Iranian Journal of Operations Research
Vol. 1, No. 2, 2009, pp. 56-67

Real Time Study of a *k-out-of-n* System: *n* Identical Elements with Increasing Failure Rates

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Reliability models based on Markov chain (Except in queuing systems) have extensive applications in electrical and electronic devices. Here, we consider a k-out-of-n:G system with n parallel and identical elements with increasing failure rates (failure rates are Weibull distributed) and non repairable elements. The failure rate of remaining elements increases when some elements fail. The system works until at least k elements work. The system of equations are established and the exact equations are sought for the parameters like MTTF and the probability of system working at the time t. A numerical example is solved to demonstrate the procedure clarifying the theoretical development.

Keywords: *Reliability, Markov chain; k-out-of-n:G System; Weibull distribution.*

Nomenclature

The notations to be used are as follows:

n : Number of elements,

$\lambda_i \times t$: Failure rate of the elements in category i at time t ,

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$P_i(t)$: Probability that the system is in state i at time t,
$R_P(t)$: Probability that system works at time t,
$MTTF$: Mean time to failure of the system,
X_j : Interval between $(j-1)$ and j failures in state $A_1A_2...A_n$.

1. Introduction

k-out-of-n models, are among the most useful models to calculate the reliability of electrical and electronic devices and systems. In the literature, there are many studies in this area. We try to categorically classify them. At the first glance, they may be classified into two main groups, namely *k-out-of-n:F* and *k-out-of-n:G* systems. If the failure of a system with n components is abandoned to the failure of at least k components $(k \le n)$, then the system is called *k-out-of-n:F*. On the other hand, if the working of the system is abandoned to the working of at least k components $(k \le n)$, then the system is called *k-out-of-n:G*. Both systems can be considered in steady state and real time situations. The elements in both situations may be repairable or non repairable. The failure rates of the elements may be constant, increasing or decreasing, whereas the repair rate is constant. Each component can be expressed in binary or multiple states. Table 1 presents a classification of the methods for *k-out-of-n* systems in the literature.

Binary models: Boland and Papastavridis [3], study a situation where there are k distinct components with failure probabilities q_i, for $i = 1,2,...,k$, where the failure probability of the jth component $(j = mk + i),(1 \le i \le k)$ is q_i. They obtained exact expressions for the failure probability of an r consecutive *k-out-of-n:F* system. Gera [9] studies the reliability of a consecutive *k-out-of-n:G* system and the problem is solved via a matrix formulation using state space method. Lam and Tony [18] introduce a general model for the consecutive *k-out-of-n:F* repairable system with exponential distribution and $(k-1)$-step Markov dependence. Sarhan and Abouammoh [22] investigate the reliability of the non repairable *k-out-of-n* system with non identical elements subject to independent and common shock. Dutuit and Rauzy [7] study the performance of binary decision diagram for all *k-out-of-n* systems and propose a new approximation scheme. Krishnamoorthy and Ushakumarti [17] obtain the system state distribution, system reliability and several other measures of performance for a *k-out-of-n:G* system with repair under D-policy. Gupta [11] calculates reliability function and the failure rate of the *k-out-of-n* system, with and without incorporating the environmental effect. Cui [6] presents a bound for n_k for which the system does not preserve IFR, when $n > n_k$. Arulmozhi [1] propose an expression for reliability of *k-out-of-n:G* system and develop an algorithm for computing reliability of *k-out-of-n* system. Yam Zuo [24] derive the state transition probabilities of the repairable circular consecutive *k-out-of-n:F* system with one repairman. Milczek [20] presents a class of limit reliability functions of homogeneous series *k-out-of-n* systems. Chen [4] develops a method for analyzing component reliability and system reliability of *k-out-of-n* systems with independent and identically distributed components based on system lifetime data. Koucky [16] deals with reliability of general *k-out-of-n* systems whose component failures need not be independent and identically distributed and the gives the approximations for the system reliability. Arulmozhi [2] gives simple and efficient computational method for determining the system reliability of *k-out-of-n* systems having unequal and equal reliabilities for components. Smith-Destombes [23] present both an exact and an approximate approach to analyze a *k-out-of-n* system with identical and repairable components. Flynn and Chung [8] present a heuristic algorithm for determining replacement policies in a discrete-time, infinite horizon, dynamic programming model of a binary coherent system with *n* statistically independent components, and then specialize the algorithm to consecutive *k-out-of-n* systems. Hsieh and Chen [19] present a simple formula for the reliability lower bound of the two-dimensional consecutive *k-out-of-n:F* system. Jalali Hawkes [14] prove two theorems for the optimal consecutive *k-out-of-n:G* line with $n \le 2k$. Da Costa Bueno [5] defines minimal standby redundancy and use the reverse rule of order $2(RR_2)$ property among compensator processes to investigate the problem of where to allocate a spare in a *k-ut-of-n:F* system and dependent components through minimal standby redundancy. Li Zuo [23] derive formulas for various reliability indices of the *k-out-of-n* system with independent exponential components including mean time between failures and some other parameters. Guan and Wu [10] study a repairable consecutive *k-out-of-n:F* system with fuzzy states.

Multi state models: Moustafa [21] using the markov method, develop a closed form availability solution for two *k-out-of-n* systems with M failure modes. Huang and Zuo [13] investigate two types of multi-state *k-out-of-n:G* systems; i.e., increasing and decreasing systems. The authors develop an analytical model on the properties of the binary-state *k-out-of-n* system. Jenab and Dhillon [15] present a flow-graph-based approach to analyze a multi state *k-out-of-n:G/F/* load-sharing systems. The multi-state *k-out-of-n:G/F/* load-sharing systems comprise ‘*n*’ identical units, that are under state monitoring and recovery function.

Table 1. Classification of on *k-out-of-n* systems

Unit Method	*k-out-of-n* Configuration	Achievement	Reference
Binary	Consecutive F	Exact expression for the failure probability	[2]
	Consecutive G	Matrix formulation and solution	[3]
	Repairable F	Priority repair rule	[5]
	F/G	Makes use of the components subject to common shocks	[6]
	Consecutive F/G	Computational algorithm	[7]
	G	Compute the optimal D value in D-Policy	[8]
	F/G	Consider the environmental effects	[9]
	Consecutive F	Presents a bound for n_k	[10]
	G	Computational algorithm	[11]
	Repairable Circular Consecutive F	Matrix formulation and solution	[12]
	F/G	Presents a limit reliability function of homogeneous series	[13]
	F/G	Develops a data completion procedure	[14]
	F/G	Achieves exact formulas and bounds	[15]
	G	Computational algorithm	[16]
	F/G	Presents exact and approximate approach	[17]
	Consecutive F/G	Presents a heuristic algorithm for replacement policies	[18]
	Consecutive F	Presents a formula for two-dimensional lower bound	[19]
	Consecutive G	Proves some theorems	[20]
	F	Investigates where to allocate the spares	[21]
	F/G	Derives some formulas	[23]
	Consecutive F	Assumes that the states are fuzzy	[24]
Multi-stale	G	Makes use of the markov chain	[1]
	G	Presents an analytic model	[4]
	F/G	Approves analytical approach	[22]

Here, we work on a system with n parallel and identical binary elements with increasing failure rates for real time conditions. The system works until more than

$(n-k)$ elements fail (less than k elements work). The remainder the paper is organized a fallow. Section 2 explores the models. Numerical examples are presented in the Section 3 and the final section deals with the conclusions.

2. Modeling

Assume a system with n parallel and identical elements. The system works until at least k elements work. Therefore, each element has two states and consequently the system will have 2^n states. Let $A_1A_2...A_k$ be the state where the elements A_1, A_2,…, and A_k are working and other $(n-k)$ elements fail. Also $A_1A_2...A_k\eta_j$ is the state that the element η_j works in addition to other k elements. The state O indicates that all the elements fail. The state structure of the system is shown in Figure1. In this figure, the category i, $i=0,1,2,...,n$, indicates that the states in this category have i working elements and $(n-i)$ failed elements. Each state is closely related to the states in the antecedent and precedent category; i.e., if an element is fails in any state, then the state is transferred to the next category. In other words, if the system is in any states in category k, with the failure of one element, the state will be in category $(k-1)$.

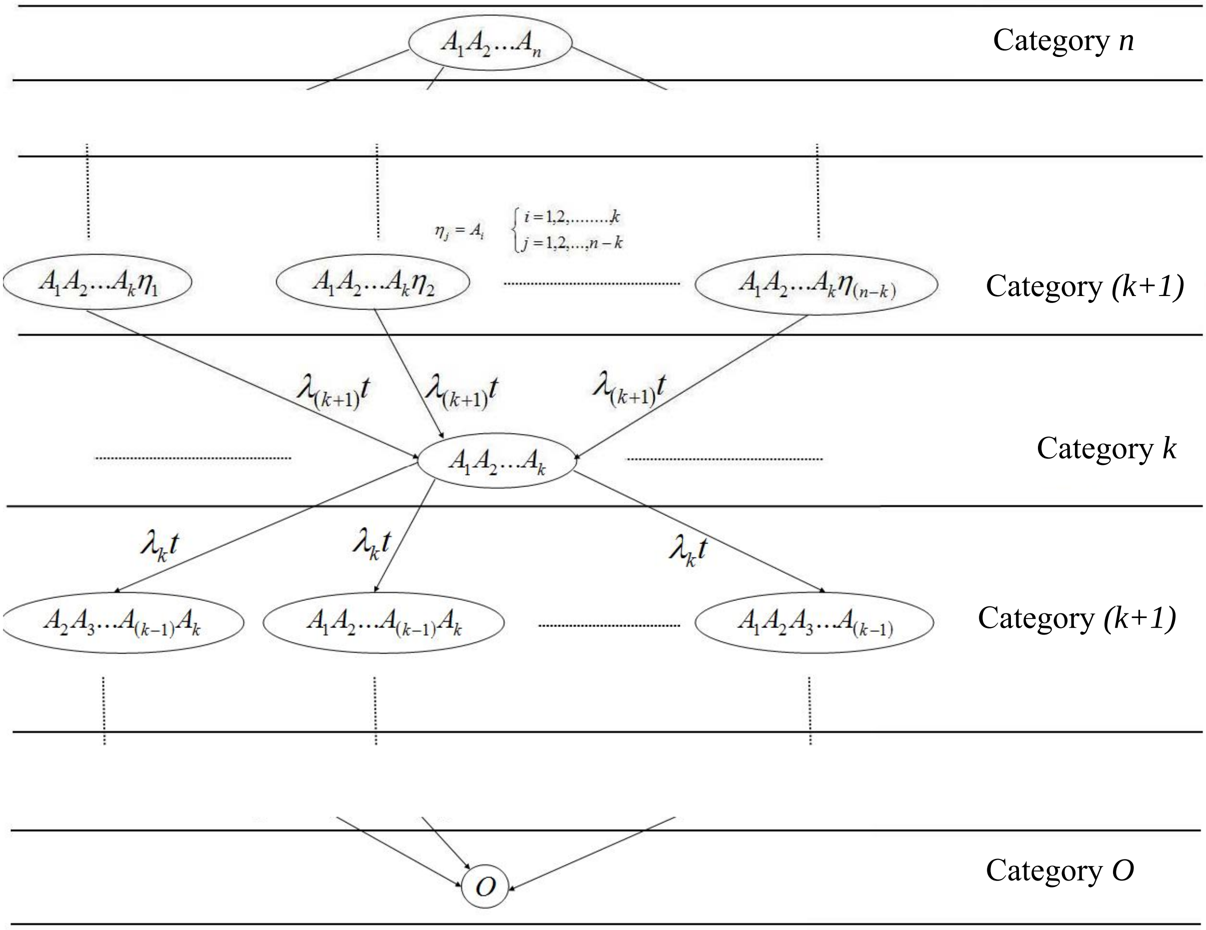

Figure 1: The structure of the model

When some elements fail, other elements work with more load and therefore the failure rate is increased for the remaining elements. Then, we have $\lambda_n < \lambda_{(n-1)} < \ldots < \lambda_1$. The system works if at least k elements work. Therefore, we have:

$$R_P(t) \;=\; \sum_{\substack{i=k \\ 1 \le A_1 < A_2 < \ldots < A_i \le n}}^{n} P_{A_1A_2\ldots A_i}(t). \tag{1}$$

We know,

$$\sum_{\substack{i=1 \\ 1 \le A_1 < A_2 < \ldots < A_i \le n}}^{k-1} P_{A_1A_2\ldots A_i}(t) \;+\; \sum_{\substack{i=k \\ 1 \le A_1 < A_2 < \ldots < A_i \le n}}^{n} P_{A_1A_2\ldots A_i}(t) = 1. \tag{2}$$

The first part of equation(2) is related to states with less than k elements working and the second part deals with two states with at least k elements working. In order to find $R_P(t)$ from equation(2), we must calculate $P_i(t)$ for each state. From the state $A_1A_2...A_n$ through O in Figure 1, we have,

$$P_{A_1A_2...A_k}(t+\Delta t)=P_{A_1A_2...A_k}(t)\ -\sum_{i=1}^{k}\lambda_k\times t\times\Delta t\times P_{A_1A_2...A_k}(t)\ +\sum_{\substack{j=1\\ \eta_j\neq A_i\\ i=1,2,...,k}}^{n}\lambda_{(k+1)}\times t\times\Delta t\times P_{A_1A_2...A_k\eta_j}(t) \quad (3)$$

Where $P_{A_1A_2...A_k}(t)$ is the probability that the system is in the state $A_1A_2...A_k$ at time t.

Also $\sum_{\substack{j=1\\ \eta_j\neq A_i\quad i=1,2,...,k}}^{n}\lambda_{(k+1)}\times\Delta t\times t\times P_{A_1A_2...A_k\eta_j}(t)$ is the rate of transfer from category $(k+1)$ to this state and $\sum_{i=1}^{k}\lambda_k\times t\times\Delta t\times P_{A_1A_2...A_k}(t)$ is the rate of transfer from this state to the states of category $(k-1)$ at time Δt. By solving equation (3), we can calculate the values of $P_i(t)$ as follows:

$$\begin{cases} P_{A_1A_2...A_n}(t)=e^{-\frac{n}{2}\times\lambda_n\times t^2} & k=n\\ P_{A_1A_2...A_k}(t)=(n-k)\ !\times\left(\prod_{v=k+1}^{n}\lambda_v\right)\times\sum_{j=k}^{n}\left[\left(\prod_{\substack{\theta=k\\ \theta\neq j}}^{n}\frac{1}{\theta\times\lambda_\theta-j\times\lambda_j}\right)\times e^{-\frac{j}{2}\times\lambda_j\times t^2}\right] & k<n,\end{cases} \quad (4)$$

and,

$$\begin{cases} R_p(t)=e^{-\frac{n}{2}\times\lambda_n\times t^2} & k=n\\ R_p(t)=\sum_{\substack{i=k\\ 1\le A_1<A_2<...<A_i\le n}}^{n}P_{A_1A_2...A_i}(t)=\left(\prod_{j=k}^{n}\lambda_j\right)\times\sum_{i=k}^{n}\left[\frac{n!}{i}\left(\prod_{\substack{\theta=1\\ \theta\neq i}}^{n}\frac{1}{\theta\times\lambda_\theta-i\times\lambda_i}\right)\times\frac{e^{-\frac{i}{2}\times\lambda_i\times t^2}}{\lambda_i}\right] & k<n,\end{cases} \quad (5)$$

Solution of this equation is provided in Appendix 1. The $MTTF$ of the system is also calculated as follows:

$$MTTF=\int_{t=0}^{+\infty}P(t)\ dt=\begin{cases}\sqrt{\frac{\pi}{2\times n\times\lambda_n}} & k=n\\ \left(\prod_{j=k}^{n}\lambda_j\right)\times\sum_{i=k}^{n}\left[\frac{n!}{i\times\lambda_i}\left(\prod_{\substack{\theta=k\\ \theta\neq i}}^{n}\frac{1}{\theta\times\lambda_\theta-i\times\lambda_i}\right)\sqrt{\frac{\pi}{2\times i\times\lambda_i}}\right] & k<n.\end{cases} \quad (6)$$

When a component fails in each category, the other components must work harder and the failure rate of these components increases. We can calculate the failure rate of each category as follows:

$$\lambda_k=\frac{n}{n-\gamma(n-k)}\lambda_n, \quad (7)$$

Where, $0 \le \gamma \le 1$. If $\gamma = 0$, then the failure rates are equal and constant and if $\gamma = 1$, then $\lambda_k = \frac{n}{k}\lambda_n$.

3. A Numerical Example

In this example, we consider a *k-out-of-3* system. Assume that λ_3 is the failure rate of all elements in category 3. Let $\gamma = 0.5$ and $\beta = 0.05$, and based on an independent sample, let the intervals between consequent failures (in state 123) be measured with the result $\{135,605,195,160,325\}$, and the expected value of the failure time intervals, $\overline{X^2} = \left(135^2 + 605^2 + 195^2 + 160^2 + 325^2\right)/5 = 110700$. Also, $\lambda_3 = 1/110700 = (9.33E-6)$, the states of the system are shown in Figure 2:

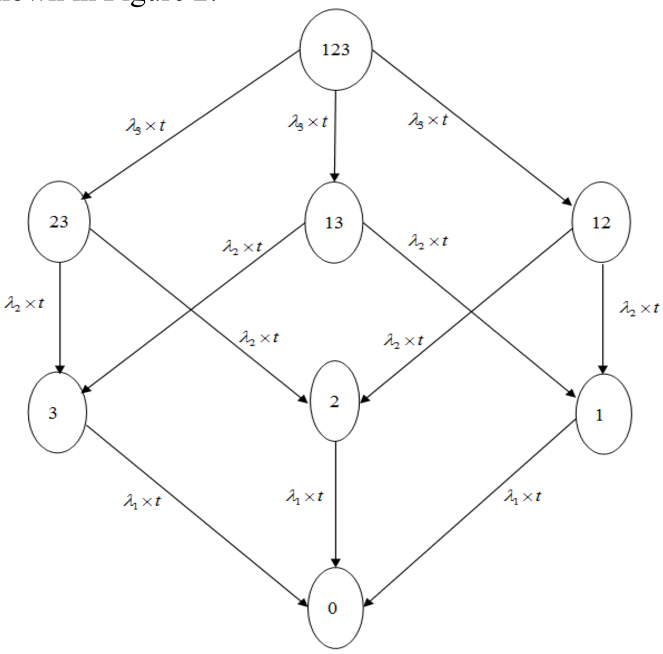

Figure 2: The states of Example 1

If $k = 2$, then the $R_p(t)$ and $MTTF$ are calculated as follows:

$$R_p(t) = \sum_{\substack{i=2 \\ 1 \le A_1 < A_2 < \ldots < A_i \le n}}^{3} P_{A_1 A_2 \ldots A_i}(t) = \left(\prod_{j=2}^{3} \lambda_j\right) \times \sum_{i=2}^{3}\left[\frac{3!}{i}\left(\prod_{\substack{\theta=2 \\ \theta \ne i}}^{3} \frac{1}{\theta \times \lambda_\theta - i \times \lambda_i}\right) \times \frac{e^{-\frac{i}{2} \times \lambda_i \times t^2}}{\lambda_i}\right]$$

$$= (\lambda_2 \times \lambda_3) \times \left[\frac{3!}{2}\left(\prod_{\substack{\theta=2 \\ \theta \ne i}}^{3} \frac{1}{\theta \times \lambda_\theta - 2 \times \lambda_2}\right) \times \frac{e^{-\frac{2}{2} \times \lambda_2 \times t^2}}{\lambda_2} + \frac{3!}{3}\left(\prod_{\substack{\theta=2 \\ \theta \ne i}}^{3} \frac{1}{\theta \times \lambda_\theta - 3 \times \lambda_3}\right) \times \frac{e^{-\frac{3}{2} \times \lambda_3 \times t^2}}{\lambda_3}\right] \qquad (8)$$

$$= \frac{3\lambda_3}{3\lambda_3 - 2\lambda_2} e^{-\lambda_2 \times t^2} + \frac{2\lambda_2}{2\lambda_2 - 3\lambda_3} e^{-\frac{3}{2} \times \lambda_3 \times t^2}$$

$$MTTF = \int_{t=0}^{+\infty} P(t)dt = \left(\prod_{j=2}^{2} \lambda_j\right) \times \sum_{i=2}^{3} \left[\frac{3!}{i \times \lambda_i} \left(\prod_{\substack{\theta=2 \\ \theta \neq i}}^{3} \frac{1}{\theta \times \lambda_\theta - i \times \lambda_i} \right) \sqrt{\frac{\pi}{2 \times i \times \lambda_i}} \right]$$

$$(\lambda_2 \times \lambda_3) \times \left[\frac{3!}{2 \times \lambda_2} \times \left(\frac{1}{3\lambda_3 - 2\lambda_2} \right) \sqrt{\frac{\pi}{4 \times \lambda_2}} + \frac{3!}{3 \times \lambda_3} \times \left(\frac{1}{2\lambda_2 - 3\lambda_3} \right) \sqrt{\frac{\pi}{6 \times \lambda_3}} \right] \tag{9}$$

$$= \frac{3 \times \lambda_3}{3\lambda_3 - 2\lambda_2} \sqrt{\frac{\pi}{4 \times \lambda_2}} + \frac{2 \times \lambda_2}{2\lambda_2 - 3\lambda_3} \sqrt{\frac{\pi}{6 \times \lambda_3}} = \frac{1}{3\lambda_3 - 2\lambda_2} \left(3 \times \lambda_3 \sqrt{\frac{\pi}{4 \times \lambda_2}} - 2 \times \lambda_2 \sqrt{\frac{\pi}{6 \times \lambda_3}} \right).$$

The value of λ_2 is:

$$\lambda_2 = \frac{3}{3 - 0.5(3-2)} \lambda_3 = 1.2\lambda_3 .$$

Then:

$$MTTF = \frac{1}{3\lambda_3 - 2\lambda_2} \left(3 \times \lambda_3 \sqrt{\frac{\pi}{4 \times \lambda_2}} - 2 \times \lambda_2 \sqrt{\frac{\pi}{6 \times \lambda_3}} \right) = 382.83, \tag{10}$$

and $R_p(250)$ is:

$$R_p(250) = \frac{3\lambda_3}{3\lambda_3 - 2\lambda_2} e^{-\lambda_2 \times 250^2} + \frac{2\lambda_2}{2\lambda_2 - 3\lambda_3} e^{-\frac{3}{2} \times \lambda_3 \times 250^2}$$
$$= 5e^{-0.6775} - 4e^{-0.8469} = 0.8244. \tag{11}$$

4. Conclusions

We discussed a real time *k-out-of-n* system, where the failure rate of the elements is increasing. That is, by failure of one element, the load on the remaining elements increases the chance of failure of the remaining elements. Necessary relations were developed for the failure rates. The contribution of the present study can be summarized as:

1) The model was considered in real time, whereas in almost all the preceding works.
2) The model is considered to be more appropriate for sophisticated systems.
3) Logical relationship existing between failure rates of two consecutive states could be accordingly tuned.

As an extension to this work, the failure rates may be considered differently for each element. In this case, either a system of *k-out-of-n* model should be taken into consideration or for the *k-out-of-n* model, a certain policy should be developed when the whole system fails. Moreover, the elements may be repairable. Hence, the cost for procurement, repair of the elements and the failure of the whole system may be taken into account in such a way that the optimum number of elements with minimum cost and maximum reliability will be determined. These variations could be studied for the systems with increasing failure rates of the elements as well.

Appendix 1: Solution of Equation (5)

In a *k-out-of-n* system, for the state in category n, we have,

$$P_{12\ldots n}(t+\Delta t)=P_{12\ldots n}(t)-n\times\lambda_n\times t\times\Delta t\times P_1(t) \tag{12}$$

$$\Rightarrow Lim_{\Delta t\to 0}\frac{P_{12\ldots n}(t+\Delta t)-P_{12\ldots n}(t)}{\Delta t}=\frac{P'_{12\ldots n}(t)}{P_{12\ldots n}(t)}=-n\times\lambda_n\times t\Rightarrow\left\{\begin{array}{l}P_1(t)=e^{-\frac{n}{2}\times\lambda_n\times t^2}+c\\ P_1(0)=1\end{array}\right\}\Rightarrow P_{12\ldots n}(t)=e^{-\frac{n}{2}\times\lambda_n\times t^2}, \tag{13}$$

Also, for the states in category $(n-1)$, we have,

$$P_{A_1A_2\ldots A_{n-1}}(t+\Delta t)=P_{A_1A_2\ldots A_{n-1}}(t)+\lambda_n\times t\times\Delta t\times P_{12\ldots n}(t)-(n-1)\times\lambda_{(n-1)}\times t\times\Delta t\times P_{A_1A_2\ldots A_{n-1}}(t) \tag{14}$$

$$\begin{aligned}&\Rightarrow Lim_{\Delta t\to 0}\frac{P_{A_1A_2\ldots A_{n-1}}(t+\Delta t)-P_{A_1A_2\ldots A_{n-1}}(t)}{\Delta t}=P'_{A_1A_2\ldots A_{n-1}}(t)+(n-1)\times\lambda_{(n-1)}\times t\times P_{A_1A_2\ldots A_{n-1}}(t)\\&=\lambda_n\times t\times P_{12\ldots n}(t)=\lambda_n\times t\times e^{-\frac{n}{2}\times\lambda_n\times t^2}\end{aligned} \tag{15}$$

$$e^{\frac{(n-1)}{2}\times\lambda_{(n-1)}\times t^2}\times P'_{A_1A_2\ldots A_{n-1}}(t)+e^{\frac{(n-1)}{2}\times\lambda_{(n-1)}\times t^2}\times(n-1)\times\lambda_{(n-1)}\times t\times P_{A_1A_2\ldots A_{n-1}}(t)=\lambda_n\times e^{\left[-\frac{n}{2}\times\lambda_n+\frac{(n-1)}{2}\times\lambda_{(n-1)}\right]\times t^2} \tag{16}$$

$$\frac{d}{dt}\left[e^{\frac{(n-1)}{2}\times\lambda_{(n-1)}\times t^2}\times P_{A_1A_2\ldots A_{n-1}}(t)\right]=\lambda_n\times e^{\left[-\frac{n}{2}\times\lambda_n+\frac{(n-1)}{2}\times\lambda_{(n-1)}\right]\times t^2} \tag{17}$$

$$\Rightarrow e^{\frac{(n-1)}{2}\times\lambda_{(n-1)}\times t^2}\times P_{A_1A_2\ldots A_{n-1}}(t)=\frac{\lambda_n}{(n-1)\times\lambda_{(n-1)}-n\times\lambda_n}\times e^{\left[-\frac{n}{2}\times\lambda_n+\frac{(n-1)}{2}\times\lambda_{(n-1)}\right]\times t^2}+C \tag{18}$$

$$\Rightarrow\left.\begin{array}{l}P_{A_1A_2\ldots A_{n-1}}(t)=\dfrac{\lambda_n}{(n-1)\times\lambda_{(n-1)}-n\times\lambda_n}\times e^{-\frac{n}{2}\times\lambda_n\times t^2}+C\times e^{-\frac{(n-1)}{2}\times\lambda_{(n-1)}\times t^2}\\ P_{A_1A_2\ldots A_{n-1}}(0)=0\end{array}\right\}\Rightarrow C=\frac{-\lambda_n}{(n-1)\times\lambda_{(n-1)}-n\times\lambda_n} \tag{19}$$

And the result is:

$$P_{A_1A_2\ldots A_{n-1}}(t)=\frac{\lambda_n}{(n-1)\times\lambda_{(n-1)}-n\times\lambda_n}\times\left[e^{-\frac{n}{2}\times\lambda_n\times t^2}-e^{-\frac{(n-1)}{2}\times\lambda_{(n-1)}\times t^2}\right], \tag{20}$$

And for the states in category $(k+1)$,

$$P_{A_1A_2\ldots A_{k+1}}(t)=(n-k-1)!\times\left(\prod_{v=k+2}^{n}\lambda_v\right)\times\sum_{j=k}^{n}\left[\left(\prod_{\substack{\theta=k+1\\ \theta\neq j}}^{n}\frac{1}{\theta\times\lambda_\theta-j\times\lambda_j}\right)\times e^{-\frac{j}{2}\times\lambda_j\times t^2}\right], \tag{21}$$

We can solve the differential equation of the states in category k as follows:

$$P_{A_1A_2\ldots A_k}(t+\Delta t)=P_{A_1A_2\ldots A_k}(t)-\sum_{i=1}^{k}\lambda_k\times t\times\Delta t\times P_{A_1A_2\ldots A_k}(t)+\sum_{\substack{j=1\\ \eta_j\neq A_i\ \ i=1,2,\ldots k}}^{n}\lambda_{(k+1)}\times t\times\Delta t\times P_{A_1A_2\ldots A_k\eta_j}(t) \tag{22}$$

$$\Rightarrow \lim_{\Delta t\to\infty}\frac{P_{A_1A_2\ldots A_k}(t+\Delta t)-P_{A_1A_2\ldots A_k}(t)}{\Delta t}=P'_{A_1A_2\ldots A_k}(t)=\sum_{i=1}^{k}\lambda_k\times t\times P_{A_1A_2\ldots A_k}(t)+\sum_{\substack{j=1\\ \eta_j\neq A_i\ \ i=1,2,\ldots k}}^{n}\lambda_{(k+1)}\times t\times P_{A_1A_2\ldots A_k\eta_j}(t) \tag{23}$$

$$P'_{A_1A_2\ldots A_k}(t)-\left(\sum_{i=1}^{k}\lambda_k\times t\right)P_{A_1A_2\ldots A_k}(t)=\sum_{\substack{j=1\\ \eta_j\neq A_i\ \ i=1,2,\ldots,k}}^{n}\lambda_{(k+1)}\times t\times P_{A_1A_2\ldots A_k\eta_j}(t) \tag{24}$$

$$\Rightarrow e^{\frac{\left(\sum_{i=1}^{k}\lambda_k\right)}{2}\times t^2}\times P'_{A_1A_2\ldots A_k}(t)-e^{\frac{\left(\sum_{i=1}^{k}\lambda_k\right)}{2}\times t^2}\times\left(\sum_{i=1}^{k}\lambda_k\times t\right)P_{A_1A_2\ldots A_k}(t)=e^{\frac{\left(\sum_{i=1}^{k}\lambda_k\right)}{2}\times t^2}\times\sum_{\substack{j=1\\ \eta_j\neq A_i\ \ i=1,2,\ldots k}}^{n}\lambda_{(k+1)}\times t\times P_{A_1A_2\ldots A_k\eta_j}(t). \tag{25}$$

Also, for all values of η_j, we know that $P_{A_1A_2\ldots A_k\eta_j}(t)=P_{A_1A_2\ldots A_{k+1}}(t)$, and so we have,

$$\begin{cases}\dfrac{d}{dt}\left[e^{\frac{\left(\sum_{i=1}^{k}\lambda_k\right)}{2}\times t^2}\times P_{A_1A_2\ldots A_k}(t)\right]=e^{\frac{\left(\sum_{i=1}^{k}\lambda_k\right)}{2}\times t^2}\times(n-k)\times t\times P_{A_1A_2\ldots A_{k+1}}(t)\\[2ex] P_{A_1A_2\ldots A_{k+1}}(t)=(n-k-1)!\times\left(\prod_{v=k+2}^{n}\lambda_v\right)\times\sum_{j=k+1}^{n}\left[\left(\prod_{\substack{\theta=k+1\\ \theta\neq j}}^{n}\dfrac{1}{\theta\times\lambda_\theta-j\times\lambda_j}\right)\times e^{-\frac{j}{2}\times\lambda_j\times t^2}\right].\end{cases} \tag{26}$$

And,

$$e^{\frac{\left(\sum_{i=1}^{k}\lambda_k\right)}{2}\times t^2}\times P_{A_1A_2\ldots A_k}(t)=\int e^{\frac{\left(\sum_{i=1}^{k}\lambda_k\right)}{2}\times t^2}\times(n-k)\times(n-k-1)!\times\left(\prod_{v=k+2}^{n}\lambda_v\right)\times\sum_{j=k+1}^{n}\left[\left(\prod_{\substack{\theta=k+1\\ \theta\neq j}}^{n}\frac{1}{\theta\times\lambda_\theta-j\times\lambda_j}\right)\times e^{-\frac{j}{2}\times\lambda_j\times t^2}\right]dt. \tag{27}$$

We subsituate the initial value of $P_{A_1A_2\ldots A_k}(0)=0$, and so the result is:

$$P_{A_1A_2\ldots A_k}(t)=(n-k)!\times\left(\prod_{v=k+1}^{n}\lambda_v\right)\times\sum_{j=k}^{n}\left[\left(\prod_{\substack{\theta=k\\ \theta\neq j}}^{n}\frac{1}{\theta\times\lambda_\theta-j\times\lambda_j}\right)\times e^{-\frac{j}{2}\times\lambda_j\times t^2}\right]. \tag{28}$$

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