

Using Nesterov's Excessive Gap Method as Basic Procedure in Chubanov's Method for Solving a Homogeneous Feasibility Problem

Z. Wei¹, C. Roos^{2,*}

We deal with a recently proposed method of Chubanov [1], for solving linear homogeneous systems with positive variables. We use Nesterov's excessive gap method in the basic procedure. As a result, the iteration bound for the basic procedure is reduced by the factor $n\sqrt{n}$. The price for this improvement is that the iterations are more costly, namely $O(n^2)$ instead of $O(n)$. The overall gain in the complexity hence becomes a factor of \sqrt{n} .

Keywords: Linear homogeneous systems, Algorithm, Polynomial-time.

Manuscript was received on 01/07/2018 revised on 12/09/2018 and accepted for publication on 20/10/2018.

1. Introduction

We deal with the problem

$$\begin{aligned} &\text{find } x \in \mathbb{R}^n \\ &\text{subject to } Ax = 0, \quad x > 0, \end{aligned} \quad (1)$$

where A is an integer (or rational) matrix of size $m \times n$ and $\text{rank}(A) = m$.

Recently Chubanov [1] proposed a polynomial-time algorithm for solving this problem. He explored the fact that (1) is homogeneous as follows. If x is feasible for (1), then also $x' = x/\max(x)$ is feasible for (1), and this solution belongs to the unit cube, i.e., $x' \in [0,1]^n$. It follows that (1) is feasible if and only if the system

$$Ax = 0, \quad x \in (0,1]^n \quad (2)$$

is feasible. Moreover, if $d > 0$ is a vector such that $x \leq d$ holds for every feasible solution of (2), then $x'' = x/d \in (0,1]^n$, where x/d denotes the entry-wise quotient of x and d , and so $x''_i = x_i/d_i$ for each i . This means that x'' is feasible for the system

$$ADx = 0, \quad x \in (0,1]^n, \quad (3)$$

* Corresponding Author.

1 National University of Singapore, Singapore, Email: lindelfeel@gmail.com.

2 Department of Electrical Engineering, Computer Science and Mathematics, Delft University of Technology, Delft, The Netherlands, Email: c.roos@tudelft.nl.

where $D = \text{diag}(d)$. Obviously, problem (3) is of the same type as problem (2), since it arises from (2) by rescaling A to AD . Moreover, if x solves (3), then Dx solves (2). The main algorithm starts with $d = \mathbf{1}$, with $\mathbf{1}$ denoting the all-one vector, and successively improves d .

A key ingredient in Chubanov's algorithm is the so-called Basic Procedure (BP). The BP generates one of the following three outputs:

- case 1: a feasible solution of (1);
- case 2: a certificate for the infeasibility of (1);
- case 0: a cut for the feasible region of (2).

In case 0, the cut has the form $x_k \leq \frac{1}{2}$ for some index k and is used to update d by dividing d_k by 2. The rescaling happens in the main algorithm, which sends the rescaled matrix AD to the BP until the BP returns case 1 or case 2.

Since A has integer (or rational) entries, the number of calls of the BP is polynomially bounded by $O(nL)$, where L denotes the bit size A . This follows from a classical result of Khachiyan [2] that gives a positive lower bound on the positive entries of a solution of a linear system of equations.

The BP of [1] needs at most $O(n^3)$ iterations per call and $O(n)$ time per iteration. So, per call the BP needs $O(n^4)$ time and hence the overall time complexity becomes $O(n^5L)$. By performing a more careful analysis, Chubanov reduced this bound by a factor n to $O(n^4L)$ [1, Theorem 2.1].

Other BPs have been proposed in, e.g., [4, 5]. These BPs also need $O(n^3)$ iterations per call and $O(n)$ time per iteration, and so they also yield an overall time complexity of $O(n^5L)$.

In [6], we proposed a BP based on the Mirror-Prox method of Nemirovski. It improves the iteration bound per call with a factor $n\sqrt{n}$ and leads to an overall time complexity of $O(n^{4.5}L)$, because it requires $O(n^2)$ time per iteration.

Here, we analyze a BP based on the Excessive Gap technique of Nesterov [3]. The outline of the remainder of the paper is as follows. We present some preliminary results in Section 2. In Section 3 we describe the new BP and prove the iteration bound of $O(n\sqrt{n})$. Since the time complexity per iteration is $O(n^2)$, the overall time complexity is the same as the one given in [6].

2. Preliminaries

Let \mathcal{N}_A denote the null space of the $m \times n$ matrix A and \mathcal{R}_A denote its row space, that is,

$$\mathcal{N}_A := \{x \in \mathbb{R}^n : Ax = 0\}, \quad \mathcal{R}_A := \{A^T u : u \in \mathbb{R}^m\}.$$

We denote the orthogonal projections of \mathbb{R}^n onto \mathcal{N}_A and \mathcal{R}_A as P_A and Q_A , respectively:

$$P_A := I - A^T(AA^T)^{-1}A, \quad Q_A := A^T(AA^T)^{-1}A.$$

Our assumption $\text{rank}(A) = m$ implies that the inverse of AA^T exists. Obviously, we have

$$I = P_A + Q_A, \quad P_A Q_A = 0, \quad A P_A = 0, \quad A Q_A = A.$$

Now, let $y \in \mathbb{R}^n$. In the sequel, we use the notation

$$z = P_A y, \quad v = Q_A y.$$

So, z and v are the orthogonal components of y in the spaces \mathcal{N}_A and \mathcal{R}_A , respectively:

$$y = z + v, \quad z \in \mathcal{N}_A, \quad v \in \mathcal{R}_A.$$

These vectors play a crucial role in our approach. This is due to the following lemma.

Lemma 2.1. (Lemma 2.1 of [5]) If $z > 0$, then z solves the *primal* problem (1) and if $0 \neq v \geq 0$, then v provides a certificate for the infeasibility of (1).

As usual, we always assume $y \in \Delta$, where Δ denotes the unit simplex in \mathbb{R}^n . So,

$$\Delta = \{u : \mathbf{1}^T u = 1, u \geq 0\}.$$

In the literature we nowadays have several ways to derive from y, z and v an upper bound for the k -th coordinate of every x that is feasible for (3). For example,

$$x_k \leq \begin{cases} \frac{\sqrt{n}\|z\|}{y_k}, & \text{in [1].} \\ \frac{\mathbf{1}^T z^+}{y_k}, & \text{in [4, 5].} \\ \mathbf{1}^T \left(\frac{v}{-v_k} \right)^+, & \text{in [5, 6].} \end{cases}$$

Here, we are only interested in the so-called proper cuts, where the upper bound is smaller than or equal to $\frac{1}{2}$. If $2n\sqrt{n}\|z\| \leq 1$, then the first two cuts are proper for at least one k . This follows for the first bound simply by taking k such that $y_k \geq 1/n$, and for the second bound by also using $\mathbf{1}^T z^+ \leq \sqrt{n}\|z^+\| \leq \sqrt{n}\|z\|$. For the third bound, it seems far from trivial that we have the same property; for a proof we refer to the Appendix in [6]³.

It may also be mentioned that the third cut is always at least as tight as the other two cuts; this is shown in [5]. In the rest of the paper, we use this cut, denoting the upper bound as $\sigma_k(y)$ and defining

$$\sigma(y) = \min_k \sigma_k(y).$$

Next, the BP based on Nesterov's Excessive Gap method is described as in Algorithm 1.

³ There is also a ‘proof’ of this statement in [5], but unfortunately there is a gap in that proof that has been overlooked.

Algorithm 1: $[y, v, z, J, \text{case}] = \text{EXCESSIVE GAP BP}(P_A)$

```

1: INITIALIZE:  $k = 0$ ;  $\bar{u} = \frac{1}{n}\mathbf{1}$ ;  $\text{case} = 0$ ;  $J = \emptyset$ ;  $u_0 = \bar{u}$ ;  $\mu_0 = 2$ ;
    $y_0 = y_{\mu_0}(u_0)$ 
2: while  $\sigma(y_k) > \frac{1}{2}$  and  $\sigma(u_k) > \frac{1}{2}$  and  $\text{case} = 0$  do
3:    $z_k = P_A y_k$ 
4:   if  $z_k > 0$  then
5:     case = 1 ( $z_k$  is primal feasible); return
6:   else
7:      $v_k = y_k - z_k$ 
8:     if  $v_k \geq 0$  then
9:       case = 2 ( $u_k$  is dual feasible); return
10:    else
11:       $\theta_k = \frac{2}{k+3}$ 
12:       $u_{k+1} = (1 - \theta_k)u_k + \theta_k((1 - \theta)y_k + \theta_k y_{\mu_k}(y_k))$ 
13:       $\mu_{k+1} = (1 - \theta_k)\mu_k$ 
14:       $y_{k+1} = (1 - \theta_k)y_k + \theta_k y_{\mu_{k+1}}(u_{k+1})$ 
15:       $k = k + 1$ 
16:    end
17:  end
18: end
19: if  $\text{case} = 0$  then
20:   find a nonempty set  $J$  such that
    $J \subseteq \{j : \text{bound}_j(y_k) \leq \frac{1}{2}\} \cup \{j : \text{bound}_j(u_k) \leq \frac{1}{2}\}$ 
21: end

```

In Algorithm 1, k serves as the iteration counter. We also use the following notation:

$$y_\mu(v) := \operatorname{argmin}_{u \in \Delta} \left\{ u^T P_A v + \frac{\mu}{2} \|u - \bar{u}\|^2 \right\}, \quad v \in \Delta, \quad (4)$$

where $\bar{u} = \mathbf{1}/n$. Note that in each iteration two problems of this type need to be solved, in line 12 and line 14, respectively. In Section 4 we show that if $P_A v$ is known, then problem (4) can be solved in $O(n)$ time. But, we first show in the next section that the number of iterations of Algorithm 1 never exceeds $O(n\sqrt{n})$.

3. Iteration Bound

Recall that y yields a solution of problem (2) if $z = P_A y > 0$. If $u \in \Delta$ then $u^T z \geq \min z$, for each $z \in \mathbb{R}^n$ and $\min_{u \in \Delta} u^T z = \min z$. Hence, $P_A y > 0$ holds if and only if $\psi(y) > 0$, where

$$\psi(y) := \min_{u \in \Delta} u^T P_A y.$$

This certainly holds if y solves the problem

$$\max_{y \in \Delta} \psi(y) = \max_{y \in \Delta} \min_{u \in \Delta} u^T P_A y > 0.$$

In order to deal with this problem, we use an adapted version of the excessive gap technique of Nesterov [3] by considering a smoothed version of the above problem. For decreasing values of the parameter μ , we consider instead the problem of maximizing the function $\phi_\mu(y)$, where

$$\phi_\mu(y) = -\frac{1}{2} \|P_A y\|^2 + \min_{u \in \Delta} \left\{ u^T P_A y + \frac{\mu}{2} \|u - \bar{u}\|^2 \right\},$$

with $\bar{u} = \mathbf{1}/n$ and $\mu \geq 0$.

In this section, we show that the algorithm needs at most $O(n\sqrt{n})$ iterations to generate a vector $y \in \Delta$ such that $2n\sqrt{n}\|z\| \leq 1$. For the proof, we consider a run of the BP during which z has always a nonpositive entry and v a negative entry. So, the BP does not halt in line 5 or line 9. In that case, the algorithm stops after at most $2n\sqrt{2n}$ iterations, as we show below. We start with a relatively simple lemma.

Lemma 3.1. $0 \leq \phi_\mu(y) - \phi_0(y) \leq \mu$.

Proof. Let $u \in \Delta$. Then $\|u\|^2 = \sum_{i=1}^n u_i^2 \leq \sum_{i=1}^n u_i = 1$. Similarly, $\|\bar{u}\|^2 \leq 1$. Hence,

$$\frac{1}{2} \|u - \bar{u}\|^2 = \frac{1}{2} (\|u\|^2 + \|\bar{u}\|^2 - 2u^T \bar{u}) \leq 1,$$

where we also used $u \geq 0$ and $\bar{u} \geq 0$. Using this, we write

$$\phi_\mu(y) \leq -\frac{1}{2} \|P_A y\|^2 + \min_{u \in \Delta} u^T P_A y + \mu = \mu + \phi_0(y),$$

It remains to show that $\phi_\mu(y) \geq \phi_0(y)$. This follows since $\phi_\mu(y)$ is increasing in μ . Hence the proof is complete. \blacksquare

Lemma 3.2. $\frac{1}{2} \|P_A y_k\|^2 \leq \phi_{\mu_k}(u_k)$.

Proof. We start with the case where $k = 0$. Then, we have $u_0 = \bar{u} = \frac{1}{n}$, $\mu_0 = 2$, $y_0 = y_{\mu_0}(u_0)$. We simplify the notation by denoting P_A simply as P . Then, we may write

$$\begin{aligned} \frac{1}{2} \|P_A y_0\|^2 &= \frac{1}{2} \|P(y_0 - \bar{u}) + P(\bar{u})\|^2 \\ &= \frac{1}{2} \|P(y_0 - \bar{u})\|^2 + \frac{1}{2} \|P\bar{u}\|^2 + (P y_0)^T P \bar{u} - \|P\bar{u}\|^2 \\ &\leq -\frac{1}{2} \|P\bar{u}\|^2 + y_0^T P \bar{u} + \frac{1}{2} \|y_0 - \bar{u}\|^2 \\ &\leq -\frac{1}{2} \|P\bar{u}\|^2 + y_0^T P \bar{u} + \frac{\mu_0}{2} \|y_0 - \bar{u}\|^2 \\ &= -\frac{1}{2} \|P\bar{u}\|^2 + \min_{u \in \Delta} \left\{ u^T P u_0 + \frac{\mu_0}{2} \|u - \bar{u}\|^2 \right\} = \phi_{\mu_0}(u_0), \end{aligned}$$

where the last but one equality is due to the definition of y_0 . We proceed with induction on k . To simplify the notation further, we denote $y = y_k$, $\mu = \mu_k$, $u = u_k$, $y' = y_{k+1}$, $\mu' = \mu_{k+1}$, $u' = u_{k+1}$ and

$$\hat{y} = (1 - \theta)y + \theta y_\mu(y). \quad (5)$$

Then, we have

$$\begin{aligned} u' &= (1 - \theta)(u + \theta y) + \theta^2 y_\mu(y) \\ &= (1 - \theta)u + \theta[(1 - \theta)y + \theta y_\mu(y)] \\ &= (1 - \theta)u + \theta \hat{y}. \end{aligned}$$

Moreover,

$$\mu' = (1 - \theta)\mu,$$

and

$$y' = (1 - \theta)y + \theta y_{\mu'}(u'). \quad (6)$$

Under the assumption that $\frac{1}{2}\|Py\|^2 \leq \phi_\mu(u)$ we need to show that $\frac{1}{2}\|Py'\|^2 \leq \phi_{\mu'}(u')$.

We have

$$\begin{aligned} \phi_{\mu'}(u') &= -\frac{1}{2}\|Pu'\|^2 + \min_{u \in \Delta} \left\{ u^T Pu' + \frac{\mu'}{2} \|u - \bar{u}\|^2 \right\} \\ &= -\frac{1}{2}\|Pu'\|^2 + y_{\mu'}(u')^T Pu' + \frac{\mu'}{2} \|y_{\mu'}(u') - \bar{u}\|^2. \end{aligned}$$

Due to the definition of u' and since $\|z\|^2$ is convex in z , we get

$$\|Pu'\|^2 = \|(1 - \theta)Pu + \theta P\hat{y}\|^2 \leq (1 - \theta)\|Pu\|^2 + \theta\|P\hat{y}\|^2.$$

Hence,

$$\begin{aligned} \phi_{\mu'}(u') &\geq -\frac{1}{2}(1 - \theta)\|Pu\|^2 - \frac{1}{2}\theta\|P\hat{y}\|^2 + y_{\mu'}(u')^T Pu' + \frac{\mu'}{2} \|y_{\mu'}(u') - \bar{u}\|^2 \\ &= -\frac{1}{2}(1 - \theta)\|Pu\|^2 - \frac{1}{2}\theta\|P\hat{y}\|^2 + y_{\mu'}(u')^T P((1 - \theta)u + \theta \hat{y}) + \frac{\mu'}{2} \|y_{\mu'}(u') - \bar{u}\|^2 \\ &= (1 - \theta) \left[-\frac{1}{2}\|Pu\|^2 + y_{\mu'}(u')^T Pu + \frac{\mu}{2} \|y_{\mu'}(u') - \bar{u}\|^2 \right] + \theta \left[-\frac{1}{2}\|P\hat{y}\|^2 + y_{\mu'}(u')^T P\hat{y} \right]. \end{aligned}$$

Let us denote the two bracketed expressions shortly by T_1 and T_2 , respectively. We proceed by evaluating T_1 , the first bracketed expression. This can be reduced as follows:

$$T_1 = -\frac{1}{2}\|Pu\|^2 + y_{\mu'}(u')^T Pu + \frac{\mu}{2} \|y_{\mu'}(u') - \bar{u}\|^2$$

$$\begin{aligned}
&= \left(\phi_\mu(u) - y_\mu(u)^T P u - \frac{\mu}{2} \|y_\mu(u) - \bar{u}\|^2 \right) + y_{\mu'}(u')^T P u + \frac{\mu}{2} \|y_{\mu'}(u') - \bar{u}\|^2 \\
&= \phi_\mu(u) + (P u)^T (y_{\mu'}(u') - y_\mu(u)) + \frac{\mu}{2} (\|y_{\mu'}(u') - \bar{u}\|^2 - \|y_\mu(u) - \bar{u}\|^2).
\end{aligned}$$

Putting $a = y_{\mu'}(u')$ and $b = y_\mu(u)$, we have

$$\begin{aligned}
\|a - \bar{u}\|^2 - \|b - \bar{u}\|^2 &= \|a\|^2 - \|b\|^2 - 2a^T \bar{u} + 2b^T \bar{u} \\
&= \|a - b\|^2 - 2\|b\|^2 + 2a^T b - 2a^T \bar{u} + 2b^T \bar{u} \\
&= \|a - b\|^2 + 2(b - \bar{u})^T (a - b)
\end{aligned} \tag{7}$$

Using this, we obtain

$$T_1 = \phi_\mu(u) + \left(P u + \mu(y_\mu(u) - \bar{u}) \right)^T (y_{\mu'}(u') - y_\mu(u)) + \frac{\mu}{2} \|y_{\mu'}(u') - y_\mu(u)\|^2.$$

From (5) and (6), we deduce

$$\theta (y_{\mu'}(u') - y_\mu(u)) = y' - \hat{y}.$$

We also use that the definition of $y_\mu(u)$ implies that this vector minimizes $y^T P u + \frac{\mu}{2} \|y - \bar{u}\|^2$ over all $y \in \Delta$. Hence, at $y = y_\mu(u)$ the vector $\nabla_y (y^T P u + \frac{\mu}{2} \|y - \bar{u}\|^2)$ has nonnegative inner product with $u - y_\mu(u)$, for all $u \in \Delta$. Since $y_{\mu'}(u') \in \Delta$, we get

$$\left(P u + \mu(y_\mu(u) - \bar{u}) \right)^T (y_{\mu'}(u') - y_\mu(u)) \geq 0.$$

Therefore, by using the induction hypothesis, we obtain

$$T_1 \geq \phi_\mu(u) + \frac{\mu}{2\theta^2} \|y' - \hat{y}\|^2 \geq \frac{1}{2} \|P y\|^2 + \frac{\mu}{2\theta^2} \|y' - \hat{y}\|^2.$$

Due to (7), with $\bar{u} = 0$, we get

$$\|a\|^2 \geq \|b\|^2 + 2b^T (a - b), \tag{8}$$

where a and b are arbitrary vectors. Using this and $P^2 = P$, we obtain

$$\frac{1}{2} \|P y\|^2 \geq \frac{1}{2} \|P \hat{y}\|^2 + (P \hat{y})^T P (y - \hat{y}) = \frac{1}{2} \|P \hat{y}\|^2 + (y - \hat{y})^T P \hat{y}.$$

It follows that

$$T_1 \geq \frac{1}{2} \|P \hat{y}\|^2 + (y - \hat{y})^T P \hat{y} + \frac{\mu}{2\theta^2} \|y' - \hat{y}\|^2.$$

For the second bracketed term we write

$$T_2 = -\frac{1}{2}\|P\hat{y}\|^2 + y_{\mu'}(u')^T P\hat{y} = \frac{1}{2}\|P\hat{y}\|^2 + (y_{\mu'}(u') - \hat{y})^T P\hat{y}.$$

Substitution yields, while also using $(1 - \theta)\mu = \mu'$,

$$\begin{aligned} \phi_{\mu'}(u') &\geq (1 - \theta)T_1 + \theta T_2 \\ &\geq \frac{1}{2}\|P\hat{y}\|^2 + (1 - \theta)(y - \hat{y})^T P\hat{y} + \theta[y_{\mu'}(u') - \hat{y}]^T P\hat{y} + \frac{\mu'}{2\theta^2}\|y' - \hat{y}\|^2 \\ &= \frac{1}{2}\|P\hat{y}\|^2 + [(1 - \theta)(y - \hat{y})^T + \theta(y_{\mu'}(u') - \hat{y})^T] P\hat{y} + \frac{\mu'}{2\theta^2}\|y' - \hat{y}\|^2 \\ &= \frac{1}{2}\|P\hat{y}\|^2 + [-\hat{y} + [(1 - \theta)y + \theta y_{\mu'}(u')]]^T P\hat{y} + \frac{\mu'}{2\theta^2}\|y' - \hat{y}\|^2 \\ &= \frac{1}{2}\|P\hat{y}\|^2 + (y' - \hat{y})^T P\hat{y} + \frac{\mu'}{2\theta^2}\|y' - \hat{y}\|^2. \end{aligned}$$

According to the definition of k in Algorithm 1, the iteration number is given by $k + 1$. We claim that

$$\mu_k = \frac{4}{(k + 1)(k + 2)}. \quad (9)$$

This is true if $k = 0$, because $\mu_0 = 2$. We proceed with induction on k . Suppose that the claim holds for some $k \geq 0$. Since $\theta_k = 2/(k + 3)$, we get

$$\mu_{k+1} = (1 - \theta_k)\mu_k = \left(1 - \frac{2}{k + 3}\right)\mu_k = \frac{k + 1}{k + 3} \frac{4}{(k + 1)(k + 2)} = \frac{4}{(k + 2)(k + 3)},$$

as desired. As a consequence, we have

$$\frac{\mu'}{\theta^2} = \frac{\mu_{k+1}}{\theta_k^2} = \frac{\frac{4}{(k + 2)(k + 3)}}{\frac{4}{(k + 3)^2}} = \frac{k + 3}{k + 2} > 1.$$

By also using that P is a projection matrix, we obtain

$$\phi_{\mu'}(u') \geq \frac{1}{2}\|P\hat{y}\|^2 + (y' - \hat{y})^T P\hat{y} + \frac{1}{2}\|P(y' - \hat{y})\|^2 = \frac{1}{2}\|Py'\|^2.$$

Hence the proof of the lemma is complete. ■

Lemma 3.3. If Algorithm 1 does not halt after $k \geq 1$ iterations, then

$$\|Py_k\|^2 \leq \frac{8}{(k + 1)^2} - \frac{1}{n^3}.$$

Proof. Since the algorithm does not halt after k iterations, we have $\|P_A y_k\|^2 \leq 2\phi_{\mu_k}(u_k)$ by Lemma 3.2 and $\|P_A u_k\|^2 \geq \frac{1}{n^3}$ by the Appendix in [6]. Also, using Lemma 3.1, we get

$$\|P_A y_k\|^2 \leq 2\phi_{\mu_k}(u_k) \leq 2(\phi_0(u_k) + \mu_k) \leq 2\mu_k - \frac{1}{n^3},$$

where we also used

$$\phi_0(u_k) = -\frac{1}{2}\|Pu_k\|^2 + \min_{u \in \Delta} u^T Pu_k \leq -\frac{1}{2}\|Pu_k\|^2 \leq -\frac{1}{2n^3},$$

since Pu_k has at least one entry less than or equal to zero (otherwise, u_k would solve the problem and the algorithm would halt with case 1). Due to (9) it follows that

$$\|P_A y_k\|^2 \leq \frac{8}{(k+1)(k+2)} - \frac{1}{n^3} \leq \frac{8}{(k+1)^2} - \frac{1}{n^3},$$

proving the lemma. ■

Lemma 3.4. Algorithm 1 requires at most $2n\sqrt{n}$ iterations.

Proof. As we established in Section 2, y_k gives rise to a proper cut if $n^3\|P_A y_k\|^2 \leq 1$. This certainly holds if $4n^3 \leq (k+1)^2$, which is equivalent to $k+1 \geq 2n\sqrt{n}$. Hence, the proof is complete. ■

4. Time Complexity per Iteration

In this section, we prove that problem (4) can be solved in $O(n)$ time, provided that $z = P_A y$ has been computed. The problem can then be restated as

$$\min_u \left\{ u^T z + \frac{\mu}{2} \|u - \bar{u}\|^2 : \mathbf{1}^T u = 1, u \geq 0 \right\}. \quad (10)$$

The Lagrange dual of this problem can be simplified to

$$\max_{v, \xi} \left\{ \xi - \frac{\mu}{2} \|v\|^2 : \mu v - \xi \mathbf{1} \geq w \right\}, \quad (11)$$

where

$$w = \frac{\mu}{2n} \mathbf{1} - z.$$

Indeed, as we next show we have weak duality. Let u be feasible for (10) and the pair (v, ξ) for (11). Then, the duality gap, i.e., the primal objective value minus the dual objective value, can be reduced as follows:

$$\begin{aligned} u^T z + \frac{\mu}{2} \|u - \bar{u}\|^2 - \left(\xi - \frac{\mu}{2} \|v\|^2 \right) &= u^T \left(\frac{\mu}{2n} \mathbf{1} - w \right) + \frac{\mu}{2} \|u\|^2 + \frac{\mu}{2} \|\bar{u}\|^2 - \mu u^T \bar{u} - \left(\xi - \frac{\mu}{2} \|v\|^2 \right) \\ &= \frac{\mu}{2n} - u^T w + \frac{\mu}{2} \|u\|^2 + \frac{\mu}{2n} - \frac{\mu}{n} u^T \mathbf{1} - \left(\xi - \frac{\mu}{2} \|v\|^2 \right) \end{aligned}$$

$$\begin{aligned}
&= -u^T w + \frac{\mu}{2} \|u\|^2 - \left(\xi - \frac{\mu}{2} \|v\|^2 \right) \\
&\geq u^T (\xi \mathbf{1} - \mu v) + \frac{\mu}{2} \|u\|^2 - \left(\xi - \frac{\mu}{2} \|v\|^2 \right) \\
&= -\mu u^T v + \frac{\mu}{2} \|u\|^2 + \frac{\mu}{2} \|v\|^2 \\
&= \frac{\mu}{2} \|u - v\|^2 \geq 0.
\end{aligned}$$

This makes clear that the duality gap vanishes if and only if

$$v = u, \quad u^T (\mu v - \xi \mathbf{1} - w) = 0. \quad (12)$$

Using this, the optimality conditions for $u \in \Delta$ can be expressed in u alone as follows:

$$\mu u - \xi \mathbf{1} \geq w, \quad u^T (\mu u - \xi \mathbf{1} - w) = 0, \quad (13)$$

for some ξ . Now, let $I := \{i : u_i > 0\}$. Since $u \geq 0$ and $\mu u - \xi \mathbf{1} - w \geq 0$, we deduce from $u^T (\mu u - \xi \mathbf{1} - w) = 0$ that

$$i \in I \Rightarrow \mu u_i - \xi = w_i.$$

If $j \notin I$, then $u_j = 0$, whence $\mu u - \xi \mathbf{1} \geq w$ implies $-\xi \geq w_j$. It follows that if $i \in I$ and $j \notin I$, then

$$w_i = \mu u_i - \xi \geq \mu u_i + w_j > w_j, \quad \forall i \in I, \quad \forall j \notin I. \quad (14)$$

We conclude from this that w_I consists of the $|I|$ largest entries of w and the elements outside I are strictly smaller than those in I . For the moment, assume that w is ordered in nonincreasing order, so that

$$w_1 \geq w_2 \geq \dots \geq w_n. \quad (15)$$

It then follows that I has the form $I = \{1, \dots, k\}$, for some k , and $w_j < w_k$, for each $j > k$. Now, using $\mathbf{1}^T u = 1$ and $u_j = 0$, for $j > k$, we may write

$$1 = \mathbf{1}^T u = \sum_{i=1}^k u_i = \sum_{i=1}^k \frac{w_i + \xi}{\mu} = \frac{1}{\mu} \left(k\xi + \sum_{i=1}^k w_i \right).$$

From this, we obtain an expression for the optimal value of ξ , namely,

$$\xi = \frac{1}{k} \left(\mu - \sum_{i=1}^k w_i \right), \quad (16)$$

and then

$$u_i = \begin{cases} \frac{1}{\mu}(w_i + \xi), & i \leq k \\ 0, & i > k. \end{cases} \quad (17)$$

If $k < n$, then the domain of the primal problem (10) is given by

$$\{u \in \Delta : u_{k+1} = \dots = u_n = 0\},$$

which expands if k increases. Hence, the optimal objective value occurs if k is maximal. One easily verifies that the vector u determined by (16) and (17) belongs to Δ only if

$$\mu + kw_k > \sum_{i=1}^k w_i. \quad (18)$$

Obviously, this holds for $k = 1$, because $\mu > 0$. A crucial observation is that if (18) does not hold for some k , then it does also not hold for larger values of k . Moreover, if it holds for some k , then testing (18) for $k + 1$ amounts to a comparison of $\mu + (k + 1)w_{k+1}$ and $\sum_{i=1}^k w_i + w_{k+1}$, which requires $O(1)$ operations. Hence, the largest k satisfying (18) can be found in $O(k)$ time. We then know the index set I and hence we can compute ξ and then u_i , for $i \leq k$. We conclude that if w is ordered as in (15), then the solution of (10) requires only $O(n)$ time.

The above reasoning uses the fact that the vector w is already ordered in nonincreasing order; to get w ordered in this way, takes $O(n \log n)$ time. Thus, it follows that problem (11), and also (10), can be solved in $O(n \log n)$ time. The computation of z requires $O(n^2)$ time, which dominates the time for ordering w . Hence, solving problem (4) requires $O(n^2)$ time. As a consequence, the overall time complexity of BP becomes $O(nL \cdot n\sqrt{n} \cdot n^2) = O(n^{4.5}L)$ time.

References

- [1] Chubanov, S. (2015), A polynomial projection algorithm for linear feasibility problems, *Mathematical Programming*, 153(2), 687-713.
- [2] Khachiyan, L. G. (1979), A polynomial algorithm in linear programming, *Doklady Akademii Nauk SSSR*, 244:1093-1096. Translated into English in *Soviet Mathematics Doklady* 20:191-194.
- [3] Nesterov, Y. (2005), Excessive gap technique in nonsmooth convex minimization, *SIAM Journal on Optimization*, 16(1), 235-249.
- [4] Peña, J., and Soheili, N. (2016), A deterministic rescaled perceptron algorithm, *Mathematical Programming*, 155(1-2), 497-510.
- [5] Roos, K. (2018), An improved version of Chubanov's method for solving a homogeneous feasibility problem, *Optimization Methods and Software*, 33(1), 26-44.
- [6] Wei, Z. and Roos, K. (2019), Using Nemirovski's Mirror-Prox method as Basic Procedure in Chubanov's method for solving homogeneous feasibility problems, Manuscript (http://www.optimization-online.org/DB_HTML/2018/04/6559.html).