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An efficient solution algorithm for fuzzy linear fractional optimization problems with an application

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In this paper, we investigate a solution procedure for a fuzzy linear fractional optimization problem in which the input parameters are considered as convex fuzzy numbers. By applying a specific fuzzy ranking method which is based on the α-cut concept, and according to Charnes and Cooper's approach of variable transformation, the solution of the original fuzzy linear fractional optimization model is transformed into the solution of at most two semi-infinite linear programs that are dissimilar among themselves via a sign in a constraint and in the objective function. An appropriate cutting plane algorithm (CPA) of Fang is utilized to obtain the optimal solution of the semi-infinite linear programs. Further, the application of our presented algorithm in improvement facility location theory is discussed properly. Finally, an illustrative example is given to clarify the solution procedure.

Keywords: *fuzzy linear fractional optimization, fuzzy ranking method, improvement facility location model.*

1. Introduction

Fractional optimization is concerned with situations where a ratio between physical and/or economical functions, for example cost/time, cost/quality, cost/ productivity, inventory/sales, output/employee, or some other quantities that measure the relative effectiveness of an underlying system, is minimized or maximized. Such optimization models have recently been a subject of wide interest in nonlinear programming and literature survey unveils numerous applications of them. These applications arise in various subjects in operational research, e.g., resource location-allocation, transportation and logistics, production, finance, stochastic processes, game theory, applied linear algebra, information theory, large scale programming and etc. (see e.g. [7], [13], [24], [25], [26]).

A fractional optimization problem with linear numerator and denominator of the objective function and linear constraints is called a linear fractional optimization (LFO) problem. Many practical problems like cutting stock problems, product planning, financial problems, blending problems, capital budgeting problems and etc. [2] are formulated as LFO models. In general case, a LFO problem is stated as:

$$\text{Maximize} \quad Z(X) = \frac{\sum_{j=1}^{n} c_j\, x_j + c}{\sum_{j=1}^{n} d_j\, x_j + d},$$

$$\text{subject to} \qquad \sum_{j=1}^{n} a_{ij}\, x_j \leq\ b_i, \quad i = 1, \dots, m, \qquad (1)$$

$$x_j \geq 0, \quad j = 1, \dots, n,$$

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where for every index $i = 1, \ldots, m, j = 1, \ldots, n$, the parameters/variables $a_{ij},\ b_i\,,\ c_j\,,\ d_j\,,\ x_j$ and d are real numbers.

There are available several methods available for solving LFO problems (see e.g. [2], [3], [26]) such that one of the very useful approaches for mathematicians is the method of variable transformation which was developed by Charnes and Cooper [5] in 1965. They showed that a LFO problem can be solved via solving two corresponding linear programming models. Later in 1968, Zionts [29] presented an idea about Charnes and Cooper's technique. He demonstrated that if the LFO problem carries out a finite optimal solution, then for all practical variants, the denominator

$$\sum_{j=1}^{n} d_j\, x_j + d$$

cannot have two different signs on the feasible solution region of the LFO problem. Consequently, only one of the two corresponding linear programs must be solved according to the sign of denominator.

In the real world decision problems, which are formulated as LFO models, we may not know the exact values of the input parameters and the uncertainty may not be of a probabilistic type. In this case, it would be suitable to interpret the uncertain parameters of the LFO problem as fuzzy numerical data which can be represented by means of fuzzy numbers. In the past years, several papers have appeared to LFO problems in the fuzzy framework, such that in 1992, Dutta, Rao and Tiwari [10] purposed a LFO problem where the coefficients of the numerator or denominator, or both, of the objective function are fuzzy numbers. They showed that this problem can be converted to an equivalent fuzzy linear programming problem. Li and Chen (1996) in the paper [19], investigated another fuzzy programming approach for the LFO problem with fuzzy coefficients. The authors presented the concept and mathematical definition of the fuzzy optimal value (FOV) and introduced an approximate algorithm for solving the FOV. A goal programming procedure has been presented by Pal, Moitra and Maulik [21] in 2003 for a multi-objective LFO problem in which an imprecise aspiration level is defined to each of the fractional objectives, such that these fuzzy objectives are termed as fuzzy goals. In 2006, Dash, Panda and Nanda [9] considered a generalized fractional 0-1 programming problem in which the objective is formed in fuzzy environment and the decision variables are restricted to be 0 or 1. They proved that this problem can be transformed to a nonlinear programming problem. In 2007, Mehra, Chandra and Bector [20] proposed two new concepts of (α,β)-acceptable optimal solution and (α,β)-acceptable optimal value for a specific class of fuzzy linear fractional optimization problem in which it is assumed that the numerator of the problem is a nonnegative fuzzy number and the denominator of the problem is a positive fuzzy number. They developed a method to obtain the (α,β)-acceptable optimal solution and (α,β)-acceptable optimal value of the problem. Their proposed method is not able to find the optimal solution and optimal objective value of the problem. Moreover, for ordering the fuzzy numbers, they utilized a model of fuzzy ranking method introduced by Wu [27]. Pop and Stancu-Minasian [22] derived a solution method to the fully fuzzified linear fractional programming problems, where all the parameters and variables are limited to be only triangular fuzzy numbers. Their method contains the transformation of the problem of maximizing a function with triangular fuzzy value to a deterministic multiple objective linear fractional programming problem with quadratic constraints. Moreover, in order to evaluate the fuzzy constraints, they applied the Kerre's ranking method. In 2024, Prasad et al. [23] provided a fuzzy goal programming approach to the fractional linear program with triangular fuzzy parameters. The the intuitionistic fuzzy linear fractional programming problem was treated by Karthick and Saraswathi [17] and a solution method was developed where it directly tackles the problem in its fuzzy domain. In 2025, using the splitting technique, an approach was proposed to fully fuzzy linear fractional transportation problem in [8].

In this paper, considering an almost different strategy, we attempt to provide an efficient solution algorithm for a fuzzy linear fractional program in which all input parameters along the constraints and numerator of the objective function are convex fuzzy numbers. In particular, an application of our results in the improvement facility location theory is also discussed properly.

Table1. The list of important notations along the paper

Notation	Description
$\tilde{Z}(\cdot)$	objective function of the fuzzy fractional program
x_j	decision variables of the fuzzy fractional program
$\tilde{M}$	convex fuzzy number
Γ	set of convex fuzzy numbers
$\tilde{M}_\alpha = [\tilde{M}^L(\alpha), \tilde{M}^R(\alpha)]$	α-cut of $\tilde{M}$
$\boldsymbol{H}_{\boldsymbol{\alpha}}(.)$	appraisal function
Ω_α	feasible region of semi-infinite fractional program
$T = (V(T), E(T))$	tree network with vertex set $V(T)$ and edge set $E(T)$
$\tilde{w}_{oi}$	weight of vertex v_i for commodity o
l_e	length of edge e
$P[v_i, v_j]$	unique path between vertices v_i and v_j
$d(v_i, v_j)$	distance between vertices v_i and v_j under original lengths
$\bar{d}(v_i, v_j)$	distance between v_i and v_j under modified lengths
x_e	modification amount of l_e in facility location model
u_e	maximum permissible amount of x_e

This paper is organized as follows. In the next section, using a specific appraisal function and fuzzy ranking method, the fuzzy linear fractional optimization problem with fuzzy parameters is transformed into a crisp semi-infinite linear fractional program. Furthermore, we define the concepts of α-optimal solution and α-optimal objective value of the problem under investigation. In section 3, we show that our fuzzy fractional model can finally be reduced to at most two semi-infinite linear programs. Then, a CPA approach of Fang is utilized to solve the semi-infinite programs and we consequently conclude a General-CPA algorithm for finding the α-optimal solution and α-optimal objective value of the original fuzzy LFO problem in section 4. Further, the application of our presented solution approach in the improvement facility location theory is expressed in section 5. In section 6, an illustrative example is finally considered in order to show the efficiency of the provided method.

Furthermore, the fundamental notations employed in the paper, are summarized in Table 1.

2. Problem formulation

In this paper, we investigate the following specific fuzzy linear fractional optimization (FLFO) problem:

$$\begin{aligned} \text{Maximize} \quad & \tilde{Z}(X) = \frac{\sum_{j=1}^{n} \tilde{c}_j x_j + \tilde{c}}{\sum_{j=1}^{n} d_j x_j + d} \\ \text{subject to} \quad & \sum_{j=1}^{n} \tilde{a}_{ij} x_j \preceq \tilde{b}_i, \quad i = 1, \dots, m, \\ & x_j \geq 0, \qquad j = 1, \dots, n. \end{aligned} \tag{2}$$

where for $\mathrm{i} = 1, \dots, m,\ j = 1, \dots, n$, the parameters $\tilde{a}_{ij}, \tilde{b}_i, \tilde{c}_j$ and $\tilde{c}$ are fuzzy numbers and d_j, x_j and d are real numbers due to the special structure of our considered application at the last of this paper. Furthermore, the notation "$\preceq$" denotes the fuzzy inequality relation between two fuzzy numbers.

In this paper, we assume that all fuzzy parameters of the FLFO problem (2) are convex fuzzy numbers which are defined as follows (see e.g. [18], [28]):

Definition 2.1. A convex fuzzy number $\widetilde{M}$ is a fuzzy set on the real line $\mathbb{R}$ with a membership function $\widetilde{M}(.)$, such that its α-cut set

$$\widetilde{M}_\alpha = \begin{cases} \{e | e \in \mathbb{R}, \quad \widetilde{M}(e) \geq \alpha\}, & if \ \ \alpha \in (0,1], \\ closure\ of\ \{e | e \in \mathbb{R},\ \widetilde{M}(e) > \alpha\}, & if \ \ \alpha = 0, \end{cases}$$

constructs a closed interval $\left[\widetilde{M}^L(\alpha), \widetilde{M}^{\mathbb{R}}(\alpha)\right]$ on the real space $\mathbb{R}$, where

$$\widetilde{M}^L(\alpha) = \min_{x \in \mathbb{R}}\{\widetilde{M}_\alpha\}$$

and

$$\widetilde{M}^R(\alpha) = \max_{x \in \mathbb{R}}\{\widetilde{M}_\alpha\}$$

are respectively the left and right real valued continuous functions in $\alpha \in [0,1]$.

In particular, if a convex fuzzy number $\widetilde{M}$ is triangular, written as triple $\widetilde{M} = (\mathrm{m}_1, \mathrm{m}_2, \mathrm{m}_3)$, then its α-cut set can clearly be obtained as

$$\widetilde{M}_\alpha = \left[\widetilde{M}^L(\alpha), \widetilde{M}^R(\alpha)\right] = [(\mathrm{m}_2 - \mathrm{m}_1)\alpha + \mathrm{m}_1, \mathrm{m}_3 - (\mathrm{m}_3 - \mathrm{m}_2)\alpha], \qquad \forall \alpha \in [0,1].$$

Now, let Γ be the set of all convex fuzzy numbers. If $\widetilde{M}_1, \dots, \widetilde{M}_n \in \Gamma$ and $\lambda_1, \dots, \lambda_n$ be real scalars, then we can observe that

$$\widetilde{M} = \sum_{j=1}^{n} \lambda_j \widetilde{M}_j \in \Gamma.$$

Furthermore, according to the extension principle in fuzzy theory (see [18], [20], [28]) we can certainly conclude the following property for all $\alpha \in [0,1]$:

$$\widetilde{M}^L(\alpha) = \sum_{j=1}^{n} \lambda_j \widetilde{M}_j^L(\alpha) \quad , \qquad \widetilde{M}^R(\alpha) = \sum_{j=1}^{n} \lambda_j \widetilde{M}_j^R(\alpha).$$

Now, according to our own experiments, we suggest an appraisal function to evaluate our solutions in order to determine a best solution, so-called the α-optimal solution for the FLFO model under investigation:

Definition 2.2. Let $\tilde{Z}(X)$ be the objective function of the FLFO model (2). For a given $\alpha \in [0,1]$, the appraisal function $\boldsymbol{H_\alpha}(.)$ is defined as follows:

$$\boldsymbol{H_\alpha}\left(\tilde{Z}(X)\right) = \frac{\sum_{j=1}^{n} \tilde{c}_j^R(\alpha)x_j + \tilde{c}^R(\alpha)}{\sum_{j=1}^{n} d_j\, x_j + d}$$

In the purposed solution approach, we first apply the appraisal function $\boldsymbol{H_\alpha}(.)$ in order to transform the problem (2) into the following linear fractional program with only fuzzy parameters on the constraints:

$$\begin{aligned} &\text{Maximize} && \boldsymbol{H_\alpha}\left(\tilde{Z}(X)\right) \\ &\text{subject to} && \sum_{j=1}^{n} \tilde{a}_{ij}x_j \preceq \tilde{b}_i, \quad i = 1, \dots, m, \\ & && x_j \geq 0, \qquad j = 1, \dots, n. \end{aligned} \tag{3}$$

Remark 2.1. A generalization is to take a finite subset $\{\alpha_1, \dots, \alpha_p\}$ of $[\alpha, 1]$ for a given $\alpha \in [0, 1]$ and maximize simultaneously the p appraisal functions $\boldsymbol{H_{\alpha_1}}(\tilde{Z}(X)), \dots, \boldsymbol{H_{\alpha_p}}(\tilde{Z}(X))$ subject to the same constraints of the problem (3). In this case, we have to solve a multiple objective FLFO problem. We are not going to discuss on this issue within this paper.

Note that we are allowed to apply the appraisal function $\boldsymbol{H_\alpha}(.)$ for both "*maximization*" and "*minimization*" models. Now, we are faced with a reduced optimization problem in which the fuzzy parameters are only seen in the constraints. Then, we are required to apply a fuzzy ranking method for evaluating the fuzzy inequalities and find the minimum of some fuzzy numbers. The various methods for ranking of fuzzy inequalities have been suggested in the literature [6]. Each method appears to have some advantages as well as disadvantages. In the context of each application, some methods seem more appropriate than others. In this paper in the position of decision maker, we are interested in applying a fuzzy ranking method which is based on the concept of α-cuts in fuzzy theory. Our fuzzy ranking method could be considered as an extension of Buckley's approach [4] and it seems to be reliable in our point of view.

Definition 2.3. *(Fuzzy ranking method)* Let $\tilde{M}_1, \tilde{M}_2 \in \Gamma$ and $\alpha \in [0,1]$ be an arbitrary comparison level. Then we say that $\tilde{M}_1 \preceq \tilde{M}_2$ at the level α, if and only if

$$\tilde{M}_1^R(s) \leq \tilde{M}_2^L(s), \quad \forall s \in [\alpha, 1].$$

Given $\alpha \in [0,1]$, If we apply Definitions 2.3 for the constraints of the FLFO problem (3), then this problem is converted to the following crisp semi-infinite linear fractional program:

$$\begin{aligned} &\text{Maximize} && \boldsymbol{H_\alpha}\left(\tilde{Z}(X)\right) = \frac{\sum_{j=1}^{n} \tilde{c}_j^R(\alpha)x_j + \tilde{c}^R(\alpha)}{\sum_{j=1}^{n} d_j\, x_j + d} \\ &\text{subject to} && \sum_{j=1}^{n} \tilde{a}_{ij}^R(s)x_j \leq \tilde{b}_i^L(s), \quad \forall s \in [\alpha, 1], \quad i = 1, \dots, m, \end{aligned} \tag{4}$$

$$x_j \geq 0, \quad j = 1, \dots, n.$$

Given $\alpha \in [0,1]$, the feasible solution region of the reduced LFO model (4) is defined as follows:

$$\Omega_\alpha = \left\{ X \in \mathbb{R}^n \mid x_j \geq 0, \ \sum_{j=1}^{n} \tilde{a}_{ij}^R(s) x_j \leq \tilde{b}_i^L(s), \forall s \in [\alpha, 1], \quad j = 1, \dots, n, i = 1, \dots, m \right\}$$

Throughout this paper, we call any member $X = (x_1, \dots, x_n) \in \Omega_\alpha$ as an "*α-feasible solution*" of the FLFO problem (2).

Definition 2.4. Given $\alpha \in [0,1]$, $X^* = (x_1^*, \dots, x_n^*) \in \Omega_\alpha$ is called an α-optimal solution of the original FLFO problem (2), if and only if for all $X = (x_1, \dots, x_n) \in \Omega_\alpha$, the following inequality holds:

$$\boldsymbol{H_\alpha}\left(\tilde{Z}(X)\right) \leq \boldsymbol{H_\alpha}\left(\tilde{Z}(X^*)\right)$$

The corresponding objective value $\boldsymbol{H_\alpha}\left(\tilde{Z}(X^*)\right)$ is called the "*α-optimal objective value*" of the FLFO problem (2).

Moreover, in this paper it is assumed that

(i) for any $\alpha \in [0,1]$, the feasible solution set Ω_α is regular, i.e., Ω_α is nonempty and bounded,

(ii) for any $X = (x_1, \dots, x_n) \in \Omega_\alpha$, $\sum_{j=1}^{n} d_j x_j + d \neq 0$

In the next section, we will attempt to show that for a given $\alpha \in [0,1]$ the α −optimal solution and α-optimal objective value of the FLFO problem (2) can be found by solving at most two crisp semi-infinite linear programming models.

3. Equivalent semi-infinite programs

Semi-infinite linear programming (SILP) deals with linear optimization problems in which the dimension of the decision space or the number of constraints (but not both) is infinite [1], [12]. Since for any $\alpha \in [0,1]$, the feasible solution region Ω_α is a regular set, then we can also apply Charnes and Cooper's technique of variable transformation (see e.g. [5], [26]) to the problem (4). In this case, we will obtain the following two crisp SILP problems with infinite number of constraints:

$$\text{Maximize} \quad \sum_{j=1}^{n} \tilde{c}_j^R(\alpha) y_j + \tilde{c}^R(\alpha)\theta$$

$$\text{s.t.} \quad \sum_{j=1}^{n} \tilde{a}_{ij}^R(s) y_j - \tilde{b}_i^L(s)\theta \leq 0 \quad \forall s \in [\alpha, 1], \quad i = 1, \dots, m,$$

$$\sum_{j=1}^{n} d_j y_j + d\theta = 1, \qquad (5)$$

$$y_j \geq 0, \quad j = 1, \dots, n,$$
$$\theta \geq 0,$$

and

$$\text{Maximize} \quad \sum_{j=1}^{n} -\widetilde{c_j}^R(\alpha)\, y_j - \tilde{c}^R(\alpha)\theta,$$
$$\text{s.t.} \quad \sum_{j=1}^{n} \widetilde{a_{ij}}^R(s) y_j - \widetilde{b_i}^L(s)\theta \leq 0 \quad \forall s \in [\alpha, 1], \quad i = 1, \dots, m,$$
$$\sum_{j=1}^{n} -d_j y_j - d\theta = 1, \tag{6}$$
$$y_j \geq 0, \quad j = 1, \dots, n,$$
$$\theta \geq 0,$$

where the crisp SILP problems (5) and (6) are obtained from (4) by taking the critical transformation

$$\theta = \frac{1}{\sum_{j=1}^{n} d_j\, x_j + d}$$
$$y_j = \theta x_j, \quad j = 1, \dots, n. \tag{7}$$

Similar to [5], we can for a given $\alpha \in [0,1]$ observe that if Ω_α is a regular set, then for solving the problem (4), it is sufficient to solve the crisp SILP problems (5) and (6). The solution which gives the largest value for objective function of the problem (4), is chosen as an optimal solution. If one knows the sign of either the numerator or the denominator of the objective function of the problem (4) at an optimal solution, then it is sufficient to solve exactly one of either the problem (5) or the problem (6). Moreover, we can conclude the following proposition in a similar way as discussed in [5]:

Proposition 3.1. For an optimal solution $(\hat{x}_1, \dots, \hat{x}_n)$ of the optimization problem (4), if either $\sum_{j=1}^{n} d_j\, \hat{x}_j + d > 0$ and $(y_1^*, \dots, y_n^*, \theta^*)$ is an optimal solution of the (5), or $\sum_{j=1}^{n} d_j\, \hat{x}_j + d < 0$ and $(y_1^*, \dots, y_n^*, \theta^*)$ is an optimal solution of the (6), then

$$(x_1^*, \dots, x_n^*) = \left(\frac{y_1^*}{\theta^*}, \dots, \frac{y_n^*}{\theta^*}\right)$$

is also an optimal solution of the problem (4).

Now for convenience and also due to describe the problems (5) and (6) in an identical format, we assume that for $j = 1, \dots, n$,

$$(p_j, r_j, p, r) = \begin{cases} \left(\widetilde{c_j}^R(\alpha), d_j, \tilde{c}^R(\alpha), d\right), & \text{if the problem (5) is considered,} \\ -\left(\widetilde{c_j}^R(\alpha), d_j, \tilde{c}^R(\alpha), d\right), & \text{if the problem(6) is considered.} \end{cases}$$

Further, let $S = [\alpha, 1]$ and

$$g_{ij}(s) = \widetilde{a_{ij}}^{R}(s), \quad i = 1, \dots, m, j = 1, \dots, n,$$
$$b_i(s) = \widetilde{b_i}^{L}(s), \qquad i = 1, \dots, m.$$

Therefore, each of the crisp SILP problems (5) and (6) can be written as the following uniform form

$$\begin{aligned} &\text{Maximize} \quad G(y,\theta) = \sum_{j=1}^{n} p_j\, y_j + p\theta, \\ &\text{s.t.} \quad \sum_{j=1}^{n} g_{ij}(s) y_j - b_i(s)\theta \le 0, \quad \forall s \in S, \quad i = 1, \dots, m, \\ &\qquad \sum_{j=1}^{n} r_j\, y_j + r\theta = 1, \\ &\qquad y_j \ge 0, \qquad j = 1, \dots, n, \\ &\qquad \theta \ge 0. \end{aligned} \tag{8}$$

where S is a compact set with an infinite cardinality and for $i = 1, \dots, m, j = 1, \dots, n$, the functions $g_{ij}(s)$ and $b_i(s)$ are real valued continuous functions defined on S. In the next section, we will introduce an efficient algorithm for solving the SILP problem (8).

4. The General-CPA algorithm for FLFO model

There may exist various solution methods for solving SILP problems (see e.g. [1], [12], [15], [16]). According to [14], algorithms for SILP problems can be classified into discretization methods, local reduction methods, exchange methods, simplex-like methods and descent methods. In this paper, the SILP problem (8) has the infinite number of constraints and finite number of variables. Then, based on a recent survey [16], the so-called " cutting plane algorithm " is a suitable technique to solve such problems. In this section, we consider a straightforward variation of the CPA approach presented in [11] and prepare it for solving the crisp SILP model (8). Then, we provide a General-CPA algorithm for finding the α-optimal solution and α-optimal objective value of the FLFO problem under investigation. Mainly, the CPA approach derives a sequence of optimal solutions of corresponding linear programming models in a symmetric manner. In any iteration of this algorithm, m new constraints are actually inserted to the previous constraints until an optimal solution is obtained. At the kth iteration of the CPA approach, if

$$S_k = \{s^1, \dots, s^k\}$$

where

$$s^t = \{s_1^t, \dots, s_m^t\} \in S^m, \qquad t = 1, \dots, k,$$

then we should solve the following crisp linear programming problem with ($km + 1$) constraints:

$$\text{Maximize} \quad G(y,\theta) = \sum_{j=1}^{n} p_j\, y_j + p\theta,$$

Subject to

$$\sum_{j=1}^{n} g_{ij}(s_i^1)y_j - b_i(s_i^1)\theta \le 0, \quad i = 1, \dots, m,$$
$$\sum_{j=1}^{n} g_{ij}(s_i^2)y_j - b_i(s_i^2)\theta \le 0, \quad i = 1, \dots, m,$$
$$\vdots$$
$$\sum_{j=1}^{n} g_{ij}(s_i^k)y_j - b_i(s_i^k)\theta \le 0, \quad i = 1, \dots, m, \quad (\mathrm{LP}^{\mathrm{k}})$$
$$\sum_{j=1}^{n} r_j y_j + r\theta = 1,$$
$$y_j \ge 0, \qquad j = 1, \dots, n,$$
$$\theta \ge 0.$$

Assume that $(y^k, \theta^k) = (y_1^k, \dots, y_n^k, \theta^k)$ is an optimal solution of the problem $(\mathrm{LP}^{\mathrm{k}})$. Then, as a critical note the "constraint violation functions" are defined as follows:

$$\varphi_i^{k+1}(s) = b_i(s)\theta^k - \sum_{j=1}^{n} g_{ij}(s)y_j^k, \quad \forall s \in S, \quad i = 1, \dots, m.$$

Since $g_{ij}(s)$ and $b_i(s)$ are continuous functions on the compact set S, then the function $\varphi_i^{k+1}(s)$ achieves its minimum value over the interval S . Therefore, we let

$$s_i^{k+1} = \arg\min_{s \in S} \varphi_i^{k+1}(s), \qquad \forall\, i = 1, \dots, m.$$

If

$$\varphi_i^{k+1}(s_i^{k+1}) = b_i(s_i^{k+1})\theta^k - \sum_{j=1}^{n} g_{ij}(s_i^{k+1})y_j^k \ge 0, \quad i = 1, \dots, m,$$

then (y^k, θ^k) is a feasible solution of the SILP problem (8) and also it will be an optimal solution. Otherwise, (y^k, θ^k) is not an optimal solution and in this case, we let

$$S_{k+1} = \{s^1, \dots, s^{k+1}\},$$

where

$$s^{k+1} = \{s_1^{k+1}, \dots, s_m^{k+1}\} \notin S_k$$

and construct the linear program $(\mathrm{LP}^{\mathrm{k+1}})$ with additional mconstraints . If this process is iterated, then the produced sequence $\{(y^k, \theta^k)\}_k$ will converge to the optimal solution of the SILP problem (8). Now, based on all explanations above, we can summarize the CPA approach for the SILP problem (8) as follows (see [11]):

CPA approach. Solves the crisp SILP problem (8)

1: **Begin**
2: Set $I = \{1, \dots, m\}$.
3: Set $k = 1$.
4: **For** all $i \in I$ choose any $s_i^1 \in S$.
5: Set $S_1 = \{s^1\}$ such that $s^1 = (s_1^1, \dots, s_m^1)$.
6: Obtain the optimal solution (y^k, θ^k) of the problem (LP^k).
7: **For** all $i \in I$, find $s_i^{k+1} = \arg\min_{s \in S} \varphi_i^{k+1}(s)$.
8: **If** there exists $i \in I$, such that $\varphi_i^{k+1}(s_i^{k+1}) < 0$ **then**
9: set $S_{k+1} = S_k \cup \{s^{k+1}\}$,
10. update $k = k + 1$ and return to step 6,
11. **else**
12: stop; (y^k, θ^k) is an optimal solution of the SILP model (8).
13: **End**

The convergence proof of the generated sequence $\{(y^k, \theta^k)\}_k$ to an optimal solution of the problem (8) is described in the following as analyzed in more details in Fang et al. [11].

Proposition 4.1. If the feasible space of the SILP problem (8) is nonempty and $\{(y^k, \theta^k)\}_k$ is a bounded sequence generated by CPA approach, then there exists a subsequence of $\{(y^k, \theta^k)\}_k$ converging to an optimal solution (y^*, θ^*) of the SILP model (8).

Now, we are ready to present the General-CPA algorithm for determining the α-optimal solution and α -optimal objective value of the FLFO problem under investigation as follows:

General-CPA algorithm: solves the FLFO model

1. **Begin**
2. **For** sgn= +1, -1 **do**
   2.1. Let $(p_j, r_j, p, r) = \text{sgn} \times (\tilde{c}_j^R(\alpha), d_j, \tilde{c}^R(\alpha), d)$
   2.2. Apply CPA approach to the SILP model (8) and get an optimal solution $(y_{sgn}^*, \theta_{sgn}^*)$ and the optimal objective value, say $G(y_{sgn}^*, \theta_{sgn}^*)$.
3. Let $G(y^*, \theta^*) = \max\{G(y_{sgn}^*, \theta_{sgn}^*): \text{ sgn} = +1, -1\}$
4. An α -optimal solution of FLFO model is derived by

$$x_j^* = \frac{y_j^*}{\theta^*}, \quad j = 1, \dots, n.$$

5. **End**

Finally, we conclude that

Proposition 4.2. The General-CPA algorithm can determine the α −optimal solution and α-optimal objective value of the FLFO problem, correctly.

Now, we are going to explain an application for our studied FLFO problem and the provided solution technique in the next section.

5. Application in improvement facility location theory

Facility location problems are well-known optimization models in operations research which have many applications in theory and practice. In a classical facility location problem on a system (network or real space), we wish to find the best locations for installing one or more facilities on the underlying system to serve the existing customers in the best possible way according to a specific criterion. On the other hand, an improvement facility location problem aims to modify some input parameters of the classical model in the cheapest possible way with respect to some modification bound, until the current location of a predetermined facility is improved as much as possible. In this section, we show that how the considered FLFO problem and our presented solution approach can be applied to the fractional improvement (objective-bounded inverse) multiple commodity median facility location problems on tree networks in fuzzy environment.

Let a tree network $T = \big(V(T), E(T)\big)$ with vertex set $V(T) = \{v_1, \dots, v_n\}$ and the edge set $E(T)$ be given. For any vertex $v_i \in V(T)$, corresponding to the commodity o, a fuzzy weight $\widetilde{w}_{oi}$ is associated which is interpreted as demand for commodity o at the customer site v_i. Any edge $e \in E(T)$ has a nonnegative length l_e. Based on the edge lengths of the underlying tree network, the travel distance required for transporting any commodity $o, o = 1, \dots, O,$ from v_j to the demand center v_i

$$d\big(v_i, v_j\big) = \sum_{e \in P[v_i, v_j]} l_e$$

where $P\big[v_i, v_j\big]$ denotes the unique path between vertices v_i, v_j on tree network T. In the classical multiple commodity median facility location problem on the tree T in the fuzzy environment, the goal is to find the optimal solution of the following optimization model as the best location for establishing our facility, where we have O type of commodities on the system:

$$\begin{aligned} \text{Minimize} \quad & \sum_{o=1}^{O} \sum_{v_i \in V(T)} \widetilde{w}_{oi} d(v_i, x), \\ \text{subject to} \quad & x \in V(T). \end{aligned}$$

In contrast to the above classical model, the fuzzy fractional improvement (objective-bounded inverse) multiple commodity median facility location problem is stated as follows: Consider the underlying tree network T with vertex weights $\widetilde{w}_{oi}$ and the edge lengths l_e . Suppose that a predetermined vertex $\boldsymbol{v_s}$ is given as the current facility location which has already been established on the system. The goal is to modify the edge lengths of T in the best possible way with respect to modification restrictions such that the position of the facility location $\boldsymbol{v_s}$ is improved as much as possible under the perturbed edge lengths.

For every$e \in E(T)$l , et $\widetilde{c_e}$ be the cost coefficient for reducing the length l_e which is uncertain due to the economic instability and then it is considered as a fuzzy number. Furthermore, let d_e ,$e \in E(T)$, be the quality coefficient corresponding to the improvement of the lengths l_e . We wish to modify the edge lengths such that the optimal objective value of the classical multiple commodity median location model becomes less than or equal to a given fuzzy value $\widetilde{\Delta}$. Note that we are not permitted to reduce the edge lengths arbitrarily. Hence, any reduction on the length l_e should obey the specific bound u_e. If x_e denotes the reduction amount of the length l_e , then, we should modify the lengths $l_e, e \in E(T)$ to $\widetilde{l_e} = l_e - x_e$ such that all the following statements are satisfied:

1. The overall modification ratio of cost/quality

$$\tilde{Z}(X) = \frac{\sum_{e\in E(T)} \widetilde{c_e} x_e + \tilde{c}}{\sum_{e\in E(T)} d_e x_e + d}$$

is minimized, where $\tilde{c}$ and d are the initial fuzzy cost and crisp quality factor on the system.

2. The fuzzy objective value of the classical median model at the predetermined facility location $\boldsymbol{v_s}$ does not proceed the given aspiration bound $\widetilde{\Delta}$, i.e.,

$$\sum_{0=1}^{O} \sum_{v_i\in V(T)} \widetilde{w}_{oi} \bar{d}(v_i, \boldsymbol{v_s}) \leq \widetilde{\Delta}. \qquad (9)$$

where $\bar{d}(v_i, \boldsymbol{v_s})$ is the travel distance between v_i and $\boldsymbol{v_s}$ under the new length $\widetilde{l_e}, e \in E(T)$.

3. The modification amounts x_e obey the bounds

$$0 \leq x_e \leq u_e, \qquad \forall e \in E(T)$$

To the best of our knowledge, the fractional case of the inverse/improvement facility location models on the fuzzy environment have not been discussed until now. Now, let us rewrite the left side of the inequality (9) as follows, since we know that there exists unique path between any two points on tree networks:

$$\sum_{0=1}^{O} \sum_{v_i\in V(T)} \widetilde{w}_{oi} \bar{d}(v_i, \boldsymbol{v_s}) = \sum_{o=1}^{O} \sum_{v_i\in V(T)} \widetilde{w}_{oi} \sum_{e\in P[v_i,v_s]} (l_e - x_e)$$

$$= \sum_{o=1}^{O} \sum_{v_i\in V(T)} \widetilde{w}_{oi} \sum_{e\in P[v_i,v_s]} l_e - \sum_{o=1}^{O} \sum_{v_i\in V(T)} \widetilde{w}_{oi} \sum_{e\in P[v_i,v_s]} x_e .$$

Considering the above discussions, the inequality (9) can finally be written as the following fuzzy inequality:

$$\sum_{e\in E(T)} \widetilde{W}_e x_e \geq \tilde{\gamma}.$$

where we have

$$\tilde{\gamma} = \sum_{o=1}^{O} \sum_{v_i\in V(T)} \widetilde{w}_{oi} \sum_{e\in P[v_i,v_s]} l_e - \widetilde{\Delta},$$

and also

$$\widetilde{W}_e = \sum_{o=1}^{O} \sum_{t\in I_e} \widetilde{w}_{oj} ,$$

such that the index set I_e is as

$$I_e = \{t\colon\ e \in P[v_t, \boldsymbol{v_s},],\ t = 1, \ldots, n\}.$$

Altogether, the fractional improvement multiple commodity median location problem on the underlying tree network T is mathematically formulated as the following specific FLFO problem:

$$\text{Minimize} \qquad \tilde{Z}(X) = \frac{\sum_{e\in E(T)} \tilde{c}_e x_e + \tilde{c}}{\sum_{e\in E(T)} d_e x_e + d},$$

$$\text{s.t.}$$

$$\sum_{e\in E(T)} \widetilde{W}_e\, x_e \geq \tilde{\gamma},$$

$$0 \leq x_e \leq u_e \qquad \forall e \in E(T).$$

where $\tilde{c}_e$, $\widetilde{W}_e$, for all $e \in E(T)$ and $\tilde{c}$, $\tilde{\gamma}$ are convex fuzzy numbers. This FLFO model can be solved by the above provided solution approach efficiently.

6. An illustrative example

Let us consider the following FLFO problem with the given membership functions for the fuzzy parameters:

$$\begin{aligned} &\text{Maximize} \qquad \tilde{Z}(X) = \frac{\tilde{2}x_1 + \widetilde{0.5}x_2 + \tilde{1}}{x_1 + 2x_2 + 1}, \\ &\text{subject to} \qquad \tilde{3}x_1 + \tilde{2}x_2 \preceq \tilde{6}, \\ &\qquad\qquad\qquad \tilde{9}x_1 + \tilde{3}x_2 \preceq \widetilde{12}, \qquad\qquad (10) \\ &\qquad\qquad\qquad x_1, x_2 \geq 0. \end{aligned}$$

where the membership functions of the fuzzy coefficients on the objective function are defined as follows:

$$\tilde{2}(x) = \begin{cases} \frac{1}{2}x, & 0 \leq x < 2, \\ \frac{-4}{10}x + \frac{18}{10}, & 2 \leq x \leq \frac{9}{2}. \end{cases}, \qquad \widetilde{0.5}(x) = \begin{cases} 2x, & 0 \leq x < \frac{1}{2}, \\ -\frac{8}{10}x + \frac{14}{10}, & \frac{1}{2} \leq x \leq \frac{7}{4}. \end{cases}$$

and

$$\tilde{1}(x) = \begin{cases} x, & 0 \leq x < 1, \\ \frac{-4}{10}x + \frac{14}{10}, & 1 \leq x \leq \frac{7}{2}. \end{cases}$$

Moreover, the membership functions of fuzzy parameters in the constraints of the problem (10) are given as follows:

$$\tilde{3}(x) = \begin{cases} 0, & x < 2, \\ x - 2, & 2 \leq x < 3, \\ \frac{1}{2}(5 - x), & 3 \leq x < 5, \\ 0, & x \geq 5. \end{cases} \qquad \tilde{2}(x) = \begin{cases} 0, & x < 1, \\ x - 1, & 1 \leq x < 2, \\ 3 - x, & 2 \leq x < 3, \\ 0, & x \geq 3. \end{cases}$$

$$\tilde{6}(x)=\begin{cases}0, & x<3,\\ \frac{1}{3}(x-3), & 3\le x<6,\\ \frac{1}{3}(9-x), & 6\le x<9,\\ 0, & x\ge 9.\end{cases}\qquad \tilde{9}(x)=\begin{cases}0, & x<8,\\ x-8, & 8\le x<9,\\ 10-x, & 9\le x<10,\\ 0, & x\ge 10.\end{cases}$$

$$\widetilde{12}(x)=\begin{cases}0, & x<10,\\ \frac{1}{2}(x-10), & 10\le x<12,\\ \frac{1}{3}(15-x), & 12\le x<15,\\ 0, & x\ge 15.\end{cases}$$

For given $\alpha = 0.6$, if we use Definition 2.3 for evaluating the fuzzy inequalities in the constraints and apply Definition 2.2 on the objective function, then the FLFO problem (10) is transformed into the following crisp semi-infinite linear fractional program:

$$\begin{aligned}
&\text{Maximize} \quad \boldsymbol{H_\alpha}\left(\tilde{Z}(X)\right)=\frac{3x_1+x_2+2}{x_1+2x_2+1},\\
&\text{subject to} \quad (5-2s)x_1+(3-s)x_2\le(3+3s), \qquad \forall s\in[\alpha,1],\\
&\qquad\qquad\quad (10-s)x_1+(5-2s)x_2\le(10+2s),\\
&\qquad\qquad\quad x_1,x_2\ge 0, \qquad\qquad (11)
\end{aligned}$$

where, for $\alpha = 0.6$, we get

$$\tilde{2}^R(0.6)=3 \quad , \quad \widetilde{0.5}^R(0.6)=1 \quad , \quad \tilde{1}^R(0.6)=2.$$

Since for all feasible solution of the problem (11), the sign of the denominator of the objective function is positive, then based on Proposition 3.1, the solution of the problem (11) is converted to the solution of the following SILP problem:

$$\begin{aligned}
&\text{Maximize} \quad 3y_1+y_2+2\theta,\\
&\text{subject to} \quad (5-2s)y_1+(3-s)y_2-(3+3s)\theta\le 0,\\
&\qquad\qquad\quad (10-s)y_1+(5-2s)y_2-(10+2s)\theta\le 0 \quad \forall s\in[0.6,1],\\
&\qquad\qquad\quad y_1+2y_2+\theta=1,\\
&\qquad\qquad\quad y_1,y_2\ge 0. \qquad\qquad (12)\\
&\qquad\qquad\quad \theta\ge 0.
\end{aligned}$$

Now, we apply our CPA algorithm to the SILP problem (12) for given $\alpha = 0.60$ and for the initial parameter $s^1=(s_1^1,s_2^1)=(0.60,0.70)$, and therefore we get

$$y_1^*=0.543689 \;,\; y_2^*=0.000000 \;,\; \theta^*=0.456311.$$

Consequently, by considering the variable transformation (12), the α-optimal solutions and the α-optimal objective values of the FLFO problem (10) are derived as

$$x_1^*=1.191488 \;,\; x_2^*=0.000000 \;,\; \boldsymbol{H_\alpha}\left(\tilde{Z}(X^*)\right)=2.543689$$

This obtained solution yields the following corresponding fuzzy objective value

$$\tilde{Z}(X^*) = [0.000000,\ 1.315534,\ 4.281551]$$

7. Concluding remarks

In this paper, we considered a FLFO problem in which the parameters are considered as convex fuzzy numbers. Applying an appraisal function for evaluating the objective values and a fuzzy ranking method to the constraints, the original FLFO problem was transformed into to a semi-infinite linear fractional program. We demonstrated that if the Charnes and Cooper's technique of variable transformation is called, then the solution of the semi-infinite linear fractional programming model is equivalently transformed into the optimal solutions of at most two SILP models that are dissimilar among themselves through a sign in a constraint and in the objective function. Analogously, a compatible CPA approach by Fang et al. [11] was applied to obtain the optimal solution of the SILP models. Then, we provided a General-CPA algorithm for obtaining the α-optimal solution and α-optimal objective value of the FLFO problem under investigation. At the end, we discussed an application of the investigated FLFO problem and the provided solution algorithm in improvement facility location theory on tree networks, properly.

As a concluding remark, if we consider our FLFO problem with the objective function

$$\tilde{Z}(X) = \frac{\sum_{j=1}^{n} \tilde{c}_j\, x_j + \tilde{c}}{\sum_{j=1}^{n} \tilde{d}_j\, x_j + \tilde{d}}$$

where, the parameters $\tilde{d}_j, \tilde{d}$ are also convex fuzzy numbers in addition, then we can, in this case, suggest the following appraisal function

$$\boldsymbol{H}_{\boldsymbol{\alpha}}\left(\tilde{Z}(X)\right) = \frac{\sum_{j=1}^{n} \tilde{c}_j^R(\alpha) x_j + \tilde{c}^R(\alpha)}{\sum_{j=1}^{n} \tilde{d}_j^L(\alpha) x_j + \tilde{d}^L(\alpha)} \quad \text{or} \quad \boldsymbol{H}_{\boldsymbol{\alpha}}\left(\tilde{Z}(X)\right) = \frac{\sum_{j=1}^{n} \tilde{c}_j^R(\alpha) x_j + \tilde{c}^R(\alpha)}{\sum_{j=1}^{n} \tilde{d}_j^R(\alpha) x_j + \tilde{d}^R(\alpha)}$$

for a given $\alpha \in [0,1]$ according to our initiative point of view. Here, we can also apply the above provided solution procedure in order to solve our FLFO problem. Here, the idea of Remark 2.1 can also be taken into account. We know that a decision maker can define various appraisal functions and ranking methods according to the structure of the problem under investigation and based on his/her own idea for solving the optimization problems in uncertain environments. Hence, our applied procedure for the considered FLFO problem together with the concentration on the improvement facility location models may not be so eminent, but it is certainly interesting in our point of view.

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