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A Polynomial Time Algorithm to Diagnose the Solvability of Single Rate n-Pair Networks with Common Bottleneck Links

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Cai et al.(2013) and Cai and Han (2014) developed polynomial-time algorithms for two- and three-pair networks with common bottleneck links, respectively. Also, Chen and Haibin(2012) developed non-polynomial-time methods for n-pair networks with common bottleneck links, where n is an arbitrary integer. This study proposes a new sufficient and necessary condition to determine the solvability of single rate n-pair networks with common bottleneck links. It closes with a polynomial time solution for n-pair networks with common bottleneck links, where n is an arbitrary integer. Our algorithm runs in $O(|V||E|^2)$ time, where $|V|$ and $|E|$ are the number of nodes and links, respectively.

Keywords: Network coding, Single rate n-pair networks, Bottleneck links, Solvability.

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1 Introduction

The solvability and linear solvability of communication networks are an essential issues in network coding. The maximum flow minimum cut theorem [2] can be used to determine the solvability of multicast networks. Furthermore, such networks are linearly solvable [15]. Unfortunately, characterizing the solvability and linear solvability of nonmulticast networks is challenging, and the results are sporadic and incomplete. Researchers concentrated on nonmulticast networks specializations such as two-unicast networks with rate (1,1), sum-networks, two-unicast networks with rate (1,2), two unit-rate multicast sessions networks and three-unicast networks with shared bottleneck links [5,6, 17-21].

Researchers have always sought to develop efficient algorithms for solving various problem [1,16,13]. Wang and Shroff [20, 21] proposed a method for diagnosing the solvability of single rate two-pair networks based on path overlap requirements, which state that a single rate two-pair network is solvable if and only if it meets certain path overlap conditions. The algorithm suggested in [20, 21] is based on the approach in [9] for discovering k edge-disjoint pathways, which requires first calculating the levels of all nodes and then using a pebbling game to locate the paths [9].

Cai et al. [6] formulated the network structures by cut set relations and presented an algorithm to diagnose the solvability of single rate two-pair networks. The method of [6] proposes a subnetwork decomposition approach to investigate the underlying graph structure of single rate two-pair networks. Their result shows that the solvability of a single rate two-pair network is completely determined by four particular link subsets of the underlying network, which can be considered as the most important links of a

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single rate two-pair network. Comparing with the approach of [20, 21], the algorithm presented in [6] is easier to implement (see [6], Page 131).

Finding bottleneck links plays a very important role in [6]. Cai and Fan [4] presented a method to find a bottleneck link, where runs in $O(|V||E|^2)$ time (also, see [6], Page 131). The region decomposition method [10, 11, 17, 18, 19] has been found efficient for analysing network structure and finding bottleneck links, which was very successful in the 3s/nt sum networks [17], two-unicast networks with rate(1,2) [18], two-multicast networks [19], two unit-rate multicast sessions networks [11] and two-pair networks [10]. The method defined a unique graph that is called the basic region graph, which has a much simpler topological structure than the original graph.

Cai and Han considered single rate three-pair networks with common bottleneck links and derived a sufficient and necessary condition to diagnose the solvability of such networks [5]. They showed that the solvability of such networks can be determined in polynomial time. For a single rate three-pair networks with common bottleneck links, the solvability is equivalent to the linear solvability and finite fields of size 2 or 3 are sufficient to construct linear solutions [5].

In [8], the single rate three-pair networks with common bottleneck links is considered and a characterization (called Property P) is presented to diagnose the solvability of them. It is shown in [7] that, the presented characterization in [8] can be generalized and a characterization (called Property P') is presented to determine the solvability of n-pair networks, where n is an arbitrary integer. Moreover, Chen et al. [7] constructed a solvable n-pair network that has no solvable solution if its alphabet size is less than n.

This paper considers the single rate n-pair network with common bottleneck links, where n is an arbitrary integer. We present a new sufficient and necessary condition to diagnose the solvability of such networks based on previous works in [5, 6]. Furthermore, based on presented algorithm in [6], a polynomial time algorithm for determining the solvability or unsolvability of such networks is presented. The rest of the paper consists of four sections in addition to Introduction section. Section 2 provides definitions and notations for single rate n-pair networks with common bottleneck links. According to [7, 8], Section 3 introduces a new necessary and sufficient criterion for determining the solvability of single rate n-pair networks. Based on [6], a novel approach is proposed to determine the solvability of single rate n-pair networks, resulting in a polynomial time algorithm. Section 4 finishes the paper.

1.1 Contribution of this paper

In this paper, based on [5,7,8], we present a new necessary and sufficient condition for characterizing the solvability of n-pair networks with common bottleneck links, where n is an arbitrary integer that admits a polynomial-time algorithm with running time $O(|V||E|^2)$. Characterizing the solvability and linear solvability of nonmulticast networks is challenging, and the results are sporadic and incomplete. Researchers concentrated on nonmulticast networks specializations such as two- and three-pair networks with common bottleneck links. By [5], there exists a necessary and sufficient condition for diagnosing the solvability of two-pair networks without bottleneck links, but no necessary and sufficient condition has yet been established for determining the solvability of n-pair networks without bottleneck links, where $n \geq 3$.

2 Preliminaries

2.1 Single rate n-pair networks with common bottleneck links

A communication network $G = (V, E, S, T)$ is modelled as a directed, acyclic, finite graph $G = (V, E)$, where V is the node set, E is the link set, $S \subseteq V$ and $T \subseteq V$ are the set of source nodes and sink nodes, respectively. A single rate n-pair network is a communication network with source node set $S = \{s_1, s_2, \ldots, s_n\}$, sink node set $T = \{t_1, t_2, \ldots, t_n\}$ and n desired unit flows from s_i to t_i for $i \in \{1,2, \ldots, n\}$. The n desired unit flows from s_i to t_i are considered as independent random variables with unit entropies and denoted by X_i for $i \in \{1,2, \ldots, n\}$. It is assumed that each source s_i generates a message $X_i \in F$ and each terminal t_i wants to get the message X_i, where F is a finite field. We suppose $s_i \neq s_j$ and $t_i \neq t_j$, for each $i \neq j$.

For a communication network $G = (V, E, S, T)$, if $S = \{s\}$ and $T = \{t\}$, then G is a point-to-point network. Let $G = (V, E, \{s\}, \{t\})$ be a point-to-point network and let $V = W \cup \overline{W}$ be a vertex partition of $G = (V, E)$ such that $s \in W$ and $t \in \overline{W} = V \backslash W$. An $s - t$ cut $\mathcal{C}$ is the collection of all the edges from W to $\overline{W}$. The capacity of $\mathcal{C}$ is defined as $\sum_{e \in C} C(e)$, where $C(e)$ is nonnegative capacity of link e. The minimum of the cut capacities for all $s - t$ cuts is called the minimum cut capacity and denoted by $C(s, t)$. A minimum cut is a cut with the minimum cut capacity.

Suppose that $G = (V, E, S, T)$ is a single rate n-pair network. There are $|S| \times |T| = n^2$ point to point networks. For a given $s_i \in S$ and $t_j \in T$, there is a point to point network $G_{i,j} = (V, E, \{s_i\}, \{t_j\})$. The $A_{i,j}$-set of $G_{i,j}$ is defined as the union of all $s_i - t_j$ minimum cuts and denoted by $A_{i,j}$. For a single rate n-pair network G, *The bottleneck links* are defined as follows:

$$A(1,2, \ldots, n) \triangleq A_{1,1} \cap A_{2,2} \cap \ldots A_{n,n}.$$

In this paper, the single rate n-pair networks with common bottleneck links are considered which concludes $A(1,2, \ldots, n) \neq \emptyset$.

For the sake of simplification, each link e of G is further assumed to be error-free, delay-free and can carry one symbol in each use, i.e., $C(e) = 1$, where $C(e)$ is nonnegative capacity of link e. For any link $e = (u, v) \in E$, node u is called the *tail* of e and node v is called the *head* of e, and are denoted by $u = tail(e)$ and $v = head(e)$, respectively. Moreover, we call e an incoming link of v and an outgoing link of u. For two links $e, e' \in E$, we call e an incoming link of e' (or e' an outgoing link of e) if $tail(e') = head(e)$. For each $e \in E$, the set of incoming links of e denotes by $In(e)$.

We assume that each source s_i has an imaginary incoming link, called X_i *source link* $s(i)$, and each terminal t_j has an imaginary outgoing link, called *terminal link* $t(j)$. Note that the source links have no tail and the terminal links have no head. For the sake of convenience, if $e \in E$ is not a source link (resp. terminal link), we call e a *non-source link* (resp. *non-terminal link*). Also, we assume that each non-source non-terminal link e of G is on a path from some source to some terminal. Otherwise, link e is removed from G.

The transmitted information over an edge $e \in E$ and an edge set $A \subseteq E$ are denoted by X_e and X_A, respectively. Also, $H(e)$ and $H(A)$ are the entropies of X_e and X_A, respectively. A code over an alphabet F for a single rate n-pair network is a collection of functions $\{f_e : e \in E\}$ such that

1. $X_e = f_e(X_{In(e)})$,
2. $X_{s(i)} = X_i$ for $i = 1,2, \ldots, n$.

A code over an alphabet F is a solvable solution for a single rate n-pair network if $H(s(i)|t(i)) = 0$ for $i = 1,2, \ldots, n$. In other words, if each source s_i can send a unit rate of information flow to t_i, for each $i \in \{1,2, \ldots, n\}$, then, the single rate n-pair network is solvable.

We always suppose there exists at least one path from s_i to t_i, for each $i = 1,2, \ldots, n$, otherwise, the given n-pair network is unsolvable. A $u - v$ *path* $P_{u,v}$ is a string of ordered edges $(e_1, e_2, \cdots, e_n)$ such that $u = tail(e_1)$, $v = head(e_n)$ and $head(e_i) = tail(e_{i+1})$, for $i = 1,2, \cdots, n-1$. In the following, the single rate n-pair networks with $C(s_i, t_i) = 1$ for $i \in \{1,2, \ldots, n\}$ are considered. If $C(s_i, t_i) = 0$, then there is no path from s_i to t_i or and the single rate n-pair problem is unsolvable.

Definition 2.1. *[12, 6] Suppose G is a single rate n-pair network and A and B are two subsets of E. Moreover, let X_A and X_B are transmitted information over an edge set A and B, respectively. If X_B is a function of X_A for all network coding solutions, then A informationally dominates B and is denoted by $A \rightsquigarrow^i B$. Furthermore, the following properties are held for informational dominance:*

1. $t(i) \rightsquigarrow^i s(i)$, for $i = 1,2,\dots,n$.
2. $A \rightsquigarrow^i A$, for $A \subseteq E$.
3. If $A \rightsquigarrow^i B$, and $B \rightsquigarrow^i C$, then $A \rightsquigarrow^i C$.
4. If $A \rightsquigarrow^i B$, and $A \rightsquigarrow^i C$, then $A \rightsquigarrow^i B \cup C$.

2.2 The solvability of single rate n-pair networks with common bottleneck links

In [6], the single rate two-pair networks with common bottleneck links ($A(1,2) \neq \emptyset$) are considered and a necessary and sufficient condition to diagnose the solvability of them is presented as follows:

Theorem 2.1. *[6] Let $G = (V, E, \{s_1, s_2\}, \{t_1, t_2\})$ be a single rate two-pair network such that $A(1,2) \neq \emptyset$. Then G is solvable if and only if there exist an $s_1 - t_2$ path P_{s_1,t_2} and an $s_2 - t_1$ path P_{s_2,t_1} with $(P_{s_1,t_2} \cup P_{s_2,t_1}) \cap A(1,2) = \emptyset$.*

By Theorem 2.1, a polynomial time algorithm to diagnose the solvability of two-pair networks with $A(1,2) \neq \emptyset$ is concluded [6].

Example 2.1. Consider the network G in Fig. 1. G is an example of a two-pair network with $(v_3, v_4) \in A(1,2) \neq \emptyset$. Also, there exist $s_1 - t_2$ path $P_{s_1,t_2} = ((s_1, v_1), (v_1, v_5), (v_5, t_2))$ and $s_2 - t_1$ path $P_{s_2,t_1} = ((s_2, v_2), (v_2, v_6), (v_6, t_1))$ in G such that $(P_{s_1,t_2} \cup P_{s_2,t_1}) \cap A(1,2) = \emptyset$. Thus, G satisfies the conditions of Theorem 2.1 and is solvable.

In [8], a property, called Property P, is presented to characterize the solvability of a class of three-pair networks with $A(1,2,3) \neq \emptyset$. Let F is a finite field, π is a permutation over F and $\oplus$ is a mapping from $F \times F$ to F. Then, Property P is defined as follows:

Definition 2.2. *[8] (Property P) Let G is a single rate three-pair network with $A(1,2,3) \neq \emptyset$ such that each source s_i generates message $X_i \in F$ for $i \in \{1,2,3\}$. A code over an alphabet F has Property P, if there exist 4 edges Y_1, Y_2, Y_3, M in G, permutations $\pi_1, \pi_2, \dots, \pi_6$ of F and a mapping $\oplus$ $F \times F \to F$ such that $(F, \oplus)$ is an Abelian group and*

$$Y_1 = \pi_4(\pi_1(X_1) \oplus \pi_2(X_2)),$$

$$Y_2 = \pi_5(\pi_1(X_1) \oplus \pi_3(X_3)),$$

$$Y_3 = \pi_6(\pi_2(X_2) \oplus \pi_3(X_3)),$$

and

$$M = \pi_1(X_1) \oplus \pi_2(X_2) \oplus \pi_3(X_3).$$

Example 2.2. Let G be the depicted network in Fig. 2. G is a three-pair network with $(v_1, v_2) \in$

$A(1,2,3) \neq \emptyset$. Also, there exist permutations $\pi_1, \pi_2, \dots, \pi_6$ and edges $(v_3, v_6), (v_4, v_7), (v_5, v_8)$ such that

$$(v_3, v_6) = \pi_4(\pi_1(X_1) \oplus \pi_2(X_2)),$$

$$(v_4, v_7) = \pi_5(\pi_1(X_1) \oplus \pi_3(X_3)),$$

$$(v_5, v_8) = \pi_6(\pi_2(X_2) \oplus \pi_3(X_3)),$$

and

$$(v_1, v_2) = \pi_1(X_1) \oplus \pi_2(X_2) \oplus \pi_3(X_3).$$

Thus, G satisfies Property *P*.

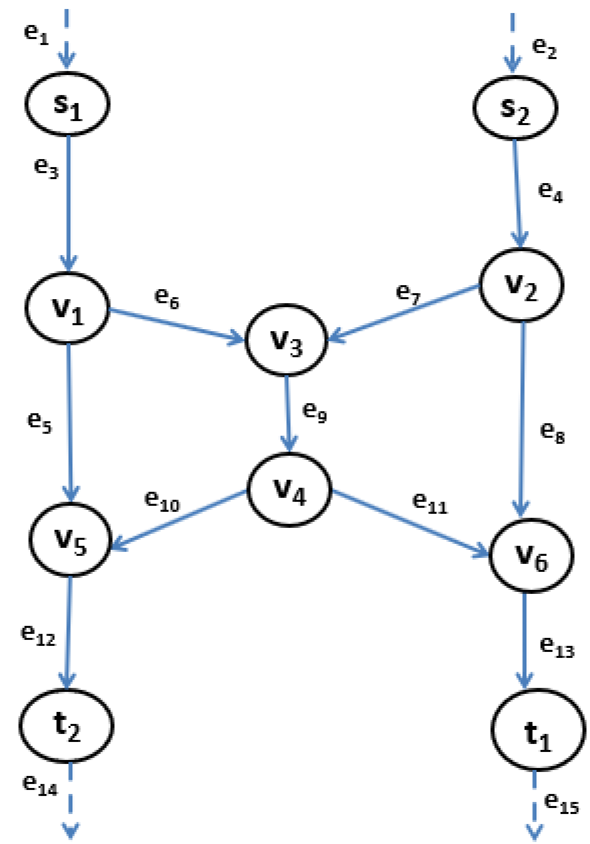

Figure 1. An example of a two-pair network with A(1,2)≠∅.

Lemma 2.1. *[8] A code over an alphabet F is a solvable solution for three-pair network G with common bottleneck links if and only if it satisfies Property P.*

In [7], to diagnose the solvability of a class of n-pair networks with $A(1,2, \dots, n) \neq \emptyset$, Property P is generalized as the next definition.

Definition 2.3. *[7] (Property P') Let G is a single rate n-pair network with* $A(1,2, \dots, n) \neq \emptyset$

such that each source s_i *generates message* $X_i \in F$ *for* $i \in \{1,2,\ldots,n\}$*. A code over an alphabet* F *has Property* P'*, if there exist* $n+1$ *edges* $Y_1, Y_2, \ldots, Y_n, M$ *in* G*, permutations* $\pi_1, \pi_2, \ldots, \pi_{2n}$ *of* F *and a mapping* $\oplus\, F \times F \to F$ *such that* $(F, \oplus)$ *is an Abelian group and*

$$Y_k = \pi_{n+k}\left(\sum_{j \neq k} \pi_j(X_j)\right), \quad k = 1,2,\ldots,n,$$

and

$$M = \pi_1(X_1) \oplus \pi_2(X_2) \oplus \ldots \oplus \pi_n(X_n).$$

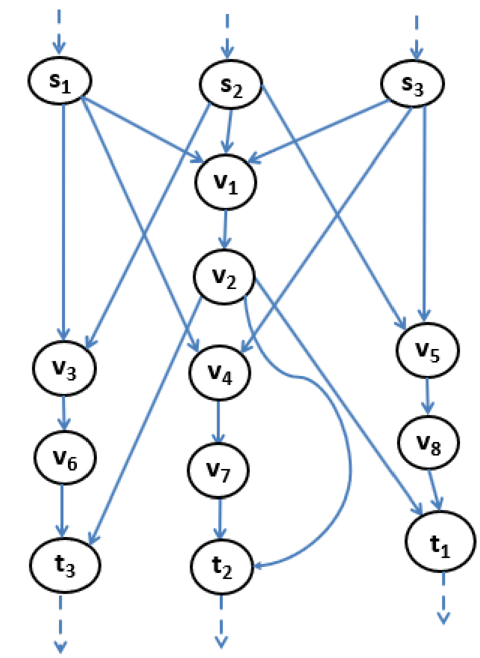

Figure 2. *An example of a network satisfies Property P.*

Lemma 2.2. *[7] A code over an alphabet F is a solvable solution for n-pair network* G *if and only if it satisfies Property* P'*.*

Lemma 2.3. *Property* P' *can be checked in factorial time.*

Proof. According to definition 2.3, a code over an alphabet F has Property P', if there exist $n+1$ edges of all the edges in G (i.e. |E|) and $2 \times n$ permutations of F such that edges and permutations satisfy in the mentioned conditions. In the worst case, $|F|!$ permutations of $|F|$ and

$$\frac{|E|!}{(n+1)! \times (|E| - (n+1))!}$$

edges of $|E|$ should be checked. Therefore, Property of P' can be checked in factorial time, which means the method of [7] is a non-polynomial time algorithm.

3 A new sufficient and necessary condition

In this section, based on Lemmas 2.1 and 2.2, a new sufficient and necessary condition to diagnose the solvability of n-pair networks with common bottleneck links is presented.

Lemma 3.1. *Let* G *be a n-pair network such that* $A(1,2,\dots,n) \neq \emptyset$. *Suppose that for each distinct* $i,j \in \{1,2,\dots,n\}$, *there is an* $s_i - t_j$ *path* P_{s_i,t_j} *such that* $P_{s_i,t_j} \cap A(1,2,\dots,n) = \emptyset$. *Then* G *satisfies Property* P'.

Proof. Let $e \in A(1,2,\dots,n) \neq \emptyset$. If $e \in A(1,2,\dots,n)$, by the definition of $A(1,2,\dots,n)$, then, there exists an $s_i - t_i$ path P_{s_i,t_i} that passes through e, for each $i \in \{1,2,\dots,n\}$. Thus, message X_i can be send to edge e from each source s_i, for $i \in \{1,2,\dots,n\}$. By defining permutation $\pi_i(x_i) = x_i$, for each $i \in \{1,2,\dots,n\}$, we conclude that there are permutations $\pi_1, \pi_2, \dots, \pi_n$ of F and a mapping $\oplus\, F \times F \to A$ such that $(F,\oplus)$ is an Abelian group and

$$e = \pi_1(x_1) \oplus \pi_2(x_2) \oplus \dots \oplus \pi_n(x_n).$$

On the other hand, by the assumption of the lemma, there is an $s_i - t_j$ path P_{s_i,t_j} such that $P_{s_i,t_j} \cap A(1,2,\dots,n) = \emptyset$ for each distinct $i,j \in \{1,2,\dots,n\}$, so, there are permutations $\pi_{n+1}, \pi_{n+2}, \dots, \pi_{2n}$ of F and $e'_k \in P_{s_i,t_j}$ such that

$$e'_k = \pi_{n+k}\left(\sum_{j\neq k}\left(\pi_j(x_j)\right)\right), \quad k = 1,2,\dots,n.$$

Thus, G satisfies Property P'.

Corollary 3.1. *(The sufficient condition) Let* G *be a n-pair network such that* $A(1,2,\dots,n) \neq \emptyset$. *Suppose that for each distinct* $i,j \in \{1,2,\dots,n\}$, *there is an* $s_i - t_j$ *path* P_{s_i,t_j} *such that* $P_{s_i,t_j} \cap A(1,2,\dots,n) = \emptyset$. *Then* G *is solvable.*

Proof. By Lemmas 2.1 and 3.1, the result is concluded.

Lemma 3.2. *Let* G *be a n-pair network such that* $A(1,2,\dots,n) \neq \emptyset$ *and* e *be an edge of* $A(1,2,\dots,n)$. *For two distinct indexes* $i,j \in \{1,2,\dots,n\}$, *if each* $s_i - t_j$ *path* P_{s_i,t_j} *is not disjoint with* $A(1,2,\dots,n)$, *then there is no* $s_i - t_j$ *path in* $G\backslash\{e\}$.

Proof. Consider two distinct indexes $i,j \in \{1,2,\dots,n\}$. Let each $s_i - t_j$ path P_{s_i,t_j} is not disjoint with $A(1,2,\dots,n)$. For the sake of contradiction, suppose that there is an $s_i - t_j$ path P_{s_i,t_j} in $G\backslash\{e\}$. By the assumption, path P_{s_i,t_j} passes through $e' \in A(1,2,\dots,n)$. We have the following two cases:

(a) Edge e is an up-link of edge e'. Then, $P_{s_i,t_j}[s_i,e'] - P_{s_i,t_i}[e',t_i]$ is an $s_i - t_i$ path that does not pass through e which is contradiction with $e \in A_{i,i}$.

(b) Edge e is a down-link of edge e'. Then, $P_{s_j,t_j}[s_j,e'] - P_{s_i,t_j}[e',t_j]$ is an $s_j - t_j$ path that does not pass through e which is contradiction with $e \in A_{j,j}$.

Lemma 3.3. *Let* G *be a n-pair network such that* $A(1,2,\dots,n) \neq \emptyset$ *and* e *be an edge of* $A(1,2,\dots,n)$. *If there are two distinct indexes* $i,j \in \{1,2,\dots,n\}$ *such that* $P_{s_i,t_j} \cap A(1,2,\dots,n) \neq \emptyset$ *for each* $s_i - t_j$ *path* P_{s_i,t_j}, *then* $\{e\} \rightsquigarrow^i t(i) \cup t(j)$.

Proof . Suppose that there are two distinct indexes $i,j \in \{1,2,\dots,n\}$ such that each $s_i - t_j$ path

P_{s_i,t_j} is not disjoint with $A(1,2,\dots,n)$. Then, by Lemma 3.2, there is no $s_i - t_j$ path in $G\backslash\{e\}$. On the other hand, by $e \in A(1,2,\dots,n) \subseteq A_{j,j}$, there is no $s_j - t_j$ path in $G\backslash\{e\}$. Thus, $t(j)$ is a down-link of $\{e\}$ for $j \in \{1,2,\dots,n\}$. Moreover, $t(i)$ is a down-link of $\{e\} \cup s(1) \cup s(2) \cup \dots \cup s(i-1) \cup s(i+1) \cup \dots \cup s(n)$. So, we have

$$\{e\} \rightsquigarrow^i t(j), \qquad j \in \{1,2,\dots,n\}. \tag{1}$$

and

$$\{e\} \cup s(1) \cup s(2) \cup \dots \cup s(i-1) \cup s(i+1) \cup \dots \cup s(n) \rightsquigarrow^i t(i), \quad i \in \{1,2,\dots,n\}. \tag{2}$$

By the first property of Definition 2.1, $t(j) \rightsquigarrow^i s(j)$, for $j \in \{1,2,\dots,n\}$. Thus, by (1) and third property of Definition 2.1, we conclude

$$\{e\} \rightsquigarrow^i s(j), \quad j \in \{1,2,\dots,n\}. \tag{3}$$

On the other hand, according to Property 2 of Definition 2.1, we have $\{e\} \rightsquigarrow^i \{e\}$, for each edge $e \in E$. So, by (3) and Property 4 of Definition 2.1, we have

$$\{e\} \rightsquigarrow^i \{e\} \cup s(1) \cup s(2) \cup \dots \cup s(i-1) \cup s(i+1) \cup \dots \cup s(n). \tag{4}$$

Then, by (2), (4) and Property 3 of Definition 2.1, we conclude that $\{e\} \rightsquigarrow^i t(i)$. Thus, by (1) and Property 4 of Definition 2.1, we conclude that $\{e\} \rightsquigarrow^i t(i) \cup t(j)$.

Corollary 3.2. *(The necessary condition) Let G be a n-pair network such that $A(1,2,\dots,n) \neq \emptyset$. If there are two distinct indexes $i,j \in \{1,2,\dots,n\}$ such that $P_{i,j} \cap A(1,2,\dots,n) \neq \emptyset$ for each $s_i - t_j$ path P_{s_i,t_j}, then G is not solvable.*

Proof. For the sake of contradiction, suppose that G is solvable. If there are two distinct indexes $i,j \in \{1,2,\dots,n\}$ such that $P_{s_i,t_j} \cap A(1,2,\dots,n) \neq \emptyset$ for each $s_i - t_j$ path P_{s_i,t_j}, then, by Lemma 3.3, there is edge $e \in A(1,2,\dots,n)$ such that $\{e\} \rightsquigarrow^i t(i) \cup t(j)$, which contradicts to that edge e has unit capacity.

By Corollaries 3.1 and 3.2, we get the next theorem, which is a new sufficient and necessary condition for the solvability of single rate n-pair networks with common bottleneck links.

Theorem 3.1. *Let G be a single rate n-pair network such that $A(1,2,\dots.n) \neq \emptyset$. Then G is solvable if and only if $P_{s_i,t_j} \cap A(1,2,\dots,n) = \emptyset$ for each distinct $i,j \in \{1,2,\dots,n\}$.*

Proof. By Corollaries 3.1 and 3.2, the result is obtained.

By Theorem 3.1, the following algorithm for diagnosing the solvability of a n-pair network with common bottleneck links is obtained.

Algorithm 1. Solvability of G with $A(1,2,\dots,n) \neq \emptyset$;
Begin

(1) Find bottleneck links $\mathcal{A} = A(1,2,\ldots,n)$;
(2) For each distinct $i,j \in \{1,2,\ldots,n\}$, check the connectivity of s_i to t_j in $G' = G\backslash\mathcal{A}$;
(3) If there is not an $s_i - t_j$ path for an $i,j \in \{1,2,\ldots,n\}$ in G', then write G is not solvable;
(4) If there is an $s_i - t_j$ path for each distinct $i,j \in \{1,2,\ldots,n\}$ in G', then write G is solvable;
End.

The next theorem computes that the running time of Algorithm 1.

Theorem 3.2 *Algorithm 1 diagnoses the solvability or unsolvability of a single rate n-pair network with* $A(1,2,\ldots,n) \neq \emptyset$ *in polynomial time.*

Proof. In Algorithm 1, by [5] and [3], Step **(1)** can be finished in time $O(|V||E|^2)$. Also, by the search algorithm, Step **(2)** can be done with time $O(|V|^2)$. Therefore, the solvability or unsolvability of a n-pair network with common bottleneck links can be determined in polynomial time.

4 Conclusion

Bottleneck links play a crucial role in diagnosing the solvability of single rate two- and three-pair networks [5,6]. Necessary and sufficient conditions have been established for determining the solvability of two-pair and three-pair networks with common bottleneck links, leading to polynomial-time algorithms for these problems. According to [6], checking the solvability of a single rate two-pair network with $A(1,2) \neq \emptyset$ can be done using a polynomial time algorithm with time complexity $O(|V||E|^2)$ (see [6], Page 131, Algorithm 4.5). Moreover, for three-pair networks with $A(1,2,3) \neq \emptyset$, [5] provides a polynomial time algorithm with a time complexity of $O(|E|^3)$. In [10], based on the region decomposition method, an $O(|E|)$-time algorithm is presented for diagnosing the solvability of single rate two-pair network with $A(1,2) \neq \emptyset$, which is faster than the presented algorithm in [6]. In [8], researchers focused on a specific class of single rate three-pair with common bottleneck links. They presented a new sufficient and necessary condition for characterizing the solvability of these networks. It was shown that presented condition in [8], can be generalized to single rate n-pair networks with common bottleneck links, where n is an arbitrary integer. However necessary and sufficient conditions were provided in [7,8], they resulted in non-polynomial-time algorithms. See Table1 for details.

Table1. Algorithms for n-pair networks with common bottleneck links.

Communication networks	**Running time of proposed algorithm**	**method to diagnose solvability**	**Contribution**
2-pair networks with common bottleneck links [6].	Polynomial time : $O(|V||E|^2)$	subnetwork decomposition/combination approach	Necessary and sufficient condition and an efficient cut-based algorithm to determine the solvability of a two-pair unicast problem is presented.

2-pair networks with common bottleneck links [10].	Polynomial time : $O(\|E\|)$	Region decomposition method.	Necessary and sufficient condition presented using region decomposition method.
3-pair networks with common bottleneck links [5].	Polynomial time: $O(\|E\|^3)$	The solvability of a single rate 3-pair network is determined by specific link subsets.	Necessary and sufficient condition to diagnose the solvability of these networks has been presented.
A class of 3-pair networks with common bottleneck links [8].	Factorial time.	Checking Property P.	Necessary and sufficient condition to diagnose the solvability of these networks has been presented by Property P.
A class of n-pair networks with common bottleneck links [7].	Factorial time.	Checking Property P'.	Necessary and sufficient condition to diagnose the solvability of these networks has been presented by Property P'.
A class of n-pair networks with common bottleneck links [This paper].	Polynomial time.	Merging specific link subsets and Property P'.	Necessary and sufficient condition to diagnose the solvability of these networks has been presented based on previous works.

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