number of degrees of freedom in the discretization under the preserved order of accuracy, which actually contribute to the decrease of the computational time. However, this advantage of the F-transform observed mainly in solving the BS equation containing several underlying factors, in which the complexity of the calculation grows exponentially. Regarding the discontinuous Galerkin method, its main advantages lie in the possibility of an easy usage of the discontinuous payoff functions and discrete sampling.

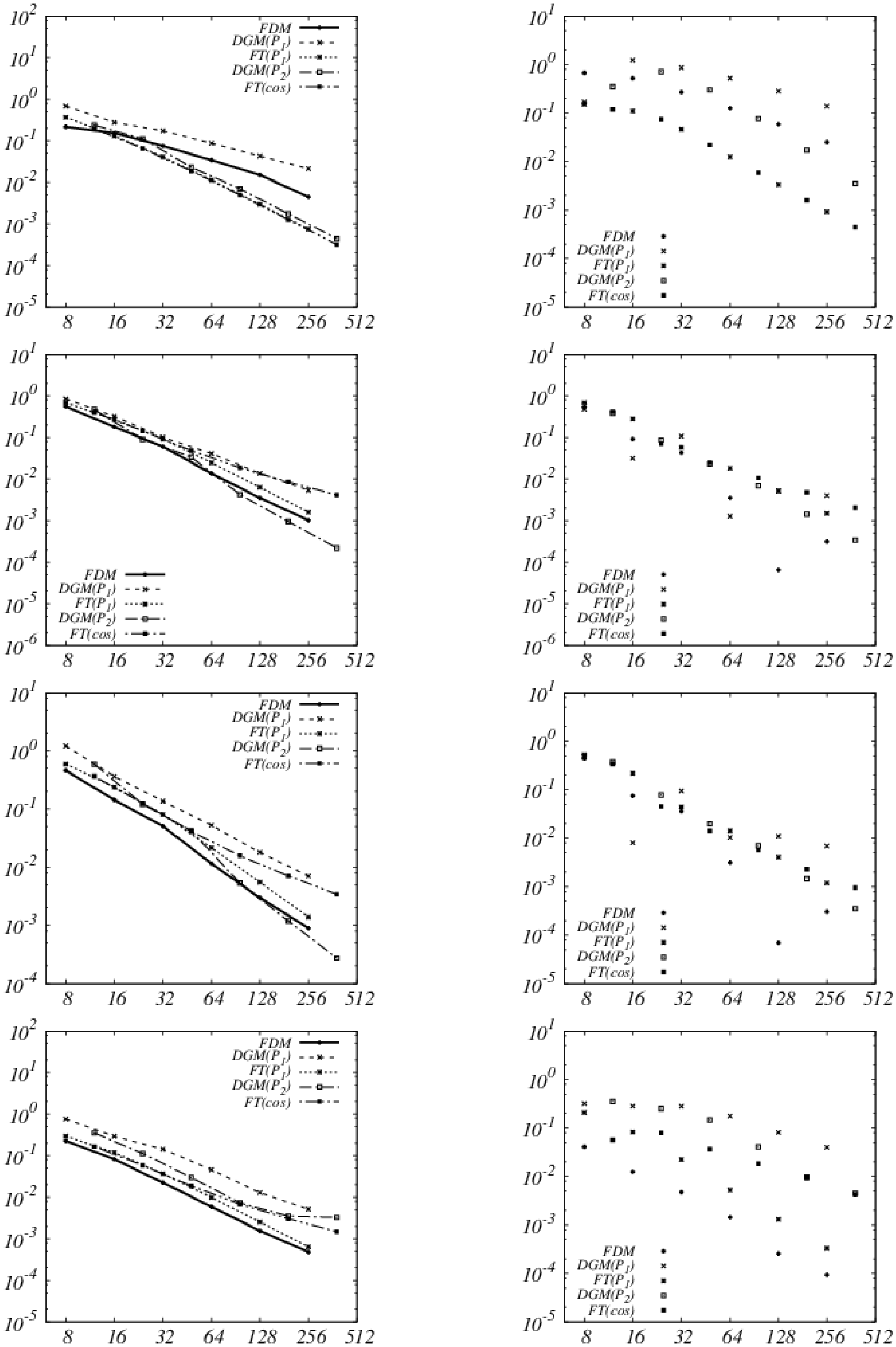

Figure 1. Comparison of relative L^2-errors (*left*) and pointwise errors (*right*) of the Delta, Theta, Vega, and Rho values for particular methods. The horizontal axis represents the degrees of freedom.

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