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Some Results on Necessary Conditions for Two Quasidifferentiable Optimization Problems*

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Two main results of necessary conditions of optimality for two kinds of problems, bilevel optimization and quasidifferentiable MPEC, are presented via Demyanov sum of quasidifferentials. The result that Lagrange multipliers are independent of the choices of quasidifferentials and supergradients is given.

Keywords: *Nonsmooth optimization; quasidifferentiable function; bilevel optimization; MPEC; optimality conditions; Demyanov difference (sum).*

1 – Introduction

We consider the following two classes of problems,

$$(\mathrm{P}_1)\qquad \begin{array}{ll} \min & \theta(x) = f(x, v_1(x), \cdots, v_m(x)) \\ \mathrm{s.t.} & v_i(x) = \max\{\varphi_i(x, y_i) \,|\, G_i(x, y_i) \le 0, H_i(x, y_i) = 0\}, \quad i = 1, \cdots, m, \end{array}$$

where $f : R^{n+m} \to R^1$ is quasidifferentiable, $x \in R^n, y_i \in R^{s_i}$, and

$$\begin{array}{l} \varphi_i, g_{ij}, h_{ik} \in C^2, \quad i = 1, \cdots, m, j = 1, \cdots, p_i, k = p_i + 1, \cdots, q_i, \\ G_i(x, y_i) = (g_{i1}(x, y_i), \cdots, g_{ip_i}(x, y_i))^T, \\ H_i(x, y_i) = (h_{i(p_i+1)}(x, y_i), \cdots, h_{iq_i}(x, y_i))^T, \end{array}$$

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and

$$(\mathrm{P}_2) \qquad \begin{array}{ll} \min & f(x,y) \\ \text{s.t.} & z_j(x,y) \le 0, \quad j=1,\cdots,r, \\ & y=(y_1,\cdots,y_m)^T \in S(x)=\prod_{i=1}^m S_i(x), \end{array}$$

where $f, z_j : R^n \times R^{s_1} \times \cdots \times R^{s_m} \to R^1$, $j=1,\cdots,r$, are quasidifferentiable, $x \in R^n, y_i \in S_i(x) \subseteq R^{s_i}$, and

$$\begin{array}{l} \varphi_i, g_{ij}, h_{ik} \in C^2, \quad i=1,\cdots,m, j=1,\cdots,p_i, k=p_i+1,\cdots,q_i, \\ G_i(x,y_i)=(g_{i1}(x,y_i),\cdots,g_{ip_i}(x,y_i))^T, \\ H_i(x,y_i)=(h_{i(p_i+1)}(x,y_i),\cdots,h_{iq_i}(x,y_i))^T, \\ S_i(x)=\arg\max\{\varphi_i(x,y_i) \,|\, G_i(x,y_i)\le 0, H_i(x,y_i)=0\}, i=1,\cdots,m. \end{array}$$

A function $f : R^n \to R^1$ is said to be quasidifferentiable at x in the sense of Demyanov and Rubinov (1980), if f is directionally differentiable at $x \in R^n$ and there exists a pair of compact convex sets, $\underline{\partial} f(x), \overline{\partial} f(x) \subseteq R^n$, such that

$$f'(x;d) = \max_{v \in \underline{\partial} f(x)} \langle v,d \rangle + \min_{w \in \overline{\partial} f(x)} \langle w,d \rangle, \quad \forall d \in R^n;$$

see [2]. $Df(x) = [\underline{\partial} f(x), \overline{\partial} f(x)]$ is called a quasidifferential of f at x, $\underline{\partial} f(x)$ and $\overline{\partial} f(x)$ are called subdifferential and superdifferential of f at x, respectively. Elements of a subdifferential and a superdifferential are called subgradients and supergradients, respectively.

The following assumptions will be used in this paper for ensuring the quasidiffentiability of $v_i(\cdot), i=1,\cdots,m$, at x^0 and the validity of optimality conditions. Define:

$$Y_i(x) = \operatorname{Arg}\max\{\varphi_i(x,y_i) \,|\, G_i(x,y_i)\le 0, H_i(x,y_i)=0\}, \quad i=1,\cdots,m.$$

Assumption 1.1. (Uniform Boundedness) $Y_i(x)$ is uniformly bounded in some neighborhood of x^0; i. e., there exists a neighborhood N_i of x^0 and a bounded set $T_i \subseteq R^{s_i}$ such that $Y_i(x) \subseteq T_i$, for any $x \in N_i$.

Assumption 1.2. (M-F Constraint Qualification) For every $y_i^0 \in Y_i(x^0)$, the lower problem,

$$\begin{array}{ll} \max & \varphi_i(x,y_i) \\ \text{s.t.} & G_i(x,y_i)\le 0, H_i(x,y_i)=0, \quad i=1,\cdots,m, \end{array} \tag{1}$$

satisfies M-F constraint qualification:

1. The vectors $\nabla_y h_{ij}(x^0,y_i^0), j=p_i+1,\cdots,q_i$, are linearly independent.

2. There exists a $w_i \in R^{s_i}$ satisfying,

$$w_i^T \nabla_y g_{ij}(x^0,y_i^0) < 0, \quad \forall j \in \{\nu \,|\, g_{i\nu}(x^0,y_i^0)=0, \nu=1,\cdots,p\}, \tag{2}$$

$$w_i^T \nabla_y h_{ij}(x^0,y_i^0) = 0, \quad j=p+1,\cdots,q. \tag{3}$$

Assumption 1.3. (Second Order Sufficient Conditions) For any $\alpha_i \in A_i(x^0, y_i^0)$ and $d \neq 0$ satisfying the following conditions,

$$\nabla_y g_{ij}(x^0, y_i^0)^T d = 0, \quad \forall j \in J(\alpha_i) = \{j \,|\, \alpha_{ij} > 0, j = 1, \cdots, p\}, \tag{4}$$

$$\nabla_y h_{ij}(x^0, y_i^0)^T d = 0, \quad j = p+1, \cdots, q, \tag{5}$$

one has,

$$d^T \nabla_{yy}^2 L_i(x^0, y_i^0, \alpha_i) d > 0, \tag{6}$$

where,

$$A_i(x^0, y_i^0) = \left\{ \alpha \in R^q \;\middle|\; \begin{array}{ll} \nabla_y L_i(x^0, y_i^0, \alpha) = 0 & \\ \alpha_j \geq 0, & j = 1, \cdots, p_i \\ \alpha_j g_{ij}(x^0, y_i^0) = 0, & j = 1, \cdots, p_i \end{array} \right\}, \tag{7}$$

$$L_i(x, y_i, \alpha_i) = \varphi_i(x, y_i) + \sum_{k=1}^{p_i} \alpha_k g_{ik} + \sum_{k=p_i+1}^{q_i} \alpha_k h_{ik}, \tag{8}$$

$$\alpha_i = (\alpha_1, \cdots, \alpha_{p_i}, \alpha_{p_i+1}, \cdots, \alpha_{q_i}). \tag{9}$$

Here, co$C$ denotes the convex hull of C. In the next section, necessary conditions for problem (P_1), i. e., for a class of quasidifferentiable bilevel optimization, are given, and necessary conditions for problem (P_2), i. e., for a class of quasidifferentiable MPEC problems, are presented in Section 3.

2 – The Case of (P_1)

For every $y_i^0 \in Y_i(x^0)$, the set of Lagrange multiplier vectors of lower level problem is nonempty if and only if M-F constraint qualification holds at y_i^0; see $[6, 10]$. Moreover, the following theorem holds.

Theorem 2.1. $[1, 7, 8, 11, 12, 13]$ *Suppose that the M-F constraint qualification and second order sufficient conditions hold for every $y_i^0 \in Y_i(x^0)$. Then, $v_i(\cdot)$ is (local) Lipschitzian, directionally differentiable and*

$$v_i'(x^0; d) = \sup_{y_i \in Y_i(x^0)} \inf\nolimits_{\alpha \in A_i(x^0, y_i)} d^T \nabla_x L_i(x^0, y_i, \alpha). \tag{1}$$

If for every i, $i = 1, \cdots, m$, $Y_i(x^0)$ is finite, i. e., $Y_i(x^0) = \{y_i^1, \cdots, y_i^{\beta_i}\}$, then (1) can be written as:

$$v_i'(x^0; d) = \max\{\langle d, e\rangle \,|\, e \in C_i^1\} - \max\{\langle d, e\rangle \,|\, e \in C_i^2\}, \tag{2}$$

i. e., $v_i(\cdot)$ is quasidifferentiable at x^0, and $[C_i^1, -C_i^2]$ is a quasidifferential of v_i at x^0, where,

$$C_i^1 = \text{co}(\bigcup_{l=1}^{\beta_i}(\sum_{\mu\neq l} B_i^{\mu})), \qquad C_i^2 = \sum_{l=1}^{\beta_i} B_i^l,$$
$$B_i^l = -\text{co}(\bigcup_{\alpha\in A(x^0,y_i^l)}\{\nabla_x L_i(x^0, y_i^l, \alpha)\}), \quad \forall l = 1, \cdots, \beta_i,$$
$$y_i^j \in R^{S_i}, \quad \forall j = 1, \cdots, \beta_i. \qquad \square$$

In the remainder of the paper, we assume that for every $i,\ i = 1, \cdots, m$, $Y_i(x^0)$ is finite. Consider problem (P_1). Since $v(\cdot)$ is Lipschitzian, it is uniformly directionally differentiable. By the quasidifferential calculus of composition functions [3], one has that θ is quasidifferentiable at x^0, and $[\underline{\partial}\theta(x^0), \overline{\partial}\theta(x^0)]$ is formulated by:

$$\begin{aligned}\underline{\partial}\theta(x^0) = \ &\{w \mid w = (\underline{v}^{(1)}, \cdots, \underline{v}^{(n)}) + \textstyle\sum_{i=n+1}^{n+m}[v^{(i)}(\lambda_i + \mu_i) - v'^{(i)}\lambda_i - v''^{(i)}\mu_i],\\ &\qquad v = (v^{(1)}, \cdots, v^{(n+m)}) \in \underline{\partial} f(y^0), \lambda_i \in C_{i-n}^1, \mu_i \in -C_{i-n}^2\},\\ \overline{\partial}\theta(x^0) = \ &\{w \mid w = (\overline{v}^{(1)}, \cdots, \overline{v}^{(n)}) + \textstyle\sum_{i=n+1}^{n+m}[u^{(i)}(\lambda_i + \mu_i) + v'^{(i)}\lambda_i + v''^{(i)}\mu_i],\\ &\qquad u = (u^{(1)}, \cdots, u^{(n+m)}) \in \overline{\partial} f(y^0), \lambda_i \in C_{i-n}^1, \mu_i \in -C_{i-n}^2\},\end{aligned}$$
$$v' \leq v \leq v'',\ v' \leq 0,\ v'' \geq 0,\ y^0 = (x^0, v_1(x^0), \cdots, v_m(x^0))^T,$$
$$\underline{v}^{(i)} = v^{(i)} - v'^{(i)}, \overline{v}^{(i)} = u^{(i)} + v'^{(i)}, i = 1, \cdots, n.$$

Let $A, B \subseteq R^n$ be convex compact. The Demyanov difference of A and B, our basic operation here, is defined by:

$$A \dot{-} B = \text{clco}\{\nabla\delta^*(h \mid A) - \nabla\delta^*(h \mid B) \mid h \in T\},$$

where, $T = \{h \in R^n \mid \nabla\delta^*(\cdot \mid A)(h)$ and $\nabla\delta^*(\cdot \mid B)(h)$ exist$\}$. The form of Demyanov difference, $\underline{\partial} f(x) \dot{-} (-\overline{\partial} f(x))$, will play the main role and is also denoted by $\partial^{\dot{+}} f(x)$.

Lemma 2.1. [4] *Let $f : R^n \to R^1$ be quasidifferentiable. If $x^0 \in \arg\min_{x\in R^n} f(x)$, then* $0 \in \partial^{\dot{+}} f(x^0)$. $\square$

The following theorem can be obtained in terms of Lemma 2.1.

Theorem 2.2. *Suppose Assumptions 1, 2 and 3 hold, and for $i,\ i = 1, \cdots, m$, $Y_i(x^0)$ is finite. If x^0 is a minimizer of (P_1), then $0 \in \partial^{\dot{+}}\theta(x^0)$.* $\square$

In what follows, assume that θ is a maximal function, i. e.,

$$\begin{aligned}&\min \quad \theta(x) = \max\{v_1(x), \cdots, v_m(x)\}\\ &\text{s. t.} \quad v_i(x) = \max\{\varphi_i(x, y_i) \mid G_i(x, y_i) \leq 0, H_i(x, y_i) = 0\}, \quad i = 1, \cdots, m.\end{aligned} \tag{3}$$

One has from the quasidifferential calculus of maximal functions that the quasidifferential of $\theta(\cdot)$ at x^0 is given by:

$$\underline{\partial}\theta(x^0) = \text{co} \bigcup_{k\in R(x^0)} (C_k^1 + \sum_{i\in R(x)\setminus\{k\}} C_i^2),$$
$$\overline{\partial}\theta(x^0) = -\sum_{i\in R(x^0)} C_i^2,$$

where, $R(x^0) = \{i \in 1 : m \mid \theta(x^0) = v_i(x^0)\}$.

Lemma 2.2. [3] *Let $A, B \subseteq R^n$ be convex compact. Then, $(A+B)\dot{-}B = A$.* □

Lemma 2.3. *Let $A_i, B \subseteq R^n, i = 1, \cdots, m$, be convex compact. Then,* $\mathrm{co}(\bigcup_{i=1}^m A_i)\dot{-}B \subseteq \mathrm{co}\bigcup_{i=1}^m (A_i\dot{-}B)$.

Proof: straightforward. □

Theorem 2.3. *Suppose Assumptions $1, 2$ and 3 hold, and for $i,\ i = 1, \cdots, m$, $Y_i(x^0)$ is finite. If x^0 is a minimizer of (3), then there exists a finite number of $\alpha \in A_0(x^0, y_k^l) \subseteq A(x^0, y_k^l), l = 1, \cdots, \beta_k, k \in R(x^0)$ and $\lambda(\alpha, l, k) \geq 0$, such that*

$$\sum_{\alpha\in A_0(x^0,y_k^l),l=1,\cdots,\beta_k,k\in R(x^0)} \lambda(\alpha, l, k)\nabla_x L_k(x^0, y_k^l, \alpha) = 0.$$

Proof: One has from Theorem 2.2 that $0 \in \partial^{\dot{+}}\theta(x^0)$. Compute $\partial^{\dot{+}}\theta(x^0)$,

$$\begin{aligned}
\partial^{\dot{+}}\theta(x^0) =\ & \underline{\partial}\theta(x^0)\dot{-}(-\overline{\partial}\theta(x^0)) \\
=\ & \mathrm{co}[\textstyle\bigcup_{k\in R(x^0)}(C_k^1 + \sum_{i\in R(x)\setminus\{k\}} C_i^2)]\dot{-}(\sum_{i\in R(x^0)} C_i^2) \\
\subseteq\ & \mathrm{co}\{(C_k^1 + \textstyle\sum_{i\in R(x)\setminus\{k\}} C_i^2)\dot{-}\sum_{i\in R(x^0)} C_i^2 \mid k \in R(x^0)\} && \text{(Lemma 2.3)} \\
=\ & \mathrm{co}\{C_k^1\dot{-}C_k^2 \mid k \in R(x^0)\} && \text{(Lemma 2.2)}
\end{aligned}$$

while,

$$\begin{aligned}
C_k^1\dot{-}C_k^2 =\ & \mathrm{co}(\textstyle\bigcup_{l=1}^{\beta_k}\sum_{\nu\neq l} B_k^\nu)\dot{-}\sum_{l=1}^{\beta_k} B_k^l \\
\subseteq\ & \mathrm{co}\textstyle\bigcup_{l=1}^{\beta_k}(\sum_{\nu\neq l} B_k^\nu\dot{-}\sum_{l=1}^{\beta_k} B_k^l) && \text{(Lemma 2.3)} \\
=\ & \mathrm{co}\textstyle\bigcup_{l=1}^{\beta_k}(0\dot{-}B_k^l) && \text{(Lemma 2.2)} \\
=\ & -\mathrm{co}\textstyle\bigcup_{l=1}^{\beta_k} B_k^l.
\end{aligned}$$

As a consequence, one has:

$$\begin{aligned}
0 \in\ & \mathrm{co}\{-\mathrm{co}\textstyle\bigcup_{l=1}^{\beta_k} B_k^l \mid k \in R(x^0)\} \\
=\ & \mathrm{co}\{\textstyle\bigcup_{l=1}^{\beta_k}(\bigcup_{\alpha\in A(x^0,y_k^l)}\{\nabla_x L_k(x^0, y_k^l, \alpha)\}) \mid k \in R(x^0)\} \\
=\ & \mathrm{co}\{\nabla_x L_k(x^0, y_k^l, \alpha) \mid \alpha \in A(x^0, y_k^l), l = 1, \cdots, \beta_k, k \in R(x^0)\},
\end{aligned}$$

that is, there exists a finite number of $\lambda(\alpha, l, k) \geq 0$, such that

$$\sum_{\alpha\in A_0(x^0,y_k^l),l=1,\cdots,\beta_k,k\in R(x^0)} \lambda(\alpha, l, k)\nabla_x L_k(x^0, y_k^l, \alpha) = 0,$$

where, $A_0(x^0, y_k^l)$ is a finite subset of $A(x^0, y_k^l)$. □

For (P_1), if $\varphi_i, i = 1, \cdots, m$, are strictly concave, $g_{ij}, i = 1, \cdots, m, j = 1, \cdots, p_i$, are convex, $h_{ik}, i = 1, \cdots, m, k = p_i + 1, \cdots, q_i$, are affine, and *Slate* constraint qualification holds, then (1) holds and the solution of lower level problem (1) is unique, that is, $Y_i(x^0) = \{y_i^0\}$; see [9]. Therefore, one has:

$$v_i'(x^0; d) = \inf_{\alpha \in A_i(x^0, y_i^0)} d^T \nabla_x L_i(x^0, y_i^0, \alpha). \tag{4}$$

In other words, v_i is superdifferentiable at x^0, and $\overline{\partial} v_i(x^0) = \mathrm{co}\{\nabla_x L(x^0, y_i^0, \alpha) \mid \alpha \in A_i(x^0, y_i^0)\}$. By the quasidifferential calculus of composition functions, the quasidifferential of $\theta(\cdot)$ at x^0 is formulated as:

$$\begin{aligned} \underline{\partial}\theta(x^0) = \ & \{w \mid w = (\underline{v}^{(1)}, \cdots, \underline{v}^{(n)}) + \textstyle\sum_{i=n+1}^{n+m} (v^{(i)} - v''^{(i)})\mu_i, \\ & \qquad (v^{(1)}, \cdots, v^{(n+m)}) \in \underline{\partial} f(y^0), \mu_i \in B_{i-n}\}, \\ \overline{\partial}\theta(x^0) = \ & \{w \mid w = (\overline{v}^{(1)}, \cdots, \overline{v}^{(n)}) + \textstyle\sum_{i=n+1}^{n+m} (u^{(i)} + v''^{(i)})\mu_i, \\ & \qquad (u^{(1)}, \cdots, u^{(n+m)}) \in \overline{\partial} f(y^0), \mu_i \in B_{i-n}\}, \end{aligned}$$

where,

$$\begin{aligned} & v' \le v \le v'', \ v' \le 0, \ v'' \ge 0, \ y^0 = (x^0, v_1(x^0), \cdots, v_m(x^0))^T, \\ & \underline{v}^{(i)} = v^{(i)} - v'^{(i)}, \ \overline{v}^{(i)} = u^{(i)} + v'^{(i)}, \\ & B_{i-n} = \mathrm{co}\{\nabla_x L_{i-n}(x^0, y_{i-n}^0, \alpha) \mid \alpha \in A_{i-n}(x^0, y_{i-n}^0)\}. \end{aligned}$$

Similarly, if f is a maximal function, then we have the following theorem.

Theorem 2.4. *Assume that $\varphi_i, i = 1, \cdots, m$, are strictly concave, $g_{ij}, i = 1, \cdots, m, j = 1, \cdots, p_i$, are convex, $h_{ik}, i = 1, \cdots, m, k = p_i + 1, \cdots, q_i$, are affine, and Slate constraint qualification holds. If x^0 is a minimizer of (3), then there exists a finite number of $\lambda(\alpha, k) \ge 0$, such that*

$$\sum_{\alpha \in A_0(x^0, y_k^0), k \in R(x^0)} \lambda(\alpha, k) \nabla_x L_k(x^0, y_k^0, \alpha) = 0,$$

where, $A_0(x^0, y_0^l)$ is a finite subset of $A(x^0, y_0^l)$, and $Y_k(x^0) = \{y_k^0\}$.

Proof: By Theorem 2.2 one has that $0 \in \partial^{\dot{+}}\theta(x^0)$. We only need to compute $\partial^{\dot{+}}\theta(x^0)$,

$$\begin{aligned} & \partial^{\dot{+}}\theta(x^0) \\ & = \underline{\partial}\theta(x^0) \dot{-} (-\overline{\partial}\theta(x^0)) \\ & = \mathrm{co}[\textstyle\bigcup_{k \in R(x^0)} (\underline{\partial} v_k(x^0) - \sum_{i \in R(x) \setminus \{k\}} \overline{\partial} v_i(x^0))] \dot{-} (-\sum_{k \in R(x^0)} \overline{\partial} v_k(x^0)) \\ & \subseteq \mathrm{co}\{(\underline{\partial} v_k(x^0) - \textstyle\sum_{i \in R(x) \setminus \{k\}} \overline{\partial} v_i(x^0)) \dot{-} \\ & \qquad (-\textstyle\sum_{k \in R(x^0)} \overline{\partial} v_k(x^0)) \mid k \in R(x^0)\} \quad \text{(Lemma 2.3)} \\ & = \mathrm{co}\{\underline{\partial} v_k(x^0) \dot{-} (-\overline{\partial} v_k(x^0)) \mid k \in R(x^0)\} \qquad \text{(Lemma 2.2)} \\ & = \mathrm{co}\{\nabla_x L(x^0, y_k^0, \alpha) \mid k \in R(x^0), \alpha \in A_k(x^0, y_k^0)\}. \end{aligned}$$

Hence, it follows from the definition of convex hull that there exists a finite number of $\lambda(\alpha, k) \geq 0$, such that

$$\sum_{\alpha \in A_0(x^0, y_k^0), k \in R(x^0)} \lambda(\alpha, k) \nabla_x L_k(x^0, y_k^0, \alpha) = 0,$$

where, $A_0(x^0, y_k^0)$ is a finite subset of $A(x^0, y_k^0)$, and $Y_k(x^0) = \{y_k^0\}$. □

3 – The Case of (P_2)

Here, we consider problem (P_2), which is equivalent to the following problem,

$$\begin{aligned} \min \quad & f(x, y) \\ \text{s. t.} \quad & z_j(x, y) \leq 0, \quad j = 1, \cdots, r, \\ & v_i(x) \leq \varphi_i(x, y_i), \quad i = 1, \cdots, m, \\ & G_i(x, y_i) \leq 0, \quad i = 1, \cdots, m, \\ & H_i(x, y_i) = 0, \quad i = 1, \cdots, m, \end{aligned} \tag{1}$$

where, $v_i(x) = \max\{\varphi_i(x, y_i) \mid G_i(x, y_i) \leq 0, H_i(x, y_i) = 0\}$.

Lemma 3.1. *Let $A \subseteq R^n, B \subseteq R^m, d = (d_1, d_2) \in R^n \times R^m$. Then, $G_d(A \times B) = G_{d_1}(A) \times G_{d_2}(B)$, where $G_d(A)$ denotes the maximal face of A determined by d.*

Proof: According to the definition of maximal face one has that $(\overline{x}, \overline{y}) \in G_d(A \times B)$ if and only if

$$\langle (\overline{x}, \overline{y}), (d_1, d_2) \rangle = \max_{(x,y) \in A \times B} \langle (x, y), (d_1, d_2) \rangle,$$

that is,

$$\langle \overline{x}, d_1 \rangle + \langle \overline{y}, d_2 \rangle = \max_{x \in A} \langle x, d_1 \rangle + \max_{y \in B} \langle y, d_2 \rangle.$$

Therefore,

$$\max_{x \in A} \langle x - \overline{x}, d_1 \rangle + \max_{y \in B} \langle y - \overline{y}, d_2 \rangle = 0. \tag{2}$$

Since $(\overline{x}, \overline{y}) \in G_d(A \times B)$, we have $(\overline{x}, \overline{y}) \in A \times B$, that is, $\overline{x} \in A$ and $\overline{y} \in B$. Hence,

$$\max_{x \in A} \langle x - \overline{x}, d_1 \rangle \geq 0$$

and

$$\max_{y \in B} \langle y - \overline{y}, d_2 \rangle \geq 0.$$

Using the last two inequalities in (2), one has:

$$\max_{x \in A} \langle x - \overline{x}, d_1 \rangle = 0$$

and

$$\max_{y \in B} \langle y - \overline{y}, d_2 \rangle = 0,$$

that is,

$$\langle \overline{x}, d_1 \rangle = \max_{x \in A} \langle x, d_1 \rangle, \quad \langle \overline{y}, d_2 \rangle = \max_{y \in B} \langle y, d_2 \rangle.$$

This leads to $(\overline{x}, \overline{y}) \in G_{d_1}(A) \times G_{d_2}(B)$. □

Corollary 3.1. *Let $A \subseteq R^n, B \subseteq R^m$ and $d = (d_1, d_2) \in R^n \times R^m$. If $\delta^*(\cdot \mid A \times B)$ is differentiable at d, then $\delta^*(\cdot \mid A)$ is differentiable at d_1 and $\delta^*(\cdot \mid B)$ is differentiable at d_2. Moreover,*

$$\nabla \delta^*(d \mid A \times B) = \nabla \delta^*(d_1 \mid A) \times \nabla \delta^*(d_2 \mid B).$$

□

Lemma 3.2. *Let $T \subseteq R^n \times R^m$ be a full measure. Then, $P_{R^n}(T)$ and $P_{R^m}(T)$ are full measures with respect to R^n and R^m, respectively, where $P_X(T)$ denotes the projection of T onto X.*

Proof: By contradiction, assume that $P_{R^n}(T)$ is not a full measure subset of R^n. Then, there exists $A \subseteq R^n$, not a zero measure, and $A \subseteq R^n \backslash P_{R^n}(T)$. Therefore, $A \times R^m \subseteq R^n \times R^m$ is not a zero measure, which contradicts the fact that $A \times R^m \subseteq (R^n \times R^m) \backslash T$ and T is a full measure with respect to $R^n \times R^m$. Hence, $P_{R^n}(T)$ is a full measure with respect to R^n. In a similar way, we can prove that $P_{R^m}(T)$ is a full measure with respect to R^m. □

Lemma 3.3. *Let $A, C \subseteq R^n, B, D \subseteq R^m$ be convex compact. Then, $(A \times B) \dot{-} (C \times D) \subseteq (A \dot{-} C) \times (B \dot{-} D)$.*

Proof: Let $T = \{d \in R^n \times R^m \mid \delta^*(\cdot \mid A \times B)$ and $\delta^*(\cdot \mid C \times D)$ are differentiable at $d\}$. One has from the definition of Demyanov difference, Corollary 3.1 and Lemma 3.2,

$$\begin{aligned}
&(A \times B) \dot{-} (C \times D)\\
&\quad = \text{clco}\{\nabla \delta^*(d \mid A \times B) - \nabla \delta^*(d \mid C \times D) \mid d \in T\}\\
&\quad = \text{clco}\{\nabla \delta^*(d_1 \mid A) \times \nabla \delta^*(d_2 \mid B) - \nabla \delta^*(d_1 \mid C) \times \nabla \delta^*(d_2 \mid D) \mid d \in T\}\\
&\quad \subseteq \text{clco}\left\{\nabla \delta^*(d_1 \mid A) \times \nabla \delta^*(d_2 \mid B) - \nabla \delta^*(d_1 \mid C) \times \nabla \delta^*(d_2 \mid D) \;\middle|\; \begin{array}{l} d_1 \in P_{R^n}(T) \\ d_2 \in P_{R^m}(T) \end{array} \right\}\\
&\quad = \text{clco}\{\nabla \delta^*(d_1 \mid A) - \nabla \delta^*(d_1 \mid C) \mid d_1 \in P_{R^n}(T)\} \times\\
&\qquad\qquad\qquad \text{clco}\{\nabla \delta^*(d_2 \mid B) - \nabla \delta^*(d_2 \mid D) \mid d_2 \in P_{R^m}(T)\}\\
&\quad = (A \dot{-} C) \times (B \dot{-} D).
\end{aligned}$$

□

Theorem 3.1. *Suppose Assumptions* $1, 2$ *and* 3 *hold, and for* $i,\ i = 1, \cdots, m$, $Y_i(x^0)$ *is finite. If* x^0 *is a minimizer of* (P_2)*, then there exist* $\lambda_j \geq 0, j = 0, \cdots, r$, $\mu_i \geq 0, i = 1, \cdots, m$, $\nu_{ij} \geq 0, i = 1, \cdots, m, j = 1, \cdots, p_i$ *and* $\omega_{ik}, i = 1, \cdots, m, k = p_i + 1, \cdots, q_i$, *not all zero, such that*

$$\begin{aligned} 0_n \in\ & \lambda_0 \partial^{\dotplus} f_x(x^0, y^0) + \textstyle\sum_{j=1}^r \lambda_j \partial^{\dotplus} z_{jx}(x^0, y^0) + \\ & \textstyle\sum_{i=1}^m \mu_i (\mathrm{co} \bigcup \{\nabla_x L_i(x^0, y_i^l, \alpha) \,|\, \alpha \in A(x^0, y_i^l), l = 1, \cdots, \beta_i\} - \nabla \varphi_{ix}(x^0, y_i)) + \\ & \textstyle\sum_{i=1}^m \sum_{j=1}^{p_i} \nu_{ij} \nabla_x g_{ij}(x^0, y_i^0) + \sum_{i=1}^m \sum_{k=p_i+1}^{q_i} \omega_{ik} \nabla_x h_{ik}(x^0, y_i^0), \\ 0_{s_i} \in\ & \lambda_0 \partial^{\dotplus} f_{y_i}(x^0, y^0) + \textstyle\sum_{j=1}^r \lambda_j \partial^{\dotplus} z_{jy_i}(x^0, y^0) + \sum_{i=1}^m \mu_i (-\nabla_{y_i} \varphi_i(x^0, y_i^0)) \\ & + \textstyle\sum_{i=1}^m \sum_{j=1}^{p_i} \nu_{ij} \nabla_{y_i} g_{ij}(x^0, y_i^0) + \sum_{i=1}^m \sum_{k=p_i+1}^{q_i} \omega_{ik} \nabla_{y_i} h_{ik}(x^0, y_i^0), \end{aligned}$$

where, $y^0 = (y_1^0, \cdots, y_m^0)^T$.

Proof: Problem (1) is equivalent to:

$$\begin{aligned} \min \quad & f(x, y) \\ \mathrm{s.t.} \quad & z_j(x, y) \leq 0, \quad j = 1, \cdots, r, \\ & v_i(x) - \varphi_i(x, y_i) \leq 0, \quad i = 1, \cdots, m, \\ & G_i(x, y_i) \leq 0, \quad i = 1, \cdots, m, \\ & H_i(x, y_i) = 0, \quad i = 1, \cdots, m. \end{aligned} \tag{3}$$

If x^0 is a minimizer of (P_2), then there exists $y^0 \in R^m$ such that (x^0, y^0) is a minimizer of (3). Consider v_i as a function of (x, y). We have from Theorem 2.1 that

$$\begin{aligned} \underline{\partial} v_i(x^0, y^0) &= C_i^1 \times 0_m, \\ \overline{\partial} v_i(x^0, y^0) &= -C_i^2 \times 0_m. \end{aligned}$$

Similarly, we have that $v_i - \varphi_i$ is quasidifferentiable at (x^0, y^0) and

$$\begin{aligned} \underline{\partial}(v_i - \varphi_i)(x^0, y^0) &= (C_i^1 - \nabla \varphi_{ix}(x^0, y_i^0)) \times \\ & \qquad \underbrace{0 \times \cdots \times 0 \times (-\nabla \varphi_{iy_i}(x^0, y_i^0)) \times 0 \times \cdots \times 0}_{m}, \\ \overline{\partial}(v_i - \varphi_i)(x^0, y^0) &= -C_i^2 \times 0_m. \end{aligned}$$

By virtue of the necessary conditions of constrained quasidifferentiable optimization due to Gao [5], there exist $\lambda_j \geq 0, j = 0, \cdots, r$, $\mu_i \geq 0, i = 1, \cdots, m$, $\nu_{ij} \geq 0, i = 1, \cdots, m, j = 1, \cdots, p_i$ and $\omega_{ik}, i = 1, \cdots, m, k = p_i + 1, \cdots, q_i$, not all zero, such that

$$\begin{aligned} 0 \in\ & \lambda_0 \partial^{\dotplus} f(x^0, y^0) + \textstyle\sum_{j=1}^r \lambda_j \partial^{\dotplus} z_j(x^0, y^0) \\ & + \textstyle\sum_{i=1}^m \mu_i \{[(C_i^1 - \nabla \varphi_{ix}(x^0, y_i^0)) \dot{-} C_i^2] \times 0 \times \cdots \times \\ & (-\nabla \varphi_{iy_i}(x^0, y_i^0)) \times \cdots \times 0\} + \textstyle\sum_{i=1}^m \sum_{j=1}^{p_i} \nu_{ij} \nabla_{(x,y)} g_{ij}(x^0, y_i^0) \\ & + \textstyle\sum_{i=1}^m \sum_{k=p_i+1}^{q_i} \omega_{ik} \nabla_{(x,y)} h_{ik}(x^0, y_i^0). \end{aligned} \tag{4}$$

Computing $(C_i^1 - \nabla\varphi_{ix}(x^0, y_i^0))\dot{-}C_i^2$, one has:

$$\begin{aligned}
&(C_i^1 - \nabla\varphi_{ix}(x^0, y_i^0))\dot{-}C_i^2 \\
&\qquad = -\nabla\varphi_{ix}(x^0, y_i^0) + (C_i^1\dot{-}C_i^2) \\
&\qquad = -\nabla\varphi_{ix}(x^0, y_i^0) - \mathrm{co}\bigcup_{l=1}^{\beta_i} B_i^l \\
&\qquad = -\nabla\varphi_{ix}(x^0, y_i^0) - \mathrm{co}\bigcup_{l=1}^{\beta_i}(-\mathrm{co}\bigcup\{\nabla_x L_i(x^0, y_i^l, \alpha \mid \alpha \in A(x^0, y_i^l))\}) \\
&\qquad = -\nabla\varphi_{ix}(x^0, y_i^0) + \mathrm{co}\bigcup\{\nabla_x L_i(x^0, y_i^l, \alpha \mid \alpha \in A(x^0, y_i^l), l = 1, \cdots, \beta_i\}.
\end{aligned}$$

Combining the above formula with (4), the conclusion is obtained from Lemma 3.3. □

If in (P_2), $f, z_j, j = 1, \cdots, r$, are differentiable, then the necessary conditions given in Theorem 3.1 turns to qualities.

Corollary 3.2. *Suppose that $f, z_j, j = 1, \cdots, r$, are differentiable in (P_2), Assumptions $1, 2$ and 3 hold, and for $i,\ i = 1, \cdots, m$, $Y_i(x^0)$ is finite. If x^0 is a minimizer of (P_2), then there exist $\lambda_j \geq 0, j = 0, \cdots, r$, $\mu_i \geq 0, i = 1, \cdots, m$, $\nu_{ij} \geq 0, i = 1, \cdots, m, j = 1, \cdots, p_i$ and $\omega_{ik}, i = 1, \cdots, m, k = p_i + 1, \cdots, q_i$, not all zero, such that*

$$\begin{aligned}
0_n = {} & \lambda_0 \nabla f_x(x^0, y^0) + \textstyle\sum_{j=1}^r \lambda_j \nabla z_{jx}(x^0, y^0) + \\
& \textstyle\sum_{i=1}^m \mu_i (\mathrm{co}\bigcup\{\nabla_x L_i(x^0, y_i^l, \alpha) \mid \alpha \in A(x^0, y_i^l), l = 1, \cdots, \beta_i\} - \nabla\varphi_{ix}(x^0, y_i)) + \\
& \textstyle\sum_{i=1}^m \sum_{j=1}^{p_i} \nu_{ij} \nabla_x g_{ij}(x^0, y_i^0) + \sum_{i=1}^m \sum_{k=p_i+1}^{q_i} \omega_{ik} \nabla_x h_{ik}(x^0, y_i^0),
\end{aligned}$$

$$\begin{aligned}
0_{s_i} = {} & \lambda_0 \nabla f_{y_i}(x^0, y^0) + \textstyle\sum_{j=1}^r \lambda_j \nabla z_{jy_i}(x^0, y^0) + \sum_{i=1}^m \mu_i (-\nabla_{y_i}\varphi_i(x^0, y_i^0)) + \\
& \textstyle\sum_{i=1}^m \sum_{j=1}^{p_i} \nu_{ij} \nabla_{y_i} g_{ij}(x^0, y_i^0) + \sum_{i=1}^m \sum_{k=p_i+1}^{q_i} \omega_{ik} \nabla_{y_i} h_{ik}(x^0, y_i^0),
\end{aligned}$$

where, $y^0 = (y_1^0, \cdots, y_m^0)^T$. □

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