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Inverse Data Envelopment Analysis for Two-stage Network Systems with Uncertain Input and Uncertain Undesirable Outputs

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Evaluation of healthcare systems, as a key organization providing different health services, is essential. This issue becomes more crucial when occurring crises such as a pandemic. They need to keep track of their success in the face of the crisis to assess the effects of policy changes and their capability to respond to new challenges. The Inverse Data Envelopment Analysis technique is an applicable method in order to estimate the input/output levels of decision-making units to preserve predetermined technical efficiency scores. In classic studies of Inverse Data Envelopment Analysis, decision-Making units as black boxes, ignoring their internal structure. This paper estimates input levels and new intermediate products to achieve a predetermined efficiency score set by the decision maker. In traditional Inverse Data Envelopment Analysis models, precise data are required to determine the input and/or output levels of each decision-making unit. However, in many scenarios, such as system flexibility, social and cultural contexts information may be indeterminate. In these cases, experts' opinions are used to model uncertainty. Uncertainty theory, a branch of mathematics, logically deals with degrees of belief. This paper aims to develop an inverse Network DEA model incorporating uncertainty theory. We assume that inputs and outputs of decision-making units are based on experts' belief degrees. To demonstrate the model is performance, we explore efficiency of healthcare systems during COVID-19 pandemic.

Keywords: *Network Inverse Data Envelopment Analysis, Uncertainty, Multi Objective Programming, Efficiency*.

1. Introduction

One of the famous techniques for the efficiency evaluation of a set of Decision-Making Units (DMUs) with multiple inputs and outputs is Data Envelopment Analysis (DEA). DEA was initially introduced by Charnes, Cooper, and Rhodes (CCR) (Charnes et al., 1978) and then was extended by Banker, Charnes, and Cooper (BCC) (Banker et al., 1984). In the real world, there are many cases that the decision-makers (or managers) of DMUs intend to estimate the appropriate inputs(outputs) when the outputs (inputs) are increased such that the efficiency scores are kept constant or set to a desired predefined target value. In these cases, DEA is not capable of identifying the appropriate inputs or outputs. Hence, to overcome this issue,(Wei et al., 2000) introduced an Inverse Data Envelopment Analysis (InvDEA) model to obtain the appropriate inputs or outputs in the presence of the predetermined input- or output efficiency scores. In other words, unlike the CCR and BCC models, their InvDEA model specifies the required level of inputs or outputs for the DMU under evaluation by considering a predetermined efficiency score for it. In this vein, (Lertworasirikul et al.,

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2011) and (Ghiyasi, 2015) developed the InvDEA model for the variable returns to scale (VRS) case. Moreover,(Jahanshahloo et al., 2015) modeled InvDEA under the intertemporal dependence assumption by utilizing multi-objective programming.(Zhang & Cui, 2016) also developed different input- and output-oriented models to extend and integrate the InvDEA model. Besides,(Lim, 2016) surveyed an InvDEA problem by considering the expected changes in the production frontier. (Ghiyasi, 2017) also provided a new InvDEA model as per the cost and revenue efficiency scores. Likewise,(Zhang & Cui, 2020)introduced a general inverse non-radial model using multi-objective programming and called it the inverse non-radial DEA model.

The InvDEA approach has gained increasing popularity in recent years due to its broad range of applications in sectors such as business, supply chain management, agriculture, education, manufacturing, sustainable production, energy and environment. In fact, InvDEA models can complement DEA models, and anywhere DEA models are used, InvDEA models can be used as well. In addition, InvDEA models can be used independently for applications such as resource allocation, budgeting and planning. For instance, (Moghaddas et al., 2022) presented a network InvDEA model to evaluate supply chain sustainability, illustrating the benefits of applying InvDEA models in supply chain management and sustainability contexts. (Younesi et al., 2023) introduced an SBM InvDEA model for interval data, offering decision-makers additional tools to analyze potential mergers and acquisitions by extending InvDEA applications to different data types.(Amin & Boamah, 2023) developed a strategic business partnership framework within an InvDEA context to help decision-makers enhance competitiveness through strategic alliances and partnerships. Furthermore, in the domain of environmental efficiency, (Pourmahmoud et al., 2026) applied an uncertain InvDEA model to assess CO2 emission efficiency, demonstrating the method's utility in managing uncertain undesirable outputs for sustainable production and environmental policy-making.

In the real world, there are many DMUs (or processes) consisting of two stages. In this case, DEA is not properly able to measure the relative efficiency of these two-stage DMUs (or two-stage processes) because it does not consider the internal structures of these DMUs. In this situation, network DEA (NDEA) can help us to measure the relative efficiency of the two-stage DMUs see, e.g., (Färe & Grosskopf, 2000);(Lewis & Sexton, 2004);(Liang et al., 2008) (Cook et al., 2010);(Castelli et al., 2010); (Yang et al., 2020). (An et al., 2019) suggested a two-stage InvDEA model considering undesirable outputs for resource planning of the Chinese commercial banking system.(Farzipoor Saen & Seyedi Hosseini Nia, 2020) proposed a network-structured InvDEA model to assess the performance and sensitivity analysis of after-sale services in a car company. Furthermore, the impact of uncertainty was examined using robust optimization.(Amin & Ibn Boamah, 2021) developed a two-stage InvDEA model to investigate potential gains from bank mergers.

The traditional InvDEA employs specified amounts of inputs and outputs. However, in many instances, these quantities are not deterministic. Typically, nondeterministic data is treated as statistical, with randomness managed using probability theory .Research explains the concept of stochastic DEA in three ways:1) one approach develops DEA models to handle observed deviations from the frontier as random variations,2) another designs DEA models to manage random noise, and 3) a third perspective views the Production Possibility Set (PPS) as a random PPS (Olesen & Petersen, 2016).(Ghomi et al., 2021)explored incorporating stochastic data into InvDEA to address resource allocation and investment analysis challenges. Nonetheless, there are situations where our limited knowledge is not solely due to randomness. In some cases, data may be ambiguous and vague. Fuzzy logic offers a flexible approach to evaluating the degree of uncertainty in such instances. The use of fuzzy theory, introduced by (Zadeh, 1996), has gained significant interest in DEA literature for addressing inherent ambiguity.(Emrouznejad et al., 2014) conducted an in-depth examination of employing fuzzy techniques in DEA. Although significant findings have emerged regarding Fuzzy DEA models, they are not without limitations. Some models simplify to linear optimization problems only if fuzzy numbers are assumed to be trapezoidal (Hatami-Marbini et al., 2011). Additionally, some models may exhibit unbounded optimal values, and many Fuzzy DEA models are both computationally inefficient and costly (Guo & Tanaka, 2008). This issue is prevalent across most Fuzzy DEA models (Soleimani-Damaneh et al., 2006).Moreover, using fuzzy methods to address nondeterministic issues may lead to inconsistencies and contradictions in certain situations.

Consider data have been collected from ten different hospitals. A DEA model can be utilized to evaluate the performance of these hospitals. One of the inputs might be equipment and infrastructure, with values ranging from 60 to 100. If the expert views the input as a fuzzy variable with this membership function, the possibility of achieving a "the amount of equipment and infrastructure is 80 " is equal to one.

$$\mu(x) = \begin{cases} (x-60)/20 & 60 \leq x < 80 \\ (100-x)/20 & 80 \leq x \leq 100. \end{cases}$$

However, the degree of belief in the "exact amount of 80 for equipment and infrastructures" is near zero, and no one can believe that the "exact amount of 80 for equipment and infrastructures" is correct. On the other hand, the possibility that the system receives the score of 80 is the same as when it does not. This is a contradiction. Therefore, fuzzy logic is not appropriate to model belief degrees. Furthermore, in the lack of historical data, probability theory would be unable to generate practical results. The uncertainty theory book (Liu, 2017) might be consulted for more information.

Uncertainty theory is a mathematical approach that aims to model human opinions. This theory can be applied in four different scenarios

1) It can be used to make predictions when there is no available sample or during emergencies like war, floods, earthquakes, or a pandemic. In such situations, historical data may not provide accurate information.
2) It can be employed to analyze the past in cases where specific measurements are inaccessible, such as carbon emissions or social benefits.
3) It can be used to model certain concepts, like "young" or "warm" which are ambiguous in human language.
4) It can be utilized to model dynamic systems with continuous-time noise, such as stock prices.

In these situations, some domain experts are invited to evaluate the belief degrees and the uncertainty theory can be utilized to deal with them (Liu, 2017). Recently, this theory has been studied by several researchers in DEA literature .(Lio & Liu, 2018), introduced an uncertain CCR model that incorporates uncertain variables for both inputs and outputs. They calculated the expected value of these uncertain variables and proposed a crisp model as an equivalent (Pourmahmoud & Bagheri, 2021), Providing an uncertain model for evaluating the performance of a basic two-stage system developed a basic two-stage model to account for uncertainty in the network structure. During the COVID-19 pandemic, probabilistic statistics might not function accurately due to the lack of comparable situations.(Pourmahmoud & Bagheri, 2023),used an uncertain model to evaluate healthcare system performance during the COVID-19 outbreak. Given the significance of InvDEA in the literature, this paper concentrates on efficiency analysis in scenarios where some data are based on belief degrees. The model is input-oriented and employs VRS technology. As an example, we have analyzed data from 30 hospitals during the COVID-19 pandemic. Additionally, the number of deceased patients has been considered as an undesirable output. During the pandemic, since this output was not precisely measurable, we determined it with the help of expert opinions. In summary, and to the best of our knowledge, this study makes the following contributions.

1. To evaluate the sustainability of the healthcare system, we propose a new inverse network data envelopment analysis model based on uncertainty theory. Specifically, in this model, patient mortality is considered an undesirable uncertain output that should be determined by expert opinions and incorporated into the efficiency evaluation of the healthcare system. This model is not only methodologically innovative, but also enables policymakers to make inverse target-oriented decision rather than merely assessing efficiency.
2. Unlike the conventional application of InvDEA, which have mainly focused on industrial or financial domains, we employ a two-stage network InvDEA model to analyze resource allocation in the management of the covid-19 pandemic, where inverse decision making plays a vital role in controlling undesirable outcomes arising from mortality.
3. The proposed inverse network data envelopment analysis model identifies the minimum required changes in hospital resources to improve treatment outcomes. This feature helps hospital managers achieve the maximum reduction in undesirable outputs(mortality) with the lowest possible cost of variable adjustments. Unlike conventional models that merely focus on efficiency evaluation, the proposed model is designed for policy targeting. Specifically, it enables the determination of the required levels of resources (specialist doctor and active beds) needed to achieve specified efficiency level and reduce mortality among covid-19 patients.
4. We have validated this approach using a case study.

This paper is structured as follows: Section 2 will introduce fundamental concepts of InvDEA, basic two-stage networks and uncertainty theory. Section 3 will present the uncertain two-stage network InvDEA and some theorems are proved. The practical use of the model will be illustrated in Section 4, followed by conclusions in Section 5 and a discussion on future research.

2. Preliminarily

In this section the concepts of DEA, Inv DEA, the basic two-stage networks, and uncertain theory are reviewed.

2.1. Invers Data Envelopment Analysis

Assume DMU_j $(j = 1,2,\dots,n)$ consumes input vector $X_j = (x_{1j}, x_{2j}, \dots, x_{mj})$ to produce desirable output vector $Y_j^D = (y_{1j}^D, y_{2j}^D, \dots, y_{sj}^D)$ and undesirable output $Y_j^{ND} = (y_{1j}^{ND}, y_{2j}^{ND}, \dots, y_{s'j}^{ND})$. For the evaluated DMU_k $k \in \{1,2,\dots,n\}$ the CCR model is as follow:

$$
\begin{aligned}
& min \quad \theta_k \\
& s.t. \quad \sum_{j=1}^{n} \lambda_j x_{ij} \le \theta_k x_{ik}, i = 1,2,\dots,m \\
& \sum_{j=1}^{n} \lambda_j y_{rj}^D \ge y_{rk}^D \ , r = 1,2,\dots,s \\
& \sum_{j=1}^{n} \lambda_j y_{r'j}^{ND} \le y_{r'k}^{ND} \ , r' = 1,2,\dots,s' \\
& \lambda_j \ge 0, j = 1,2,\dots,n
\end{aligned}
\tag{1}
$$

Definition 1: The $DMU_k$is (weak) efficient when the optimal value of the model 1 is equal to one. InvDEA models aim to address queries such as: if DMU_k $k \in \{1,2,\ldots,n\}$ alters its output to how much input is needed to maintain DMU_k's relative efficiency? Assume DMU_k changes its output level from (y_k^D, y_k^{ND}) to $(\beta_k^D, \beta_k^{ND}) = (y_k^D + \Delta y_k^D, y_k^{ND} + \Delta y_k^{ND})$ .The DEA literature introduces the following Multiple Objectives Linear Programming (MOLP) model for calculating this required input.

$$\begin{aligned} min \quad & (\alpha_{1k}, \alpha_{2k}, \ldots, \alpha_{mk}) = (x_{1k} + \Delta x_1, x_{2k} + \Delta x_2, \ldots, x_{mk} + \Delta x_m) \\ s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta_k^* \alpha_{ik}, i = 1,2,\ldots,m \\ & \sum_{j=1}^{n} \lambda_j y_{rj}^D \geq \beta_{rk}^D, r = 1,2,\ldots,s \\ & \sum_{j=1}^{n} \lambda_j\, y_{r'j}^{ND} \leq \beta_{r'k}^{ND} \ , r' = 1,2,\ldots,s' \\ & \lambda_j \geq 0, \ j = 1,2,\ldots,n \end{aligned} \tag{2}$$

Where $\alpha_{ik} = (x_{1k} + \Delta x_1, x_{2k} + \Delta x_2, \ldots, x_{mk} + \Delta x_m)$ represents the required inputs to guarantee the unchanged DMU_k's relative efficiency.λ_j $(j = 1,2,\ldots,n)$ denote the intensity vectors , $(\beta_{rk}^D, \beta_{r'k}^{ND})$ $before\ define\ and$ θ_k^* represents the optimal value of the CCR model .

Definition 2: Let $(\lambda, \alpha) = (\lambda_1, \lambda_2, \ldots, \lambda_n; \alpha_1, \alpha_2, \ldots, \alpha_m)$ represent a feasible solution for model 2, If there is not any feasible solution such as $(\bar{\lambda}, \bar{\alpha})$ for model 2 where $\bar{\alpha}_i < \alpha_i, i = 1,2,\ldots,m$, then (λ, α) is considered a weakly efficient solution for the model.

Theorem 1: Assume that DMU_k changes its output level from (y_k^D, y_k^{ND}) to $(\beta_k^D, \beta_k^{ND}) = (y_k^D + \Delta y_k^D, y_k^{ND} + \Delta y_k^{ND})$. If $(\bar{\lambda}, \bar{\alpha})$ is a weak efficient solution of MOLP model (2) then the efficiency score of new DMU_k's stays unchanged.

Proof. See (Ghiyasi, 2015)

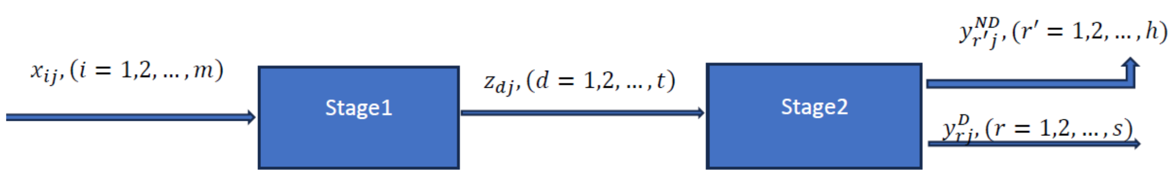

Fig 1. A two-stage system with undesirable outputs.

2.2. Two-stage InvDEA with undesirable outputs

Suppose that n DMUs need to be evaluated as shown in Fig1. For each DMU_j $(j = 1,2,\ldots,n)$ stage 1 consumes m inputs, $X_j = (x_{1j}, x_{2j}, \ldots, x_{mj})$, and produces t outputs, $Z_j = (z_{1j}, z_{2j}, \ldots, z_{tj})$,which are called intermediate measures. Afterward, these t intermediate measures are treated as inputs in stage 2, which produces s desirable outputs, $Y_j^D = (y_{1j}^D, y_{2j}^D, \ldots, y_{sj}^D)$, and h undesirable outputs, $Y_j^{ND} = (y_{1j}^{ND}, y_{2j}^{ND}, \dots, y_{h'j}^{ND})$.(An et al., 2019) proposed the input-oriented network-DEA model to measure the performance of the two-stage system by undesirable outputs. Their model is envelopment-based and can measure the overall efficiency of two-stage system.

$$
\begin{aligned}
&\min \ \theta_k \\
&s.t. \quad \sum_{j=1}^{n} \lambda_j x_{ij} \le \theta_k x_{ik}, i = 1,2,\dots,m \\
&\sum_{j=1}^{n} \lambda_j z_{dj} \ge z_{dk}, d = 1,2,\dots,t \\
&\sum_{j=1}^{n} \mu_j z_{dj} \le z_{dk}, d = 1,2,\dots,t \\
&\sum_{j=1}^{n} \mu_j y_{rj}^{D} \ge y_{rk}^{D}, \ r = 1,2,\dots,s \\
&\sum_{j=1}^{n} \mu_j y_{r'j}^{ND} \le y_{r'k}^{ND}, \ r' = 1,2,\dots,h \\
&\lambda_j \ge 0, \mu_j \ge 0, j = 1,2,\dots,n
\end{aligned} \tag{3}
$$

which optimal θ_k^* represents the overall efficiency score of the two-stage system with undesirable outputs. $\lambda_j \ (j = 1,2,\dots,n) and \ \mu_j (j = 1,2,\dots,n)$ denote the intensity vectors corresponding to stages 1 and 2.

Definition 3. If the optimal value θ_k^* of model (3) is unity, then DMU_k is (weakly) overall efficient. If the two-stage system DMU_k increases its undesirable outputs from y_k^{ND} to $\beta_k^{ND} = y_k^{ND} + \Delta y_k^{ND}$ and increases its desirable outputs from y_k^{D} to $\beta_k^{D} = y_k^{D} + \Delta y_k^{D}$ without changing its efficiency score θ_k^*, then how many new inputs and new intermediate measures will be produced for DMU_k?

Suppose that $\gamma_j = (\gamma_{1j}, \gamma_{2j}, \dots, \gamma_{tj})$ denotes the new intermediate measures of perturbed DMU_k.The DEA literature introduces the following two-stage InvDEA model with undesirable Outputs for calculating this required input.

$$
\begin{aligned}
&\min \ (\alpha_{1k}, \alpha_{2k}, \dots, \alpha_{mk}) = (x_{1k} + \Delta x_1, x_{2k} + \Delta x_2, \dots, x_{mk} + \Delta x_m) \\
&s.t. \quad \sum_{j=1}^{n} \lambda_j x_{ij} \le \theta_k^* \alpha_{ik}, i = 1,2,\dots,m \\
&\sum_{j=1}^{n} \lambda_j z_{dj} \ge \gamma_{dk}, d = 1,2,\dots,t \\
&\sum_{j=1}^{n} \mu_j z_{dj} \le \gamma_{dk}, d = 1,2,\dots,t \\
&\sum_{j=1}^{n} \mu_j y_{rj}^{D} \ge \beta_{rk}^{D}, \ r = 1,2,\dots,s \\
&\sum_{j=1}^{n} \mu_j y_{r'j}^{ND} \le \beta_{r'k}^{ND}, \ r' = 1,2,\dots,h \\
&\lambda_j \ge 0, \mu_j \ge 0, j = 1,2,\dots,n
\end{aligned} \tag{4}
$$

2.3. Uncertainty theory

Uncertainties arise from a variety of sources that can generally be categorized into uncertainties due to the nature of random phenomena and lack of knowledge or cognition. When there is not enough historic data, the experts' opinions are applied. For handling belief degrees Liu introduced uncertainty theory in 2007 (Liu & Liu, 2007). He modeled experts' opinions applying M as an m (Γ, L, M) with three axioms (normality, duality, and sub-additivity) is called an uncertainty space.
Aiming to achieve a product uncertain measure on the product σ-algebra L, the Product Axiom was defined by Liu (Liu & Liu, 2007). In the following, some concepts of uncertainty theory are presented and some of its features are reviewed.

Definition 4 :(Liu & Liu, 2007) Uncertain variable is a measurable function ξ from an uncertainty space (Γ,L,M) to the set of real numbers such that

$$\{\xi \in B\} = \{ \gamma \in \Gamma \mid \xi(\gamma) \in B\}$$

is an event for any Borel set B of real numbers?

The uncertainty distribution ϕ of an uncertain variable ξ is defined as

$$\phi(x) = M\{\xi \leq x\}, \forall x \in R$$

which is said to be regular if it is a continuous function and strictly increasing with respect to

x at which $0 < \phi(x) < 1$, $\lim_{x\to-\infty} \phi(x) = 0$ and $\lim_{x\to\infty} \phi(x) = 1$.In this case, the inverse uncertainty

distribution of ξ is the invers function of ϕ and is shown by $\phi^{-1}(\alpha)$. When ϕ is regular, the

expected value of ξ is as follow:

$$E[\xi] = \int_0^1 \phi^{-1}(\alpha)d\alpha$$

There are different uncertainty distributions mentioned in the literature (Liu & Liu, 2007) . The following
uncertainty distribution, for example, is called a linear uncertain variable.

$$\psi(x) = \begin{cases} 0, & x < w \\ \dfrac{x-w}{v-w} & w \leq x \leq v \\ 1, & x \geq v \end{cases}$$

where w and v are integers and $w < v$.

The inverse uncertainty distribution and the expected value of a function of certain independent uncertain variables will be determined by using the following theorem.

Theorem 2: (Liu, 2017) Let $\xi_1, \xi_2, \dots, \xi_n$, be independent uncertain variables with regular uncertainty distributions $\phi_1, \phi_2, \dots, \phi_n$, respectively. If f is strictly increasing with respect to $\xi_1, \xi_2, \dots, \xi_m$ and strictly decreasing with respect to$\xi_{m+1}, \xi_{m+2}, \dots, \xi_n$ then
(I) $\xi = f(\xi_1, \xi_2, \dots, \xi_n)$ is an uncertain variable and its invers uncertainty distribution is as follow:

$$\phi^{-1}(\alpha) = f(\phi^{-1}{}_1(\alpha), \dots, \phi^{-1}{}_m(\alpha), \phi^{-1}{}_{m+1}(1-\alpha), \dots, \phi^{-1}{}_n(1-\alpha))$$

(II) the uncertain variable $\xi = f(\xi_1, \xi_2, \dots, \xi_n)$ has an expected value as follows:

$E[\xi]=\int_0^1 f(\ \phi^{-1}{}_1(\alpha),\ldots,\ \phi^{-1}{}_m(\alpha),\ \phi^{-1}{}_{m+1}(1-\alpha),\ldots,\ \phi^{-1}{}_n(1-\alpha))d\alpha$

There are situations where the information should be determined by expert opinion. Uncertain theory can be used to evaluate a system in these circumstances.

3. Uncertain two-stage network Inv DEA model

In this section we propose Uncertain Network Inv DEA model in which inputs and outputs are based on expert opinion. Suppose DMU_j $(j=1,2,\ldots,n)$ stage 1 consumes m inputs, $\breve{X}_j = (x_{1j},x_{2j},\ldots,x_{mj})$, and produces t outputs, $\breve{Z}_j = (z_{1j},z_{2j},\ldots,z_{tj})$,which are called intermediate measures. Afterward, these t intermediate measures are treated as inputs in stage 2, which produces s desirable outputs, $\breve{Y}_j^D = (y_{1j}^D,y_{2j}^D,\ldots,y_{sj}^D)$, and h undesirable outputs, $\breve{Y}_j^{ND} = (y_{1j}^{ND},y_{2j}^{ND},\ldots,y_{h'j}^{ND})$. To assess the relative efficiency of DMU_k $k\in\{1,2,\ldots,n\}$ under CRS technology the following model is considered:

$$
\begin{aligned}
&min\ \ \theta_k\\
&s.t.\ \ \sum_{j=1}^{n}\lambda_j E(\breve{x}_{ij})-\theta_k E(\breve{x}_{ik})\ \le 0\ ,i=1,2,\ldots,m\\
&\quad -\textstyle\sum_{j=1}^{n}\lambda_j E(\breve{z}_{dj})+E(\breve{z}_{dk})\le 0, d=1,2,\ldots,t\\
&\quad \textstyle\sum_{j=1}^{n}\mu_j E(\breve{z}_{dj})-E(\breve{z}_{dk})\le 0,\ d=1,2,\ldots,t\\
&\quad -\sum_{j=1}^{n}\mu_j E(\breve{y}_{rj}^D)+E(\breve{y}_{rk}^D)\ \ \le 0\ ,\ r=1,2,\ldots,s\\
&\quad \sum_{j=1}^{n}\mu_j E(\breve{y}_{r'j}^{ND})-E\left(\breve{y}_{r'k}^{ND}\right)\ \le 0\ ,\ r^{'}=1,2,\ldots,h \qquad (5)\\
&\quad \lambda_j\ge 0,\mu_j\ge 0, j=1,2,\ldots,n
\end{aligned}
$$

Traditional Network InvDEA models cannot provide accurate results. to address this gap in the literature, an Uncertain Network InvDEA (UNInvDEA) model will be introduced in the following section. Model 5 is an uncertain model. To make Model 5 deterministic, the Theorem 3 is introduced in the following manner.

Theorem3: suppose $\breve{X}_j = \left(\breve{x}_{1j},\breve{x}_{2j},\ldots,\breve{x}_{mj}\right)$,$\breve{Z}_j=\left(\breve{z}_{1j},\breve{z}_{2j},\ldots,\breve{z}_{tj}\right)$,$\breve{Y}_j^D\ \left(\breve{y}_{1j}^D,\breve{y}_{2j}^D,\ldots,\breve{y}_{sj}^D\right)$, $and\ \breve{Y}_j^{ND} = \left(\breve{y}_{1j}^{ND},\breve{y}_{2j}^{ND},\ldots,\breve{y}_{h'j}^{ND}\right)$ are independent uncertain input, intermediate ,desirable output, and undesirable output variables respectively with regular distribution $\phi_j = \left(\varphi_{1j},\varphi_{2j},\ldots,\varphi_{mj}\right)$, $Z_j = \left(\xi_{1j},\xi_{2j},\ldots,\xi_{tj}\right)$, $\Psi_j = \left(\psi_{1j},\psi_{2j},\ldots,\psi_{sj}\right), and\ \Gamma_j = (\gamma_{1j},\gamma_{2j},\ldots,\gamma_{hj})$ for inputs, intermediate, desirable outputs, and undesirable outputs. The crisp form of Model 5 is as follow.

$$
\begin{aligned}
& min \quad \theta_k \\
& s.t. \quad \int_0^1 [\sum_{j=1}^{n} \lambda_j \, {\varphi_{ij}}^{-1}(\alpha) - \theta_k \, {\varphi_{ik}}^{-1}(1-\alpha)] d\alpha \; \leq 0 \, , i = 1,2, \dots , m \\
& \qquad \int_0^1 [- \textstyle\sum_{j=1}^{n} \lambda_j \, {z_{dj}}^{-1}(1-\alpha) + {\xi_{dk}}^{-1}(\alpha)] d\alpha \leq 0, d = 1,2, \dots , t \\
& \qquad \int_0^1 [\textstyle\sum_{j=1}^{n} \mu_j {z_{dj}}^{-1} (\alpha) - {\xi_{dk}}^{-1}(1-\alpha)] d\alpha \leq 0, \; d = 1,2, \dots , t \\
& \qquad \int_0^1 [- \sum_{j=1}^{n} \mu_j \, {\psi_{rj}}^{-1}(1-\alpha) + {\psi_{rk}}^{-1}(\alpha)] d\alpha \; \leq 0 \; , r = 1,2, \dots , s \\
& \qquad \int_0^1 [\sum_{j=1}^{n} \mu_j {\gamma_{r'j}}^{-1}(\alpha) - {\gamma_{r'k}}^{-1}(1-\alpha)] d\alpha \; \leq 0 \, , \; r' = 1,2, \dots , h \qquad (6) \\
& \qquad \quad \lambda_j \geq 0, \mu_j \geq 0, j = 1,2, \dots , n
\end{aligned}
$$

Proof: see (Pourmahmoud & Bagheri, 2021).

Now if the two-stage system $DMU_k \; k \in \{1,2,\dots,n\}$ perturbs its uncertain desirable Outputs y_k^D to $\breve{\beta}_k^D = \breve{y}_k^D + \widetilde{\Delta y}_k^D$ and uncertain undesirable outputs $\breve{y}_k^{ND}$ to $\breve{\beta}_k^{ND} = \breve{y}_k^{ND} + \widetilde{\Delta y}_k^{ND}$. The InvDEA models determine the uncertain input level and intermediate measures required to maintain the previous efficiency despite the mentioned perturbation. the Uncertain Network MOLP (UNMOLP) problem is proposed as a multi-objective problem as follows:

$$
\begin{aligned}
& min \quad E(\breve{\alpha}_{1k}, \breve{\alpha}_{2k}, \dots, \breve{\alpha}_{mk}) = E\big(\breve{x}_{1k} + \widetilde{\Delta x}_1, \breve{x}_{2k} + \widetilde{\Delta x}_2, \dots, \breve{x}_{mk} + \widetilde{\Delta x}_m\big) \\
& s.t. \quad E(\sum_{j=1}^{n} \lambda_j \breve{x}_{ij} - \theta_k^* \breve{\alpha}_{ik}) \leq 0, i = 1,2, \dots, m \\
& \qquad \mathrm{E}(- \textstyle\sum_{j=1}^{n} \lambda_j \, \breve{z}_{dj} + \breve{\gamma}_{ik}) \leq 0, d = 1,2, \dots , t \\
& \qquad \mathrm{E}(\textstyle\sum_{j=1}^{n} \mu_j \, \breve{z}_{dj} - \breve{\gamma}_{ik}) \leq 0, \; d = 1,2, \dots , t \\
& \qquad E\left(- \sum_{j=1}^{n} \mu_j \, \breve{y}_{rj}^D + \breve{\beta}_{rk}^D \right) \leq 0 \quad , r = 1,2, \dots , s \\
& \qquad E(\sum_{j=1}^{n} \mu_j \, \breve{y}_{r'j}^{ND} - \breve{\beta}_{r'k}^{ND}) \leq 0, \; r' = 1,2, \dots , h \qquad (7) \\
& \qquad \lambda_j \geq 0, \mu_j \geq 0, j = 1,2, \dots , n
\end{aligned}
$$

Assume that all inputs are priced, and the weights are specified values. If w_i is the weight for the i-th input the weighted sum model is proposed as follow:

$$
\begin{aligned}
& min \quad E(\textstyle\sum_{i=1}^{m} w_i \, \breve{\alpha}_i) \\
& s.t. \quad E(\sum_{j=1}^{n} \lambda_j \breve{x}_{ij} - \theta_k^* \breve{\alpha}_{ik}) \leq 0, i = 1,2, \dots, m
\end{aligned}
$$

$$\mathrm{E}\left(-\sum_{j=1}^{n}\lambda_j\,\check{z}_{dj}+\check{\gamma}_{ik}\right)\le 0,\ d=1,2,\ldots,t$$
$$\mathrm{E}\left(\sum_{j=1}^{n}\mu_j\,\check{z}_{dj}-\check{\gamma}_{ik}\right)\le 0,\ d=1,2,\ldots,t$$
$$E\left(-\sum_{j=1}^{n}\mu_j\,\check{y}_{rj}^{D}+\check{\beta}_{rk}^{D}\right)\le 0,\ r=1,2,\ldots,s$$
$$E\left(\sum_{j=1}^{n}\mu_j\,\check{y}_{r'j}^{ND}-\check{\beta}_{r'k}^{ND}\right)\le 0,\ r'=1,2,\ldots,h \tag{8}$$
$$\lambda_j\ge 0,\mu_j\ge 0, j=1,2,\ldots,n$$

Model 8 is nondeterminate. To obtain a deterministic representation of the model, the following theorem is proposed proposition.

Theorem4: suppose $\check{X}_j=\left(\check{x}_{1j},\check{x}_{2j},\ldots,\check{x}_{mj}\right)$, $\check{Z}_j=\left(\check{z}_{1j},\check{z}_{2j},\ldots,\check{z}_{tj}\right)$, $\check{Y}_j^D=\left(\check{y}_{1j}^D,\check{y}_{2j}^D,\ldots,\check{y}_{sj}^D\right)$, $\check{Y}_j^{ND}=\left(\check{y}_{1j}^{ND},\check{y}_{2j}^{ND},\ldots,\check{y}_{h'j}^{ND}\right)$, $\check{\alpha}_k=(\check{\alpha}_{1k},\check{\alpha}_{2k},\ldots,\check{\alpha}_{mk}), and\ \check{\beta}_k=(\check{\beta}_{1k},\check{\beta}_{2k},\ldots,\check{\beta}_{mk})$ are independent uncertain input, uncertain intermediate ,uncertain desirable output, uncertain undesirable output variables , increasing the level of uncertain inputs, and increasing the level of uncertain outputs respectively with regular distribution $\phi_j=(\varphi_{1j},\varphi_{2j},\ldots,\varphi_{mj})$, $Z_j=\left(\xi_{1j},\xi_{2j},\ldots,\xi_{tj}\right)$, $\Psi_j=\left(\psi_{1j},\psi_{2j},\ldots,\psi_{sj}\right),\Gamma_j=\left(\gamma_{1j},\gamma_{2j},\ldots,\gamma_{hj}\right),\Upsilon_k=(\upsilon_{1k},\upsilon_{2k},\ldots,\upsilon_{mk})$, and $H_k=(\eta_{1k},\eta_{2k},\ldots,\eta_{sk})$. The equivalent of Model 8 is as follow:

$$min\ \int_0^1\Big[\sum_{i=1}^{m}w_i\,{\upsilon_{ik}}^{-1}(\alpha)\Big]\mathrm{d}\alpha$$
$$s.t.\quad \int_0^1\Big[\sum_{j=1}^{n}\lambda_j\,{\varphi_{ij}}^{-1}(\alpha)-\theta_k^*{\upsilon_{ik}}^{-1}(1-\alpha)\Big]d\alpha\le 0,\ i=1,2,\ldots,m$$
$$\int_0^1\Big[-\sum_{j=1}^{n}\lambda_j\,{\xi_{dj}}^{-1}(1-\alpha)+{\xi_{dk}}^{-1}(\alpha)\Big]d\alpha\le 0, d=1,2,\ldots,t$$
$$\int_0^1\Big[\sum_{j=1}^{n}\mu_j{\xi_{dj}}^{-1}(\alpha)-{\xi_{dk}}^{-1}(1-\alpha)\Big]d\alpha\le 0,\ d=1,2,\ldots,t$$
$$\int_0^1\Big[-\sum_{j=1}^{n}\mu_j\,{\psi_{rj}}^{-1}(1-\alpha)+{\eta_{rk}}^{-1}(\alpha)\Big]d\alpha\le 0,\ r=1,2,\ldots,s$$
$$\int_0^1\Big[\sum_{j=1}^{n}\mu_j{\gamma_{r'j}}^{-1}(\alpha)-{\eta_{r'k}}^{-1}(1-\alpha)\Big]d\alpha\le 0,\ r'=1,2,\ldots,h \tag{9}$$
$$\lambda_j\ge 0,\mu_j\ge 0, j=1,2,\ldots,n$$

Proof: Consider the following relationships:

$$f_1(\tilde{\alpha}_i)=\sum_{i=1}^{m}w_i\,\tilde{\alpha}_i \tag{10}$$
$$f_2(\lambda_j,\tilde{x}_{ij},\tilde{\alpha}_i)=\sum_{j=1}^{n}\lambda_j\tilde{x}_{ij}-\theta_k^*\tilde{\alpha}_i \tag{11}$$

$$f_3(\lambda_j, \tilde{z}_{dj}, \tilde{\gamma}_{ik}) = -\sum_{j=1}^{n} \lambda_j \tilde{z}_{dj} + \tilde{\gamma}_{ik} \tag{12}$$

$$f_4(\mu_j, \tilde{z}_{dj}, \tilde{\gamma}_{ik}) = \sum_{j=1}^{n} \mu_j \tilde{z}_{dj} + \tilde{\gamma}_{ik} \tag{13}$$

$$f_5(\mu_j, \tilde{y}_{rj}^{D}, \tilde{\beta}_{rk}) = -\sum_{j=1}^{n} \mu_j \tilde{y}_{rj}^{D} + \tilde{\beta}_{rk}^{D} \tag{14}$$

$$f_6(\mu_j, \tilde{y}_{r'j}^{ND}, \tilde{y}_{r'k}^{ND}) = \sum_{j=1}^{n} \mu_j \, \tilde{y}_{r'j}^{ND} - \tilde{\beta}_{r'k}^{ND} \tag{15}$$

According to the initial section of Theorem 2, the inverse uncertainty distributions of the uncertain variables mentioned above are as follows:

$$f_1^{-1}(\alpha) = \sum_{i=1}^{m} w_i \, \upsilon_{ik}^{-1}(\alpha) \tag{16}$$

$$f_2^{-1}(\alpha) = \sum_{j=1}^{n} \lambda_j \phi_{ij}^{-1}(\alpha) - \theta_k^* \upsilon_{ik}^{-1}(1-\alpha) \tag{17}$$

$$f_3^{-1}(\alpha) = -\sum_{j=1}^{n} \lambda_j \, {\xi_{dj}}^{-1}(1-\alpha) + {\xi_{dk}}^{-1}(\alpha) \tag{18}$$

$$f_4^{-1}(\alpha) = \sum_{j=1}^{n} \mu_j {\xi_{dj}}^{-1}(\alpha) - {\xi_{dk}}^{-1}(1-\alpha) \tag{19}$$

$$f_5^{-1}(\alpha) = -\sum_{j=1}^{n} \mu_j \psi_{rj}^{-1}(1-\alpha) + \zeta_{rk}^{-1}(\alpha) \tag{20}$$

$$f_6^{-1}(\alpha) = \sum_{j=1}^{n} \mu_j {\gamma_{r'j}}^{-1}(\alpha) - {\eta_{r'k}}^{-1}(1-\alpha) \tag{21}$$

Furthermore, based on the latter part of Theorem 2 we can consider the following relationships:

$$E(f_1) = \int_0^1 \left[\sum_{i=1}^{m} w_i \upsilon_{ik}^{-1}(\alpha) \right] d\alpha \tag{22}$$

$$E(f_2) = \int_0^1 \left[\sum_{j=1}^{n} \lambda_j \phi_{ij}^{-1}(\alpha) - \theta_k^* \upsilon_{ik}^{-1}(1-\alpha) \right] d\alpha \quad i = 1,2,\dots,m \tag{23}$$

$$E(f_3) = \int_0^1 \left[-\sum_{j=1}^{n} \lambda_j \, {\xi_{dj}}^{-1}(1-\alpha) + {\xi_{dk}}^{-1}(\alpha) \right] d\alpha \quad d = 1,2,\dots,\mathrm{t} \tag{24}$$

$$E(f_4) = \int_0^1 \left[\sum_{j=1}^{n} \mu_j {\xi_{dj}}^{-1}(\alpha) - {\xi_{dk}}^{-1}(1-\alpha) \right] d\alpha \quad d = 1,2,\dots,t \tag{25}$$

$$E(f_5) = \int_0^1 \left[-\sum_{j=1}^{n} \mu_j \psi_{rj}^{-1}(1-\alpha) + \zeta_{rk}^{-1}(\alpha)) \right] d\alpha \quad r = 1,2,\dots,s \tag{26}$$

$$E(f_6) = \int_0^1 \left[\sum_{j=1}^{n} \mu_j \, \gamma_{r'j}^{-1}(\alpha) - \gamma_{r'k}^{-1}(1-\alpha) \right] d\alpha \quad r' = 1,2,\dots,s' \tag{27}$$

Therefore, Model 9 is the equivalent crisp form of Model 8.

Definition 5. Suppose that $(\alpha_{ik}^*, \lambda_j^*, \mu_j^*, \gamma_{ik}^*)$ is a feasible solution model (8). If there is no feasible solution $(\bar{\alpha}_{ik}, \bar{\lambda}_j, \bar{\mu}_j, \bar{\gamma}_{ik})$ such that $\bar{\alpha}_{ik} < \alpha_{ik}^*$ for i=1,2,…,m in model (8),then $(\alpha_{ik}^*, \lambda_j^*, \mu_j^*, \gamma_{ik}^*)$ is a weak pareto optimal solution for model (8).

We then check whether the efficiency of the perturbed DMU_k is consistent with that of the initial DMU_k. Suppose that DMU_{k+1} represents the perturbed DMU_k. The efficiency of DMU_{k+1} can be estimated by the following model.

$$
\begin{aligned}
&\text{Min } \theta_k \\
&s.t.\quad E\left[\sum_{j=1}^{n} \lambda_j x_{ij} + \lambda_{n+1}\alpha_{ik} - \theta_k \alpha_{ik}\right] \le 0\ , i = 1,2,\dots,m \\
&\text{E}[-\ \textstyle\sum_{j=1}^{n} \lambda_j z_{dj} - \lambda_{n+1}\gamma_{ik} + \gamma_{ik}] \le 0, d = 1,2,\dots,t \\
&\text{E}[\ \textstyle\sum_{j=1}^{n} \mu_j z_{dj} + \mu_{n+1}\gamma_{ik} - \gamma_{ik}] \le 0, d = 1,2,\dots,t \\
&E\left[-\sum_{j=1}^{n} \mu_j y_{rj}^D - \mu_{n+1}\beta_{rk}^D + \beta_{rk}^D\right] \le 0\ \ , r = 1,2,\dots,s \\
&E\left[\sum_{j=1}^{n} \mu_j y_{r'j}^{ND} + \mu_{n+1}\beta_{r'k}^{ND} - \beta_{r'k}^{ND}\right] \le 0\ \ , r' = 1,2,\dots,h \\
&\lambda_j \ge 0, \mu_j \ge 0, j = 1,2,\dots,n+1
\end{aligned}
\qquad (28)
$$

Model 28 is also nondeterminate. To obtain a deterministic representation of the model similar to the previous model, it can be derived.

Theorem 5. Suppose that θ_k^* is the optimal efficiency of model (5) for DMU_k, that the desirable outputs increase from y_{rk}^D to $\breve{\beta}_{rk}^D = \breve{y}_{rk}^D + \widetilde{\Delta y}_{rk}^D$ ($\widetilde{\Delta y}_{rk}^D \geq 0$ and $\widetilde{\Delta y}_{rk}^D \neq 0$), and that the undesirable outputs increase from $y_{r'k}^{ND}$ to $\beta_{r'k}^{ND} = y_{r'k}^{ND} + \Delta y_{r'k}^{ND}$ ($\Delta y_{r'k}^{ND} \geq 0$ and $\Delta y_{r'k}^{ND} \neq 0$). If $(\alpha_{ik}^*, \lambda_j^*, \mu_j^*, \gamma_{ik}^*)$ is a weak Pareto optimal solution of the MOLP model (8), then the optimal value of model (28) is also θ_k^* .
Proof: see (An et al., 2019).

4. Application

In COVID-19 pandemic, probabilistic statistics may be unable to perform correctly because there have not been any similar circumstances. we applied our model to assess the performance of healthcare systems during COVID-19 outbreak. In this section uses the developed model to evaluate the efficiency of 30 hospital from Iran, considering deaths of patients as an uncertain undesirable output. The data is given in Table 1. The number of operational bed and the number of physicians is two indicators which are considered as inputs, patients under treatment as an intermediate product. The output can be divided into desirable and undesirable outputs desirable outputs treated patients, while undesirable outputs include deaths of patients.

Table 1.**the data of hospital from Iran**

hospitals	The number of operational bed(x1)	The number of specialist doctor(x2)	Patients under treatment (z_1)	Treated patients (y_1^D)	deaths of patients (y_1^{ND})
1	45	90	820	740	15
2	38	70	590	530	11
3	50	100	900	860	7
4	30	60	480	420	13
5	60	110	1000	940	10
6	42	95	780	700	13
7	35	80	640	590	11
8	47	85	820	770	9
9	39	75	620	570	13
10	55	105	970	900	10
11	33	65	510	460	16
12	41	90	750	710	10
13	48	92	840	790	8
14	36	68	580	530	12
15	52	108	960	910	7
16	40	88	730	690	10
17	46	100	880	820	9
18	34	72	600	550	12
19	37	78	610	570	11
20	49	98	890	840	8
21	43	96	770	720	14
22	32	62	500	450	14
23	44	94	760	710	10
24	50	102	950	900	7
25	38	77	620	580	11
26	45	97	800	750	9

27	36	66	560	510	13
28	53	106	980	930	8
29	39	79	630	590	11
30	40	85	770	710	11

deaths of patients is an uncertain variable that should be determined by experts' opinions. This study uses uncertainty theory to apply experts' opinions and determine this index. We assume that the deaths of patients' index is an uncertain variable with a linear distribution, when the exact form of the uncertain distribution is unknown, the linear uncertain distribution is a standard and widely adopted choice in liu's theory. This distribution satisfies the least informative principle and allows the expected values of uncertain variables to be derived in closed form, there by preserving the linearity and tractability of the proposed models. expert opinions were collected using structured questionnaires and/or individual interviews. each expert provided the minimum and maximum credible values for each variable, along with their associated belief degrees. A panel of experts with comparable expertise was consulted, and the individual belief degrees were aggregated using the arithmetic mean to obtain the final uncertain parameters.

the endpoints of the uncertain variables can be approximated by ±3 standard deviations around the mean value when historical data are available. This approach provides a practical and widely accepted method to define credible bounds for uncertain inputs and outputs. In all cases, the intervals represent belief-based uncertainty in the sense of Liu's uncertainty theory, not probabilistic confidence intervals.

The result is shown in Table 2. We initially evaluate the performance of the 30 hospitals, whose data are presented in Table 3, by using the black box model and two-stage model.

Table 2. the uncertain data with linear distribution

hospitals	The number of operational bed(x1)	The number of specialist doctor(x2)	Patients under treatment (z_1)	Treated patients (y_1^D)	deaths of patients (y_1^{ND})
1	45	90	820	740	L[12,18]
2	38	70	590	530	L[8,15]
3	50	100	900	860	L[5,10]
4	30	60	480	420	L[10,17]
5	60	110	1000	940	L[7,13]
6	42	95	780	700	L[11,16]
7	35	80	640	590	L[9,14]
8	47	85	820	770	L[6,12]

9	39	75	620	570	L[10,16]
10	55	105	970	900	L[8,13]
11	33	65	510	460	L[13,19]
12	41	90	750	710	L[7,13]
13	48	92	840	790	L[6,11]
14	36	68	580	530	L[9,15]
15	52	108	960	910	L[5,10]
16	40	88	730	690	L[8,13]
17	46	100	880	820	L[6,12]
18	34	72	600	550	L[10,15]
19	37	78	610	570	L[9,14]
20	49	98	890	840	L[6,11]
21	43	96	770	720	L[11,17]
22	32	62	500	450	L[11,17]
23	44	94	760	710	L[8,13]
24	50	102	950	900	L[5,9]
25	38	77	620	580	L[9,14]
26	45	97	800	750	L[7,12]
27	36	66	560	510	L[10,16]
28	53	106	980	930	L[6,10]
29	39	79	630	590	L[8,14]
30	40	85	770	710	L[9,14]

Table 3. The results of black box model and two stage model

DMU	Black box model efficiency		Two-stage model efficiency	
	Classic model	Uncertain model	Classic model	Uncertain model
1	0.9531	0.8605	0.9404	0.9200
2	1.000	1.000	0.8078	0.8078
3	0.9467	0.9560	0.9067	0.8530

4	0.5932	0.5360	0.4820	0.4320
5	1.000	1.000	1.000	0.7990
6	0.9745	0.7982	0.8773	0.8355
7	1.000	1.000	1.000	1.000
8	1.000	1.000	1.000	1.000
9	0.8959	0.7498	0.7164	0.7164
10	0.9947	0.9333	0.9046	0.8045
11	1.000	0.5936	0.4554	0.4554
12	0.9735	0.9725	0.8405	0.8344
13	0.9311	0.9157	0.9047	0.8820
14	0.9702	0.9374	0.9188	0.9188
15	0.9600	0.9541	0.9556	0.8330
16	0.9788	0.9774	0.8455	0.8441
17	1.000	0.9889	0.8928	0.8628
18	1.000	1.000	0.9987	0.9987
19	0.7312	0.7260	0.6382	0.6382
20	0.9341	0.9312	0.9281	0.8766
21	1.000	0.8090	0.8126	0.7549
22	0.9951	0.9905	0.7274	0.7274
23	0.7753	0.7743	0.7052	0.7052
24	1.000	1.000	0.9417	0.9217
25	0.7875	0.7705	0.6809	0.6809
26	0.8239	0.8454	0.7761	0.7434
27	1.000	1.000	0.9763	0.9763
28	1.000	0.9383	0.8832	0.8832
29	0.7648	0.7298	0.6506	0.6506
30	1.000	1.000	1.000	1.000

First, the number of efficient DMUs identified by the black box classic model is 12, which is far greater than the number obtained by the two-stage classic model (4 DUM is efficient). This finding indicates that the discernment of the black box classical model is not as good as that of the two-stage classic model. Therefore, adopting the two-stage classical model to evaluate the efficiency of DMUs can obtain more authentic results that help the InvDEA model produce highly reasonable recommendations. second, by comparing the results, the efficiency of DMUs in the black box uncertain model is better than in the two-stage uncertain model. we aim to explore the relationship between inputs-intermedia, desirable output, undesirable output, and efficiency. It is crucial to note that achieving deaths of patients' reduction may not be feasible in the short term. Therefore, we explore two scenarios for deaths of patients' reduction. we increase the desirable and undesirable outputs by different percentages, with the percentage increase of desirable outputs higher than that of undesirable outputs. This target is more favored by DMUs given that people always expect to increase the desirable outputs as much as possible while minimizing the increase in the undesirable outputs. Therefore, we increase the desirable outputs by 15% and the undesirable outputs by 10% and 5%, to compare the efficiency in the classical two-stage network model and the uncertain two-stage network model.

Scenarios1: We increase desirable outputs by 15% and undesirable outputs by 10 % and analyze the new resource plans. We assume that the weight of each input is 1. Table3 reports the new number of inputs and the net percentage increase. The comparison of the results indicates increase in output requires increased inputs of all hospital in both models. According to the result, DMU5 is the most demanding hospital in the classical model. However, DMU12 is the most demanding hospital regarding the uncertain model. On the other hand, in the classical model, DMUs 22 and 11 are the least demanding hospital, with an average increase of approximately 0.16 percent while in the uncertain model, DMUs 2 and 27 are the least demanding hospitals, with an average increase of approximately 0.21 percent.

Scenarios2: The amount for increasing the desirable outputs by 15% and undesirable outputs by 5% is shown in Table 4. DMU5 is the most demanding hospital in the classical model. However, DMU11 is the most demanding hospital regarding the uncertain model. On the other hand, in the classical model, DMUs 22 and 27 are the least demanding hospital, with an average increase of approximately 0.20 percent while in the uncertain model, DMUs 4 and 27 are the least demanding hospitals, with an average increase of approximately 0.25 percent from the input–output perspective, with the same efficiency, having less outputs corresponds to having less inputs. Therefore, the new number of inputs in this scenario is less than that in scenario 1. Specifically, the resource amount of DMUs 6, 11,21 and 22 are equal or larger than those obtained in scenario 1, thereby suggesting that if these three DMUs want to minimize the increase in their undesirable outputs, they need to shoulder more costs compared with the other DMUs. Therefore, these DMUs must improve their technologies to save costs.

Table 4. the first scenario, expanding both outputs at different rates

15 percent expanding desirable output and 5 percent expanding undesirable output in classical model				15 percent expanding desirable output and 5 percent expanding undesirable output in uncertain model		
	θ	Δx	γ	θ	Δx	γ
1	0.9404	0.7139	0.6530	0.9200	1.2668	0.6110
2	0.8078	0.2209	0.2110	0.8078	0.4297	0.2161
3	0.9067	0.7824	0.8070	0.8530	1.7507	0.8368
4	0.4820	0.1724	0.1578	0.4320	0.2847	0.1087
5	1.000	1.000	0.9856	0.7990	2.0323	0.9874
6	0.8773	0.6566	0.5760	0.8355	1.2441	0.5357
7	1.000	0.3479	0.3070	1.0001	0.5365	0.3285

8	1.000	0.5320	0.6530	1.000	1.0722	0.6677
9	0.7164	0.3000	0.2690	0.7164	0.6670	0.2913
10	0.9046	0.9772	0.8840	0.8045	1.8901	0.9121
11	0.4554	0.1465	0.0572	0.4554	0.2990	0.0841
12	0.8405	0.5222	0.5190	0.8344	1.2338	0.5543
13	0.9047	0.7649	0.6920	0.8820	1.2813	0.7049
14	0.9188	0.2446	0.1920	0.9188	0.3915	0.2161
15	0.9556	0.9135	0.9230	0.8330	1.7903	0.9307
16	0.8455	0.4800	0.4800	0.8441	1.1452	0.5172
17	0.8928	0.6578	0.7690	0.8628	1.4419	0.7616
18	0.9987	0.1948	0.2300	0.9987	0.4204	0.2542
19	0.6382	0.3130	0.2500	0.6382	1.0949	0.2913
20	0.9281	0.7365	0.7880	0.8766	1.4593	0.7987
21	0.8126	0.7100	0.5770	0.7549	1.2214	0.5729
22	0.7274	0.0612	0.0380	0.7274	1.3123	0.0655
23	0.7052	0.7629	0.5380	0.7052	1.2659	0.5543
24	0.9417	0.8400	0.9030	0.9217	1.5937	0.9121
25	0.6809	0.3951	0.2690	0.6809	1.0001	0.3099
26	0.7761	0.7382	0.6150	0.7434	1.3606	0.6296
27	0.9763	0.1312	0.1530	0.9763	0.3085	0.1798
28	0.8832	0.9357	0.9610	0.8832	1.7531	0.9878
29	0.6506	0.4427	0.2880	0.6506	1.1067	0.3285
30	1.000	0.5181		1.000	0.8956	0.5543

Table 5. the second scenario, expanding both outputs at different rates

15 percent expanding desirable output and 10 percent expanding undesirable output in classical model				15 percent expanding desirable output and 10 percent expanding undesirable output in uncertain model		
	θ	Δx	γ	θ	Δx	γ
1	0.9404	0.5598	0.6324	0.9200	1.2778	0.7479
2	0.8078	0.2087	0.2728	0.8078	0.4497	0.3529
3	0.9067	0.8900	0.8903	0.8530	1.9285	0.9706
4	0.4820	0.1926	0.7057	0.4320	0.4625	0.2531
5	1.000	0.8262	1.0090	0.7990	2.1301	0.1222
6	0.8773	0.5246	0.5594	0.8355	1.2441	0.6726

7	1.000	0.2453	0.3754	1.000	0.5799	0.4653
8	1.000	0.6530	0.7137	1.000	1.1253	0.8033
9	0.7164	0.3000	0.3290	0.7164	0.7117	0.4282
10	0.9046	0.9000	0.9414	0.8045	2.0131	1.0487
11	0.4554	0.1048	0.1310	0.4554	0.7759	0.2209
12	0.8405	0.5170	0.6022	0.8344	2.3277	0.6912
13	0.9047	0.7649	0.7439	0.8820	1.3398	0.8392
14	0.9188	0.1670	0.2681	0.9188	0.4625	0.3529
15	0.9556	0.9079	0.9924	0.8330	1.9213	1.0645
16	0.8455	0.4753	0.5656	0.8441	1.3609	0.6540
17	0.8928	0.6578	0.7933	0.8628	1.4973	0.8963
18	0.9987	0.1928	0.2990	0.9987	0.4689	0.3910
19	0.6382	0.3130	0.3360	0.6382	1.1215	0.4282
20	0.9281	0.7365	0.8524	0.8766	1.5044	0.9323
21	0.8126	0.5673	0.5940	0.7549	1.2214	0.7097
22	0.7274	0.0417	0.1254	0.7274	1.3123	0.2024
23	0.7052	0.7629	0.5958	0.7052	1.3135	0.6912
24	0.9417	0.9030	0.9837	0.9217	1.7733	1.0472
25	0.6809	0.3156	0.3510	0.6809	2.2273	0.4468
26	0.7761	0.6868	0.6698	0.7434	1.4132	0.7664
27	0.9763	0.1252	0.2265	0.9763	0.3749	0.3158
28	0.8832	0.9200	1.0209	0.8832	1.7976	1.0996
29	0.6506	0.3706	0.3754	0.6506	2.0618	0.4653
30	1.000	0.4450	0.5790	1.000	0.9250	0.6912

5. Conclusion

Effective resource allocation is essential for enhancing performance across industries. Inverse Data Envelopment Analysis offers a post-DEA sensitivity analysis tool designed to address such allocation challenges. Traditional InvDEA models typically assume deterministic inputs and outputs; however, in many practical scenarios, this assumption does not hold due to inherent data uncertainties. To address this limitation, we propose an uncertain Inv network DEA model that explicitly incorporates undesirable outputs, such as deaths of patients, and accommodates uncertainty by replacing uncertain variables with their expected values, as estimated by expert opinions. This substitution allows for the conversion of the nonlinear uncertain model into an equivalent linear form, making it computationally tractable using standard linear solvers. To demonstrate the applicability of this model, we conducted a case study assessing the efficiency of hospital of Iran while considering deaths of patients as an undesirable output. Deaths of patients was treated as an uncertain variable based on expert assessments. We analyzed two scenarios to create a gradual death of patient's reduction strategy for the hospital. Furthermore, sensitivity and uncertainty analyses performed on the expected-value model allow researchers to quantify the impact of uncertainty on efficiency results. Future research could explore alternative uncertainty-handling methods, such as confidence-level or robust optimization approaches, to achieve greater precision.it is also possible to use various solution methods to solve multi-objective problems, such as uncertain Goal Programming can be studied in future work.

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