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A non-smooth multi-objective model for hub location problem

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In this paper, we consider a multi-objective hub location problem (MOHLP) to locate two constrained facilities in order to minimize the distance between these facilities and the weighted distance between each facility and related customers. For this purpose, we establish a necessary and sufficient condition of optimality for finding an efficient solution of the problem. We show that MOHLP can be reduced to a simple bi-level distance problem. Then we develop an efficient algorithm to find the optimal solution set of BDP, and provide its convergence without any assumption. Moreover, an algorithm is proposed to solve MOHLP, which converges in a finite number of iterations. Some examples are stated to clarify the proposed algorithms.

***Keywords**: Convex Analysis, Hub, Location, Multi-Objective.*

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1. Introduction

Facility location is a fundamental problem in computer science, industrial engineering and operations research, which is referred to find an optimal placement of some locations (centers or hubs) among a set of demand points (customers or clients). Based on the properties of the potential facility locations and demand points, many variations of the problem may arise, e.g., constrained or unconstrained, one objective or multi-objective, rectilinear distance or Euclidean distance. There are many different kinds of facility location problems for which various methods have been proposed; see [6,33] and references therein.

Rectilinear distances are applicable when travel is allowed only in two perpendicular directions such as North–South and East–West. This distance metric is also popular among researchers because the analysis is usually simpler than the other metrics; see [11]. The rectilinear distance is also called the Taxicab Norm distances, because it is the distance a car would drive in a city layout in square blocks (if there are no one-way streets). The rectilinear distance has widely been used in facility location problems in [2,11,15,16,17,19,20,22,24,28,31,32].

Hub Location Problems lie at the heart of network design planning in transportation and telecommunication systems. They constitute a challenging class of optimization problems that focus on the location of hub facilities and on the design of hub networks. There are numerous variants of the hub location problems such as hub network topologies, flow-dependent discounted costs, capacitated models, uncertainty, dynamic and multi-modal models, and competition and collaboration [4,5,7,8,12]. Okelly has proposed in [23] the first quadratic integer programming formulation for the classic uncapacitated single allocation p-hub median problem. The hub

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location problem with profit-oriented objectives has been studied in [5] that measure the trade-off between the revenue derived from served commodities and the overall network design and flow costs. A mixed integer programming model has presented in [30] to formulate a profit-maximizing hub location problem with the service capacity and flow capacity constraints.

Many real-world multi-objective facility location problems are evaluated with multiple, often conflicting criteria or objective functions. When there is no a priori information about the importance of each objective, the solutions to such a multi-objective optimization problem are usually compared in terms of an efficient solutions. There is also a growing tendency that, in many real-world, decision-makers are likely to pursue multiple objectives to achieve the efficient utilization of available resources [1,13,14,26,29]. This trend transforms the problem into a multi-objective facility location problem with objectives that may occasionally conflict with one another. Hence by motivating the mentioned practical facility location examples and considering the wide applications of the multi-objective problem, we focus on a multi-objective location problem. In [14], a temporary emergency service center has presented for a natural gas distribution company. A mixed-integer programming model has developed in [1] and the scenario production method is used to solve this stochastic model.

Table 1 shows various kinds of locations problems, the distance functions used to solve them and their developers.

Table 1. The literature of the developed location model.

Year	Problem	Distance	Constraint	References
2021	Multi-objective capacitated location-routing problem	Stochastic	Unconstrained	[1]
2021	The rectilinear barrier Weber location problem	Rectilinear	Unconstrained	[3]
2016	The impact of hub network	Without distance	Unconstrained	[4]
2011	Inverse p-median problem	Euclidean	Unconstrained	[6]
1996	Uncapacitated single allocation p-hub median problem	Euclidean	Unconstrained	[12]
2018	Constrained rectilinear distance location problem	Rectilinear	Constrained	[21]
2017	Generalized constrained multi-source Weber problem	Euclidean	Constrained	[22]
2009	Location and relocation problem	Euclidean	Unconstrained	[31]

Due to the wide applications of location problems in operations research, marketing, urban planning, etc., we formulate a new mathematical multi-objective model of transportation problems by combining the location and the hub problems. Consider the locations of two clusters of customers and their demands, the multi-objective hub location problem (MOHLP) is concerned with locating two hubs in the specific regions of Euclidean plane and allocating them to the demand customers in order to satisfy the distance between the hubs and their demand at the minimum total cost. MOHLP can be applied to many problems, for instance, in locating a school, a warehouse, a post office, a fire station, a hospital, an ambulance station, and so on. As an example, MHOLP can used to build the free trade zones in two neighboring countries, so that their interests are guaranteed.

Most of the location problems studied in the literature have no restriction on the location; see [1,2,4,5,8,10,11,14,30]. Although in the presence of the restricted location for the facility (see [21,22]) has more practical relevance than the unconstrained case, it has not been given much attention until lately. The choice of a suitable constraint plays a crucial role for a reasonable estimation of MOHLP in realistic environments. A city is usually partitioned into several blocks (or boxes), and the city managers may decide to locate the new facility in a determined block in order to minimize the sum of transportation costs between the new facility and the customers. Hence it is more practical that the new facility is to be placed in a box. Hence we consider the MHOLP with box constraints.

We present a necessary and sufficient condition to find a subset of the efficient solution set (or the set of all optimal solutions) of MOHLP by using the convex analysis tools. The presented necessary and sufficient condition enables us to find a subset of the efficient solutions of MOHLP between all feasible solutions simply and effectively. Moreover, it leads to design an efficient algorithm. The advantages of the proposed algorithm are as follows:

1. It uses a few parameters.
2. It finds a subset of the efficient solutions.
3. It terminates in a finite number of iterations.
4. It is simple and easy to implement.

The paper is organized as follows. In the next section, we provide the preliminary and notation results that will be used in the sequel. Section 3 is devoted to present a necessary and sufficient optimality condition for finding a subset of the efficient solution set of MOHLP. In section 4, we state an efficient algorithm for solving the MOHLP and the convergence results are discussed.

2. Notions and preliminaries

In this section, we present notation and auxiliary results that will be needed in what follows. Let $a,\ b \in \mathbb{R}^n$. We denote the line segment between a and b by $[a,b]$, where,

$$[a,b] = \{ta + (1-t)b \ \mid \ 0 \leq t \leq 1\}.$$

Let Π be a non-empty closed set in $\mathbb{R}^n$. Let y be a point in Π that is closest to $x \in \mathbb{R}^n$; we say that y lies in the projection of x onto Π. Denote by $P_\Pi(x)$ the set of points in Π closest to x, i.e.

$$P_\Pi(x) = \operatorname{argmin}_{s\in\Pi} \| x - s\| = \{y \in \Pi \mid \| x - y\| \leq \| x - s\| \ , \forall s \in \Pi\}.$$

We recall the following results from [9,27]. Let $\hat{x} \in \mathbb{R}^n$ and $\phi : \mathbb{R}^n \to \mathbb{R}$ be a convex function. In place of the gradient, we consider subgradients of $\hat{x}$, those elements ξ of $\mathbb{R}^n$ satisfying:

$$\phi(x)-\phi(\hat{x}) \geq \langle \xi, x-\hat{x} \rangle, \quad \forall x \in \mathbb{R}^n.$$

We denote the set of subgradients of ϕ at $\hat{x}$ (called the classical Fenchel subdifferential of Convex Analysis) by $\partial\phi(\hat{x})$. It is worth mentioning that when ϕ is convex, the most existing subdifferentials coincides with the classical Fenchel subdifferential of Convex Analysis.

Let S be a closed subset of $\mathbb{R}^n$. The normal cone to S at $s \in S$ is defined as:

$$N_S(s) := \{\xi \in \mathbb{R}^n \mid \langle \xi, s' - s \rangle \leq 0, \forall s' \in S\}.$$

The indicator function of S, denoted by $I_S(.)$, the extended-valued function defined by

$$I_S(x) = \begin{cases} 0, & \text{if } x \in S, \\ +\infty, & \text{otherwise.} \end{cases}$$

In the following theorem, we summarize some results, which are used in what follows.

Theorem 2.1. [9, Proposition 2.1, Corollary 1.2.7, Proposition 1.2.11, Exercise 1.10.2] Let $\phi: \mathbb{R}^n \to \mathbb{R}$ and $\psi: \mathbb{R}^n \to \mathbb{R}$ be two convex functions. Then the following assertions hold:

i. For any scalar λ, we have $\partial\lambda\phi(x) = \lambda\,\partial\phi(x)$.

ii. The point $\hat{x}$ is a (global) minimizer of ϕ, if and only if the condition $0 \in \partial\phi(\hat{x})$ holds.

iii. Let ψ be differentiable at $\hat{x}$, then $0 \in \partial(\phi+\psi)(\hat{x})$, if and only if:

$$-\nabla\psi(\hat{x}) \in \partial\phi(\hat{x}).$$

iv. For closed convex set S, $\partial I_S(x) = N_S(x)$.

Consider the following Multi-objective Optimization Problem (MOP):

$$\begin{aligned} \min \quad & f(x) = (f_1(x), f_2(x), \ldots, f_m(x)) \\ & s.t. \; x \in S, \end{aligned} \tag{1}$$

where, $f_i: \mathbb{R}^n \to \mathbb{R}$ are the continuous functions and S is a closed set.

We say $x^* \in S$ is an efficient solution for MOP (1), if there does not exist any $x \in S$ such that

$$f_i(x) \leq f_i(x^*),\ i = 1, 2, \ldots, m,$$

and $f_j(x) < f_j(x^*)$ for at least $j \in \{1, 2, \ldots, m\}$.

3. Model and solution method

The mathematical model of the multi-objective hub location problem is given as follows:

$$\begin{aligned} (\text{MOHLP}):\ \min \quad & \left(f_1(x) := \sum_{j=1}^{d} w_j \| x - a_j \|_1, f_2(y) := \sum_{j=1}^{\bar{d}} \bar{w}_j \| y - \bar{a}_j \|_1, f_3(x, y) := \| x - y \|_2 \right) \\ & x \in X,\ y \in Y, \\ & X \cap Y = \phi, \end{aligned} \tag{2}$$

where,

i. $\| \ \|_1$ isthe rectilinear distance andis Euclidean distance $\| \ \|_2$,

ii. $w_j > 0\ (j \in D := \{1, 2, ..., d\})$, and $\bar{w}_j > 0\ (j \in \bar{D} := \{1, 2, ..., \bar{d}\})$ are the corresponding weights,

iii. $a_j\ (j \in D)$ and $\bar{a}_j\ (j \in \bar{D})$,are the location of customers

iv. $X = \{x \in \mathbb{R}^n \mid \mathbf{l} \leq x \leq \mathbf{u}\}$ and $Y = \{y \in \mathbb{R}^n \mid \bar{\mathbf{l}} \leq y \leq \bar{\mathbf{u}}\}$ are the feasible solution sets, and

v. $x \in X$ $y \in Y$ and are the locations of hubs.

Next we proceed to present a necessary and sufficient condition for finding the efficient solution of MOHLP (2). We consider the following Bi-level Distance Problem (BDP):

$$\begin{aligned} (\text{BDP}):\quad \min \quad & \| x - y \|_2, \\ & x \in \Gamma_1 := \operatorname{argmin}_{x \in X} f_1(x), \\ & y \in \Gamma_2 := \operatorname{argmin}_{y \in Y} f_2(y). \end{aligned} \tag{3}$$

The following theorem presents a relation between MOHLP (2) and BDP (3).

Theorem 3.1. Suppose that (x^*, y^*) is an optimal solution of BDP (3), then (x^*, y^*) is an efficient solution of MOHLP (2).

Proof. On the contrary, suppose that (x^*, y^*) is not an efficient solution of MOHLP (2). Hence there exists $(\hat{x}, \hat{y})$ such that

$$f_1(\hat{x}) \le f_1(x^*), \tag{4}$$
$$f_2(\hat{y}) \le f_2(y^*), \tag{5}$$
$$\| \hat{y} - \hat{x} \|_2 \le \| y^* - x^* \|_2, \tag{6}$$

and one of the inequalities (4), (5) or (6) is strict. If the inequality (4) (or inequality (5)) is strict, then it contradicts to $x^* \in \Gamma_1$ (or $y^* \in \Gamma_2$). Now if $\| \hat{y} - \hat{x} \|_2 \le \| y^* - x^* \|_2$, it contradicts to optimality of (x^*, y^*) for BDP (3). Hence the proof is complete.

We now proceed to describe the solution method for finding the solution sets Γ_1 and Γ_2. Consider the following mathematical location model:

$$\min_{x \in \mathbb{R}^n} f(x) := \sum_{j=1}^{d} w_j \| x - a_j \|_1, \ s.t. \ \mathbf{l} \le x \le \mathbf{u}, \tag{7}$$

where $\mathbf{l} \in \mathbb{R}^n$ and $\mathbf{u} \in \mathbb{R}^n$ are fixed vectors and inequalities are taken componentwise and $w_j \ (j = 1, 2, \ldots, d)$ are positive multipliers. Let $x = (x_1, x_2, \ldots, x_n)^T$ and $a_j = (a_{1j}, a_{2j}, \ldots, a_{nj})^T$ for $j = 1, 2, \ldots, d$. We can rewrite Problem (7) as follows:

$$\min_{x \in \mathbb{R}^n} f(x) := \sum_{i=1}^{n} \sum_{j=1}^{d} w_j \, | x_i - a_{ij} |, \ s.t. \ \mathbf{l}_i \le x_i \le \mathbf{u}_i, i = 1, 2, \ldots, n.$$

Each quantity on the right-hand side can be treated as an independent optimization problem for each i :

$$\min_{x_i \in \mathbb{R}} f_i(x_i) := \sum_{j=1}^{d} w_j \, | x_i - a_{ij} |, \ s.t. \ \mathbf{l}_i \le x_i \le \mathbf{u}_i . \tag{8}$$

Let Γ and $\Gamma^i \ (i = 1, 2, \ldots, n)$ be the optimal solution sets of (7) and (8) respectively. It is easy to see that $\Gamma = \Gamma^1 \times \Gamma^2 \times \ldots \times \Gamma^n$. Therefore, we focus on a method for solving the subproblems (8). Throughout the paper, subscript i in (8) does not need and we consider the simplified model of the subproblems without confusion:

$$\min_{x\in\mathbb{R}} f(x) := \sum_{j=1}^{d} w_j\,|x-a_j|,\ \ s.t.\ \ l \le x \le u. \tag{9}$$

We denote the optimal solution set of BDP (3) as follows:

$$\Omega = \{(x^*, y^*) \mid \|y^* - x^*\|_2 \le \|y - x\|_2,\ x \in \Gamma_1 \text{ and } y \in \Gamma_2\}.$$

To simplify the notion, we denote $\Omega^j = \{(x_j, y_j)\}$. To present our algorithm, we need the following theorem.

Theorem 3.2. The optimal solution set of BDP (3) is as follows. For each j, if $\Gamma_1^j \cap \Gamma_2^j \neq \varnothing$, then

$$\Omega^j = \left\{(t,t) \mid t \in \Gamma_1^j \cap \Gamma_2^j\right\},$$

otherwise

$$\Omega^j = \left\{(t_1, t_2) \mid t_1 \in P_{\Gamma_1^j}(t_2) \text{ and } t_2 \in P_{\Gamma_2^j}(t_1)\right\}.$$

Proof. Based on [21], each Γ_i $(i = 1,2)$ is a box. Without loss of generality, suppose that $\Gamma_1 = \{x \in \mathbb{R}^n \mid \mathbf{a} \le x \le \mathbf{b}\}$ and $\Gamma_2 = \{y \in \mathbb{R}^n \mid \mathbf{c} \le x \le \mathbf{d}\}$. Hence BDP (3) can be rewrite as follows:

$$\begin{aligned}
&\min_{x,y\in\mathbb{R}^n} g(x,y) := \sum_{j=1}^{n} (x_j - y_j)^2,\\
&s.t.\ \ \mathbf{a}_j \le x_j \le \mathbf{b}_j,\ \ j = 1,2,\ldots,n,\\
&\qquad\ \mathbf{c}_j \le y_j \le \mathbf{d}_j,\ \ j = 1,2,\ldots,n.
\end{aligned}$$

Each quantity on the right-hand side can be treated as an independent optimization problem for each j:

$$\min_{x_j, y_j \in \mathbb{R}^n} g_j(x_j, y_j) := (x_j - y_j)^2,$$
$$s.t.\ \mathbf{a}_j,,\ x_j,,\ \mathbf{b}_j, \qquad (10)$$
$$\mathbf{c}_j,,\ y_j,,\ \mathbf{d}_j.$$

Now if $\Gamma_1^j \cap \Gamma_2^j \neq \emptyset,$ then the optimal solution set of Problem (10) is $\Omega^j = \Gamma_1^j \cap \Gamma_2^j.$ Therefore

$$\Omega^j = \{(t,t) \mid t \in \Gamma_1^j \cap \Gamma_2^j\}.$$

Next consider the other case, when $\Gamma_1^j \cap \Gamma_2^j = \emptyset.$ Suppose that (t_1,t_2) is the optimal solution of Problem (10). Hence (t_1,t_2) is the optimal solution of the following unconstrained problem:

$$\min_{x \in \mathbb{R}^n} h_j(x_j, y_j) := (x_j - y_j)^2 + I_{\Gamma_1^j}(x_j) + I_{\Gamma_2^j}(y_j),$$
$$s.t.\ x_j,\ y_j \in \mathbb{R}.$$

By Theorem 2.1, we have

$$0 \in \partial h_j(.)(t_1, t_2).$$

It follows from Theorem 2.1 that

$$2(t_2 - t_1, t_1 - t_2)^T \in \partial I_{\Gamma_1^j}(t_1) + \partial I_{\Gamma_2^j}(t_2).$$

By the assumption, we obtain $t_1 \neq t_2$ and

$$t_2 - t_1 \in N_{\Gamma_1^j}(t_1),$$
$$t_1 - t_2 \in N_{\Gamma_2^j}(t_2).$$

Hence

$$\Omega^j = \left\{ (t_1, t_2) \mid t_1 \in P_{\Gamma_1^j}(t_2) \text{ and } t_2 \in P_{\Gamma_2^j}(t_1) \right\},$$

4. Algorithm

In this section, we present an algorithm to find the optimal solution set of BDP (3). Also the global convergence of the proposed method is proved in linear time. Moreover, an algorithm is proposed for solving MOHLP (2) in $O(m \log m)$ time where $m = \max\{d, \bar{d}\}$.

A detailed description of the algorithm for solving BDP (3) states as follows.

Algorithm 1.

Input: Two boxes $\Gamma_1 := \left\{ x \in \mathbb{R}^n \mid \mathbf{a} \le x \le \mathbf{b} \right\}$ and $\Gamma_2 := \left\{ x \in \mathbb{R}^n \mid \mathbf{c} \le x \le \mathbf{d} \right\}$.

Output: The set of minimizers of distance between Γ_1 and Γ_2.

For $i = 1, 2, \ldots, n$ **do:**

set $\mathbf{e}_i := \min\{\mathbf{a}_i, \mathbf{c}_i\}$,

W.L.G. suppose that $\mathbf{e}_i = \mathbf{a}_i$,

if $\mathbf{b}_i \le \mathbf{c}_i$, **then** $\Omega^i = \{(b_i, c_i)\}$,

else if $\mathbf{b}_i \le \mathbf{d}_i$, **then** $\Omega^i = \left\{ (t, t) \mid t \in [\mathbf{c}_i, \mathbf{b}_i] \right\}$,

else $\Omega^i = \left\{ (t, t) \mid t \in [\mathbf{c}_i, \mathbf{d}_i] \right\}$.

Theorem 4.1. Algorithm 1 runs in $O(n)$ time.

Proof. The proof follows from Theorem 3.2.

The overall solution method for MOHLP (2) can be outlined as follows.

Algorithm 2.

Input: The number of customers d. The location of customers $a_j \in D$ and $\bar{a}_j \in \bar{D}$, and the positive multipliers $w_j \in D$ and $\bar{w}_j \in \bar{D}$, and the feasible solution sets X and Y.

Output: The subset of the efficient solution set Ω of MOHLP (2).

Step1. Use the presented algorithm in [21] for finding the solution sets Γ_1 and Γ_2.

Step 2. Use Algorithm 1 for finding the set of minimum distance between two boxes Γ_1 and Γ_2.

Theorem 4.2. The output set Ω generated by Algorithm 2 is a subset of the efficient solution set of MOHLP (2). Moreover, if $n \le m$, then Algorithm 2 terminates after at most n iterations, and can be implemented to run in $O(m \log m)$ time.

Proof. The first part of the proof follows from Theorem 3.1. Now consider the second part of the proof. By [21], Step 1 runs in $O(d \log d) + O(\bar{d} \log \bar{d})$. Indeed the overall running time for Step 1 is $O(m \log m)$ On the other hand, Step 2 runs in $O(n)$ time. Since $n \le m$, it follows that the overall running time is $O(m \log m)$.

To make these algorithms clear, the following example is provided.

Example 4.1. Assume in problem (2) that

$$
\begin{aligned}
&a_1^T = (0,0),\ a_2^T = (0,2),\ a_3^T = (2,2),\ a_4^T = (2,0),\\
&w_1 = w_2 = w_3 = w_4 = 1,\\
&\bar{a}_1^T = (0,0),\ \bar{a}_2^T = (0,2),\ \bar{a}_3^T = (2,2),\\
&\bar{w}_1 = \bar{w}_3 = 1,\ \bar{w}_2 = 4,\\
&X = Y = \mathbb{R}^2.
\end{aligned}
$$

By Theorem 3.1, problem (2) is equivalent the following bi-level problem:

$$
\begin{aligned}
(\text{BDP}):\quad \min\quad & \| x - y \|_2,\\
& x \in \Gamma_1 := \operatorname{argmin}_{x \in \mathbb{R}^2} \sum_{i=1}^{4} \| x - a_i \|_1, \qquad (13)\\
& y \in \Gamma_2 := \operatorname{argmin}_{y \in \mathbb{R}^2} \| y - \bar{a}_1 \|_1 + 3\| y - \bar{a}_2 \|_1 + \| y - \bar{a}_3 \|_1 .
\end{aligned}
$$

Next we apply the Algorithm 2 to solve the BDP (14). In Step 1, we have

$$
\begin{aligned}
&\Gamma_1 = \{ x \in \mathbb{R}^2 \mid 0 \le x_1 \le 2,\ 0 \le x_2 \le 2 \},\\
&\Gamma_2 = \{ (6,1)^T \}.
\end{aligned}
$$

Indeed,

$$
\begin{aligned}
&\Gamma_1^1 = \{ x \in \mathbb{R} \mid 0 \le x \le 2 \},\ \Gamma_1^2 = \{ x \in \mathbb{R} \mid 0 \le x \le 2 \},\\
&\Gamma_2^1 = \{6\},\ \Gamma_2^2 = \{1\}.
\end{aligned}
$$

In Step 2, using Algorithm 1, we have

$$i = 1,\ e_1 = 0,\ \Omega^1 = \{(2,6)\},$$
$$i = 2,\ e_2 = 1,\ \Omega^2 = \{(1,1)\}.$$

Hence $x^* = (2,1)^T$ and $y^* = (6,2)^T$.
Illustration of solution method for solving problem (14) is presented in Figure 1.

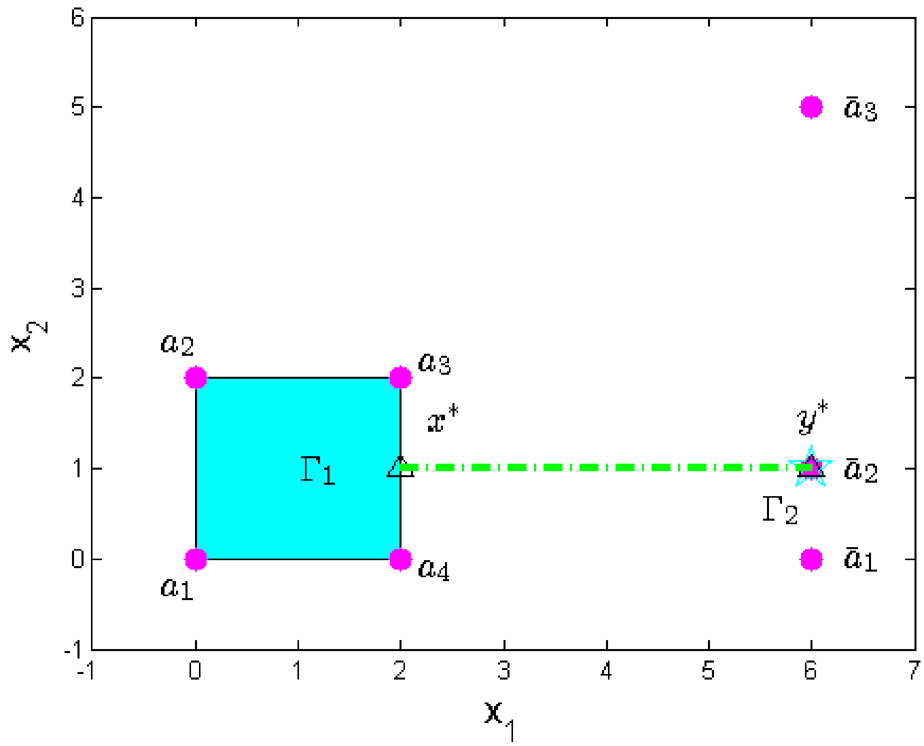

Figure 1. Illustration of Example 4.1.

Next we consider the Example 1 in the constrained case.

Example 4.2. Assume in Example 1 that $X = \mathbb{R}^2$, and $Y = \{y \in \mathbb{R}^2 \mid y_2 = 4\}$.
By Theorem 3.1, problem (2) is equivalent the following bi-level problem:

$$\text{(BDP)}: \quad \min \quad \| x - y \|_2,$$
$$x \in \Gamma_1 := \text{argmin}_{x \in \mathbb{R}^2} \sum_{i=1}^{4} \| x - a_i \|_1,$$
$$y \in \Gamma_2 := \text{argmin}_{y \in Y} \| y - \bar{a}_1 \|_1 + 3\| y - \bar{a}_2 \|_1 + \| y - \bar{a}_3 \|_1 .$$

Next we run the Algorithm 2 to solve the BDP (14). In Step 1, we get

$$\Gamma_1 = \{x \in \mathbb{R}^2 \mid 0 \leq x_1 \leq 2,\ 0 \leq x_2 \leq 2\},$$
$$\Gamma_2 = \{(6,4)^T\}.$$

Indeed,

$$\Gamma_1^1 = \{x \in \mathbb{R} \mid 0 \leq x \leq 2\},\ \Gamma_1^2 = \{x \in \mathbb{R} \mid 0 \leq x \leq 2\},$$
$$\Gamma_2^1 = \{6\},\ \Gamma_2^2 = \{4\}.$$

In Step 2, using Algorithm 1, we obtain

$$i = 1,\ e_1 = 0,\ \Omega^1 = \{(2,6)\},$$
$$i = 2,\ e_2 = 1,\ \Omega^2 = \{(2,4)\}.$$

Hence $x^* = (2,2)^T$, and $y^* = (6,4)^T$.
The solution method for solving problem (14) is illustrated in Figure 2.

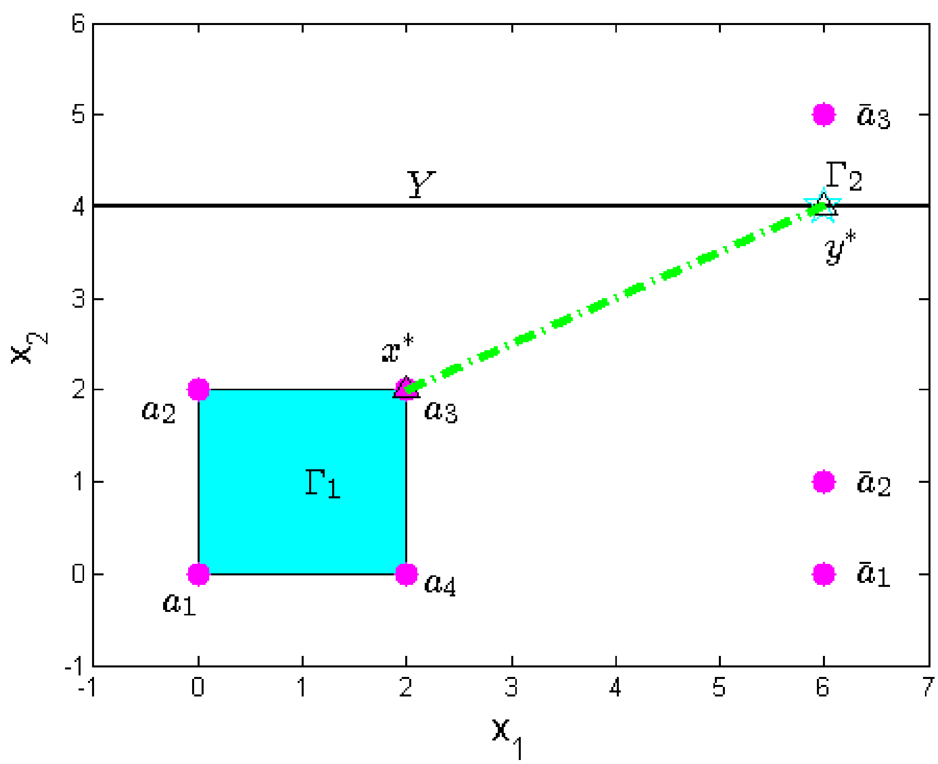

Figure 2. Illustration of Example 4.2.

Conclusion

We consider the Multi-objective hub location model which is applicable to formulate and solve the location problems in the real world. The necessary and sufficient condition is stated for finding the efficient solution. Then using the obtained result, we show that the problem can be reduced to a simple bi-level distance problem. Moreover, an efficient algorithm is presented to find the optimal solution set of the bi-level distance problem. The global convergence of the algorithm is proved and two example is provided for clarifying the proposed method. How to solve the non-convex hub location problems arising in operations research by the exact algorithms deserves further and more extensive investigation in our future research.

References

[1] Alamatsaz, K., Ahmadi, A. and Mirzapour Al-e-hashem, S.M.J. (2021), A multiobjective model for the green capacitated location-routing problem considering drivers satisfaction and time window with uncertain demand, *Environ Science Pollut Research*.

[2] Andersson, G., Francis, R.L., Normak, T. and Rayco, M.B. (1998), Aggregation method experimentation for large scale network location problems, *Location Science*, 6, 25–39.

[3] Amiri-Aref, M., Shiripour, S. and Ruiz-Hernandez, D. (2021), Exact and approximate heuristics for the rectilinear Weber location problem with a line barrier, *Computers and Operations Research*, 132.

[4] Aziz, N., Chauhan, S. and Vidyarthi, N. (2016), The impact of hub failure in hub-and-spoke networks: mathematical formulations and solution techniques, *Computers and Operations Research*, 65, 174-188.

[5] Aziz, N., Vidyarthi, N. and Chauhan, S. (2018), Modelling and analysis of hub-and-spoke networks under stochastic demand and congestion, *Annals of Operations Research*, 264, 1-2.

[6] Baroughi Bonab, F., Burkard, R.E. and Gassner, P. (2011), Inverse p-median problems with variable edge lengths, *Mathematical Method of Operations Research*, 73, 263-280.

[7] Campbell, J.F. and Okelly, M.E. (2012), Twenty-five years of hub location research, *Transportation Science*, 46, 153-169.

[8] Contreras, I., Cordeau, J.F. and Laporte, G. (2012), Exact solution of large-scale hub location problems with multiple capacity levels, *Transportation Science*, 46, 439-459.

[9] Clarke, F.H., Ledyaev, Y.S., Stern, J. and Wolenski, R.R. (1998), Nonsmooth analysis and control theory. Springer-Verlag: New York.

[10] Drezner, Z. and Wesolowsky, G.O. (1991), Facility location when demand is time dependent, *Naval Research Logistic*, 38, 763–777.

[11] Drezner, Z. and Wesolowsky, G.O. (2001), On the collection depots location problem, *European Journal of Operations Research*, 130, 510-518.

[12] Ernst, A.T. and Krishnamoorthy, M. (1996), Efficient algorithms for the uncapacitated single allocation p-hub median problem, *Location Science*, 4, 139-154.

[13] Karatas, M. and Yakici, E. (2018), An iterative solution approach to a multi-objective facility location problem, *Applied Soft Computing*, 62, 272-287.

[14] Karatas, M. and Yakici, E. (2021), A multi-objective location analytics model for temporary emergency service center location decisions in disasters, *Decision Analytics Journal*.

[15] Larson, R.C. and Sadiq, G. (1983), Facility location with the Manhattan metric in the presence of barriers to travel, *Operations Research*, 31, 652-669.

[16] Love, R.F. and Morris, J.G. (1975), Solving constrained multi-facility location problems involving ℓ_p distances using convex programming, *Operations Research*, 23, 581-587.

[17] Love, R.F., Morris, J.G. and Wesolowsky, G.O. (2018), Facilities location: models and methods, *Mathematical Method of Operations Research*, 88, 81-98.

[18] Naumov, V. (2021), Substantiation of Loading Hub Location for Electric Cargo Bikes Servicing City Areas with Restricted Traffic, *Energies*, 14.

[19] McGarvey, R.G. and Cavalier, T.M. (2003), A global optimal approach to facility location in the presence of forbidden regions, *Computers and Industrial Engineering*, 45, 1-15.

[20] Miyagawa, M. (2012), Rectilinear distance to a facility in the presence of a square barrier, *Annals of Operations Research*, 196, 443-458.

[21] Nobakhtian, S. and Raeisi Dehkordi, A. (2018), A fast algorithm for the rectilinear distance location problem, *Mathematical Methods of Operations Research*, 88, 81-98.

[22] Nobakhtian, S., Raeisi Dehkordi, A. (2017), An algorithm for generalized constrained multi-source Weber problem with demand substations, *4OR - A Quarterly Journal of Operations Research*, 16, 343-377.

[23] Okelly, M.E. (1987), A quadratic integer program for the location of interacting hub facilities, *European Journal of Operations Research*, 32, 393–404.

[24] Raeisi Dehkordi, A. (2019), The optimal solution set of the multi-source Weber problem, *Bulletin of the Iranian Mathematical Society*, 45, 495-514.

[25] Shahparvari, S.h. and Nasirian, A., et. al. (2020), A GIS-LP integrated approach for the logistics hub location problem, *Computers and Industrial Engineering*, 146.

[26] Phillip R. Jenkins, J. and Lunday, J. (2020), Robbins, Robust, multi-objective optimization for the military medical evacuation location-allocation problem, *Omega*, 97.

[27] Rockafellar, R.T. (1970), Convex analysis, Princeton: New Jersey, Princeton University Press.

[28] Vygen, J. (1997), The two-dimensional weighted median problem, *Zeitschrift für Angewandte Mathematik und Mechanik*, 77, 433-436.

[29] Wang, C., Wang, Z., Tian, Y., Zhang, X. and Xiao, J. (2021), A Dual-Population Based Evolutionary Algorithm for Multi-Objective Location Problem Under Uncertainty of Facilities, IEEE Transactions on Intelligent Transportation Systems.

[30] Wu, T., Shi, Z. and Zhang, C. (2021), The hub location problem with market selection, *Computers and Operations Research*, 127.

[31] Zanjirani-Farahani, R., Drezner, Z., and Asgari, N. (2009), Single facility location and relocation problem with time dependent weights and discrete planning horizon, *Annals of Operations Research*, 167, 353-368.
[32] Zanjirani-Farahani, R. and Hekmatfar, M. (Ed), (2009), Facility location: concepts, models, algorithms and case studies. Springer-Verlag: Berlin.