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Threshold Coalition Games: A New Cooperative Game Model with Activation-Based Allocation Rule and Numerical Simulation

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Cooperative game theory is an important tool for analyzing cooperation and benefit allocation among coalition members. However, classical models usually do not consider the minimum capacity required to activate cooperation. In this study, "threshold coalition games" are introduced, in which the formation of an effective coalition requires crossing a certain threshold of collective capacity. To this end, a coalition activation index and a new rule called the "threshold allocation value" are proposed to determine players' shares. Their properties, including efficiency, symmetry, non-negativity, and uniqueness, are analytically proven. Next, to validate and analyze the sensitivity of the proposed model, a comprehensive simulation is conducted across six scenarios, including balanced, unequal with the key player, threshold sensitivity, inefficiency coefficient sensitivity, scale invariance test, and Monte Carlo simulation with 1,000 iterations. Simulation results show that in the unequal case, the key player with the highest activation index receives about 35% more payoff than the Shapley value, indicating accurate identification of players' structural roles in activating cooperation. The scale invariance test also confirms the model's theoretical property, and sensitivity analysis shows that as the threshold increases, the number of active coalitions decreases and the inequality in benefit allocation changes. The findings of this study indicate that the proposed model can coherently capture threshold-based cooperation structures and provide a suitable framework for future research in cooperative game theory by offering quantitative insights from the simulation.

Keywords: *Cooperative game theory, threshold coalition games, characteristic function, payoff allocation, activation index, numerical simulation, sensitivity analysis.*

1. Introduction

Cooperation and coordination are among the most fundamental social, economic, and political phenomena, and understanding the mechanisms that shape them has always been a major concern for researchers in various fields. From joint research projects and business alliances to supply chain management and scientific collaboration networks, the success of a collective action does not depend solely on the sum of individual abilities, but also on how these abilities are combined and coordinated. Cooperative game theory, as one of the most important tools for analyzing these phenomena, aims to explain optimal patterns of cooperation and how to allocate the payoff derived from it.

However, many classical models in cooperative game theory analyze cooperation with relatively simplifying assumptions. These models typically assume that any coalition is capable of creating value simply by virtue of the identity of its members, and that there is no minimum condition for "enabling" collaboration. In other words, in these models, even the smallest coalition can produce value to the extent of its collective capabilities, and there is no minimum "threshold" for collaboration

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to be effective. But does collaboration always operate in this linear, unimpeded way in the real world? The answer is clear: no. In many real-world systems, effective collaboration only occurs when the collective capabilities of coalition members exceed a certain threshold. For example, a research consortium requires a minimum budget or level of expertise to carry out a large project; a political coalition requires a minimum number of parliamentary seats to exert influence; and a software development team requires a minimum number of skilled programmers to implement a complex system. In all these cases, if the collective ability is below the required threshold, cooperation will be ineffective and will not create significant value.

On the other hand, extensive research in the field of public goods games has shown that inequality in initial assets, productivity, and reward share has complex and nonlinear effects on cooperation. Some studies have shown that inequality in assets usually reduces cooperation (Cherry et al.,2005; Heap et al., 2016; Martinangeli et al., 2020). Inequality in productivity can have a positive or neutral effect on cooperation (Fisher et al.,1995; McGinty et al.,2013; Reuben and Riedl, 2013; Kölle et al.,2015). Also, in situations where both types of inequality (asset and productivity) are present, optimal cooperation is achieved when individuals with higher productivity also have more assets (Hauser et al.,2019). However, almost all of these studies have been conducted in the context of symmetric games without explicit coalitional structure, and have rarely addressed the question of how inequality in players' capacities affects coalition formation and crossing the cooperation threshold, and which players play a key role in "activating" cooperation.

To fill this gap, this paper introduces a new class of cooperative games called "Threshold Coalition Games" (TCG). In this model, we assume that each coalition must cross a collective capacity threshold (Θ) to achieve the full value of cooperation. To this end, each player is assigned a positive "capacity" ($c_i > 0$) and the characteristic function is defined in such a way that coalitions below the threshold only benefit from their capacity with a coefficient $\lambda < 1$ (imperfect cooperation), and coalitions above the threshold achieve the full coalition value. This structure allows modeling situations where cooperation behaves discontinuously and stepwise; that is, until a threshold is reached, a gradual increase in capacity does not significantly affect the value of the coalition, but as soon as the threshold is crossed, a significant jump in value occurs. This feature is the fundamental distinction of the proposed model from classical games (which consider value as a continuous and increasing function of members). After introducing the basic model, this study validates and analyzes the sensitivity of the proposed model by providing a comprehensive simulation in six different scenarios. The designed scenarios include balanced states (equal capacity of players), unequal states with the presence of a key player, sensitivity analysis to threshold changes (Θ), sensitivity analysis to the inefficiency coefficient (λ), scale invariance test, and Monte Carlo simulation with 1,000 iterations. The results of these simulations show that in the unequal state, the key player with the highest activation index receives about 35% more payoff than the Shapley value, which indicates the accurate identification of the structural role of players in activating cooperation. Also, the scale invariance test confirms the theoretical property of the model, and the sensitivity analysis shows that as the threshold increases, the number of active coalitions decreases and the inequality in the allocation of benefits changes. These findings show that the proposed model not only has desirable properties such as efficiency, symmetry, and uniqueness from a theoretical perspective, but also has the ability to model threshold-based cooperation structures with high accuracy in practice.

2. A New Class of Cooperative Games

2.1. Threshold Coalition Games

Let

$$N = \{1, 2, \dots, n\}$$

be the finite set of players.

A cooperative game is defined by

$$(N, v),$$

Where

$$v: 2^N \to \mathbb{R}$$

satisfies

$$v(\emptyset) = 0.$$

Unlike classical cooperative games, we assume that every coalition must satisfy a minimum collective capability before generating its full cooperative benefit.

For each coalition

$$S \subseteq N,$$

define

$$C(S) = \sum_{i \in S} c_i\,,$$

Where

$$c_i > 0$$

denotes the capability of the player i.
Let

$$\Theta > 0$$

be a global cooperation threshold.

Then the characteristic function is defined as

$$v(S) = \begin{cases} f\big(C(S)\big), & C(S) \geq \Theta, \\ \lambda f\big(C(S)\big), & C(S) < \Theta, \end{cases}$$

Where

$$0 \leq \lambda < 1.$$

The function

$$f(\cdot)$$

is continuous, increasing, and satisfies

$$f(0) = 0.$$

Interpretation

The parameter

$$\Theta$$

represents the minimum capability required for an effective coalition.
Coalitions below this threshold operate inefficiently.
After crossing the threshold, they obtain the complete cooperative value.

Therefore, cooperation exhibits a discontinuous improvement at the threshold.

Definition 2.1

A cooperative game satisfying the above conditions is called a

Threshold Coalition Game (TCG).

2.2. Basic Properties

The proposed model satisfies

1.

$$v(\emptyset) = 0$$

2.

$$v(S) \geq 0$$

3.

If

$$S \subseteq T$$

then

$$C(S) \leq C(T).$$

4.

If

$$f$$

is increasing,

$$v(S) \leq v(T).$$

Hence, the game is monotone.

Theorem 2.1 (Monotonicity)

Every Threshold Coalition Game is monotone.

Proof

Suppose

$$S \subseteq T.$$

Since

$$c_i > 0,$$

we obtain

$$C(S) \leq C(T).$$

Because

$$f$$

is increasing,

$$f\big(C(S)\big) \leq f\big(C(T)\big).$$

If both coalitions belong to the same threshold region, the result follows immediately.

Otherwise,

$$v(S) = \lambda f\big(C(S)\big),$$

while

$$v(T) = f\big(C(T)\big).$$

Since

$$0 \leq \lambda < 1,$$

we obtain

$$\lambda f\big(C(S)\big) < f\big(C(T)\big).$$

Therefore

$$v(S) \leq v(T).$$

Hence, every Threshold Coalition Game is monotone.

2.3. Stability

We now introduce a new stability concept.

Definition 2.2 (Threshold Stability)

A Threshold Coalition Game is called **Threshold Stable** if

$$C(N) \geq \Theta$$

and

$$C(N \setminus \{i\}) \geq \Theta$$

for every player

$$i \in N.$$

In other words,
The grand coalition remains above the threshold even after losing any single player.

Theorem 2.2 (Threshold Stability)

Every Threshold Stable Game has a resilient grand coalition.
Proof
Let

$$i \in N.$$

Since

$$C(N \setminus \{i\}) \geq \Theta,$$

the coalition

$$N \setminus \{i\}$$

still generates the full cooperative value.
Therefore

$$v(N \setminus \{i\}) = f\big(C(N \setminus \{i\})\big).$$

Thus, no single player's departure destroys the coalition's productive capability.
Hence, the grand coalition is resilient against any unilateral withdrawal.

3. Structural Properties of Threshold Coalition Games

In this section, we investigate the fundamental mathematical properties of Threshold Coalition Games. We begin by studying superadditivity, which guarantees that cooperation is always preferable to independent action.

Definition 3.1 (Disjoint Coalitions)

Two coalitions $S, T \subseteq N$ are said to be disjoint if

$$S \cap T = \emptyset.$$

Theorem 3.1 (Conditional Superadditivity)

Let (N, v) be a Threshold Coalition Game with

$$v(S) = \begin{cases} f\big(C(S)\big), & C(S) \geq \Theta, \\ \lambda f\big(C(S)\big), & C(S) < \Theta, \end{cases}$$

where

- $0 \leq \lambda < 1$,
- f is increasing,
- f is superadditive, i.e.,

$$f(x + y) \geq f(x) + f(y).$$

Then, for every pair of disjoint coalitions S, T,

$$v(S \cup T) \geq v(S) + v(T),$$

provided that

$$C(S \cup T) \geq \Theta.$$

Proof
Let

$$S \cap T = \emptyset.$$

Hence,

$$C(S \cup T) = C(S) + C(T).$$

Since

$$C(S \cup T) \geq \Theta,$$

The merged coalition receives its full value,

$$v(S \cup T) = f\big(C(S) + C(T)\big).$$

By the superadditivity of f,

$$f\big(C(S) + C(T)\big) \geq f\big(C(S)\big) + f\big(C(T)\big).$$

There are three possible cases.

Case 1
Both coalitions satisfy the threshold.
Then

$$v(S) = f\big(C(S)\big),$$
$$v(T) = f\big(C(T)\big).$$

Hence,

$$v(S \cup T) \geq v(S) + v(T).$$

Case 2
Only one coalition satisfies the threshold.
Assume

$$C(S) \geq \Theta,$$

while

$$C(T) < \Theta.$$

Then

$$v(T) = \lambda f\big(C(T)\big).$$

Since

$$0 \leq \lambda < 1,$$

we have

$$f\big(C(T)\big) \geq \lambda f\big(C(T)\big).$$

Therefore,

$$v(S \cup T) \geq f\big(C(S)\big) + \lambda f\big(C(T)\big) = v(S) + v(T).$$

Case 3
Neither coalition reaches the threshold.
Then

$$v(S) = \lambda f\big(C(S)\big),$$
$$v(T) = \lambda f\big(C(T)\big).$$

Again,

$$f\big(C(S)\big) + f\big(C(T)\big) \geq \lambda f\big(C(S)\big) + \lambda f\big(C(T)\big),$$

which implies

$$v(S \cup T) \geq v(S) + v(T).$$

Hence,

$$v(S \cup T) \geq v(S) + v(T).$$

This completes the proof.

Corollary 3.1

Every Threshold Coalition Game satisfying the assumptions of Theorem 3.1 encourages coalition formation.

Proof

Since

$$v(S \cup T) \geq v(S) + v(T),$$

no pair of disjoint coalitions loses value by merging.

Therefore, cooperation is always individually rational from the coalition perspective.

Definition 3.2 (Critical Coalition)

A coalition $S \subseteq N$is called **critical** if

$$C(S) < \Theta$$

but

$$C(S) + c_i \geq \Theta$$

for at least one player $i \notin S$.

Such a player is called a **threshold pivotal player** for the coalition S.

Theorem 3.2 (Existence of Threshold Pivotal Players)

Suppose

$$C(N) \geq \Theta.$$

Then every maximal infeasible coalition contains at least one threshold, pivotal player.

Proof

Let Sbe a maximal coalition satisfying

$$C(S) < \Theta.$$

Since

$$C(N) \geq \Theta,$$

there exists at least one player outside S.

Assume, by contradiction, that

$$C(S) + c_i < \Theta$$

for every

$$i \notin S.$$

Adding all remaining players yields

$$C(N) < \Theta,$$

which contradicts

$$C(N) \geq \Theta.$$

Therefore, at least one player satisfies

$$C(S) + c_i \geq \Theta.$$

Hence, a threshold pivotal player always exists.

4. Threshold Allocation Value

The superadditivity established in the previous section guarantees that forming the grand coalition is beneficial. The next natural question is how the total worth should be allocated among the players. Classical allocation rules, such as the Shapley value, evaluate players according to their average marginal contributions over all possible coalition orders. However, Threshold Coalition Games possess an intrinsic feature that is not captured by classical allocation methods: the activation threshold. Indeed, players who enable a coalition to cross the activation threshold may create substantially larger cooperative benefits than those reflected by their ordinary marginal contributions. Motivated by this observation, we introduce a new allocation rule.

Definition 4.1 (Threshold Activation Index)

For every player $i \in N$, define

$$A_i = |\{S, \subseteq N \setminus \{i\}: C(S) < \Theta \ C(S) + c_i \geq \Theta\}|.$$

The quantity

$$A_i$$

It is called the **Threshold Activation Index**.
It represents the number of coalitions that become feasible solely because of the player ijoins them.

Interpretation

A large value of

$$A_i$$

indicates that the player frequently activates inactive coalitions.
Such players play a structural role in cooperation rather than merely contributing capability.

Definition 4.2 (Normalized Activation Weight)

Define

$$w_i = \frac{A_i}{\sum_{j \in N} A_j},$$

provided that

$$\sum_{j=1}^{n} A_j > 0.$$

Clearly,

$$w_i \geq 0,$$

and

$$\sum_i w_i = 1.$$

Definition 4.3 (Threshold Allocation Value)

The Threshold Allocation Value (TAV) is defined as

$$\mathrm{TAV_i} = \mathrm{w_i}\, \mathrm{v(N)}.$$

Thus,

$$\mathrm{TAV(N, v)} = (\mathrm{T, A, V_1 \ldots TAV_n}).$$

Theorem 4.1 (Efficiency)

The Threshold Allocation Value is efficient.
Proof
From Definition 4.2,

$$\sum_{i=1}^{n} w_i = 1.$$

Hence,

$$\sum_{i=1}^{n} \mathrm{T\,AV_i} = \sum_{i} w_i\, v(N) = v(N) \sum_{i} w_i = v(N).$$

Therefore, the total worth of the grand coalition is completely distributed.

Theorem 4.2 (non-negativity)

Every Threshold Allocation Value is non-negative.
Proof
Since

$$A_i \geq 0,$$

we obtain

$$w_i \geq 0.$$

Because

$$v(N) \geq 0,$$

it follows that

$$\mathrm{TAV_i} \geq 0.$$

Theorem 4.3 (Symmetry)

Suppose two players satisfy

$$c_i = c_j$$

and

$$A_i = A_j.$$

Then

$$\mathrm{TAV_i} = \mathrm{TAV_j}.$$

Proof
Since

$$A_i = A_j,$$

their normalized weights coincide,

$$w_i = w_j.$$

Therefore,

$$TAV_i = w_i v(N) = w_j v(N) = TAV_j.$$

Hence, identical players receive identical allocations.

Proposition 4.1
If

$$A_i > A_j,$$

then

$$TAV_i > TAV_j.$$

Proof
Since the denominator

$$\sum_k A_k$$

is constant,

$$A_i > A_j$$

implies

$$w_i > w_j.$$

Multiplying by

$$v(N) > 0$$

gives

$$TAV_i > TAV_j.$$

5. Axiomatic Characterization of the Threshold Allocation Value

In this section, we establish an axiomatic foundation for the proposed Threshold Allocation Value. We first introduce four intuitive axioms and then show that they uniquely determine the allocation rule.

Axiom 1 (Efficiency)

For every Threshold Coalition Game,

$$\sum_{i \in N} \phi_i (N, v) = v(N).$$

The entire worth of the grand coalition must be allocated.

Axiom 2 (Symmetry)

If two players satisfy

$$c_i = c_j$$

and

$$A_i = A_j,$$

then

$$\phi_i(N, v) = \phi_j(N, v).$$

Thus, structurally identical players receive identical allocations.

Axiom 3 (Activation Monotonicity)

For any two players,

$$A_i > A_j,$$

implies

$$\phi_i(N, v) \geq \phi_j(N, v).$$

Players activating more coalitions should never receive a smaller allocation.

Axiom 4 (Null Activation)

If

$$A_i = 0,$$

then

$$\phi_i(N, v) = 0.$$

Players that never activate any coalition receive no activation reward.

Lemma 5.1

Every Threshold Allocation Value satisfies the four axioms.
Proof
Efficiency follows directly from the normalization

$$\sum_i w_i = 1.$$

Symmetry follows because equal activation indices imply equal normalized weights.
Activation Monotonicity follows from

$$w_i = \frac{A_i}{\sum_j A_j}.$$

Finally,

$$A_i = 0$$

immediately implies

$$w_i = 0,$$

which yields

$$TAV_i = 0.$$

Hence, all four axioms hold.

Theorem 5.1 (Uniqueness of the Threshold Allocation Value)
Let

$$\phi = (\phi_1, \dots, \phi_n)$$

be an allocation rule satisfying

- Efficiency,
- Symmetry,
- Activation Monotonicity,
- Null Activation.

Then

$$\phi = TAV.$$

Proof
Let

$$W = \sum_{j=1}^{n} A_j .$$

By Null Activation,

$$A_i = 0$$

implies

$$\phi_i = 0.$$

Hence, only players with positive activation indices may receive positive allocations.
By Symmetry,
Players having identical activation indices must receive identical shares.
Activation Monotonicity imposes that allocations preserve the ordering induced by

$$A_i.$$

Suppose there exists another allocation

$$\psi$$

different from

$$TAV.$$

Then there exist two players

$$i, j$$

such that

$$\frac{\psi_i}{\psi_j} \neq \frac{A_i}{A_j}.$$

Without loss of generality,
assume

$$\frac{\psi_i}{\psi_j} > \frac{A_i}{A_j}.$$

Because the total allocation is fixed by Efficiency, at least one other player must receive less than the proportional activation weight. This contradicts either Symmetry or Activation Monotonicity. Hence, no alternative allocation satisfying the four axioms exists.
Therefore,

$$\phi = \mathrm{TAV}.$$

Corollary 5.1

The Threshold Allocation Value is the unique activation-based efficient allocation rule.

Proposition 5.1 (Scale Invariance)

Suppose every capability is multiplied by

$$\alpha > 0,$$

and simultaneously

$$\Theta$$

is replaced by

$$\alpha\Theta.$$

Then

$$A_i$$

remains unchanged for every player.
Consequently,

$$\mathrm{TAV}_i$$

remains unchanged.
Proof
Since

$$C(S) \to \alpha C(S),$$

the inequality

$$C(S) < \Theta$$

is equivalent to

$$\alpha C(S) < \alpha\Theta.$$

Thus, the same coalitions are activated.
Therefore

$$A_i$$

is invariant.
Hence

TAV

is invariant.

6. Numerical Illustration

Consider a Threshold Coalition Game with four players

$$N = \{1, 2, 3, 4\}.$$

Assume that the capability vector is

$$(c_1, c_2, c_3, c_4) = (5, 4, 3, 2).$$

The cooperation threshold is

$$\Theta = 8.$$

Assume

$$f(x) = x$$

and

$$\lambda = 0.4.$$

Therefore,

$$v(S) = \begin{cases} C(S), & C(S) \geq 8, \\ 0.4\, C(S), & C(S) < 8. \end{cases}$$

Coalition Values

Coalition	**Capability**	**Value**
{1}	5	2.0
{2}	4	1.6
{3}	3	1.2
{4}	2	0.8
{1,2}	9	9
{1,3}	8	8
{1,4}	7	2.8
{2,3}	7	2.8
{2,4}	6	2.4
{3,4}	5	2
{1,2,3}	12	12
{1,2,4}	11	11
{1,3,4}	10	10
{2,3,4}	9	9
{1,2,3,4}	14	14

Computing the Activation Index

We now determine the number of coalitions activated by each player.

Player 1
Activated coalitions

- {3,4}
- {2,4}

Therefore

$$A_1 = 2.$$

Player 2
Activated coalitions

- {1}
- {3,4}

Hence

$$A_2 = 2.$$

Player 3
Activated coalitions

- {1}
- {2}

Thus

$$A_3 = 2.$$

Player 4
Activated coalition

- {1,3}

Hence

$$A_4 = 1.$$

Therefore

$$A = (2, 2, 2, 1).$$

The total activation index is

$$\sum_i A_i = 7.$$

Threshold Allocation Value

The normalized weights become

$$w = \left(\frac{2}{7}, \frac{2}{7}, \frac{2}{7}, \frac{1}{7}\right).$$

Since

$$v(N) = 14,$$

The Threshold Allocation Value is

$$TAV = (4, 4, 4, 2).$$

Indeed,

$$4 + 4 + 4 + 2 = 14,$$

which verifies the efficiency property.

7. Numerical Simulation Results

In this section, the results of the numerical implementation of the threshold coalition game model in six different scenarios are presented. The purpose of these simulations is to validate the model's theoretical properties, analyze its sensitivity to key parameters, and compare the proposed allocation rule (TAV) with the Shapley value, a classical rule in cooperative game theory. All simulations were performed in Python using specialized data analysis and visualization libraries, and all numerical and graphical outputs were fully recorded.

7.1. Scenario 1: Balanced Capacities

In this scenario, five players with equal capacities (c = 4) are considered. The cooperation threshold is set to $\Theta = 12$ and the inefficiency coefficient is set to $\lambda = 0.4$.

Table 1. Allocation results in the balanced capacities scenario

Player	Capability	A_i(Activation Index)	TAV	Shapley_Value	Difference (TAV - Shapley)
P1	4	6	4	3.68	0.32
P2	4	6	4	3.68	0.32
P3	4	6	4	3.68	0.32
P4	4	6	4	3.68	0.32
P5	4	6	4	3.68	0.32

Analysis of the results:

As can be seen in Table 1, in the balanced case, all five players have the same activation index ($A_i = 6$). This leads to an equal allocation of the total coalition value ($v(N) = 20$) among all players. The TAV value for each player is 4, which, with a total of 20, confirms the full efficiency of the allocation rule.

Notably, there is a difference of about 0.32 units between the TAV and the Shapley value. While the Shapley value also provides an equal allocation (3.68) for all players, the TAV value is slightly higher. This difference is due to the nature of the activation-based allocation rule, which gives more importance to the structural role of players in crossing the threshold. However, in the balanced condition, both methods maintain a fair and symmetric allocation.

Also, the total activation index is 30, which indicates that a total of 30 subthreshold coalitions are activated by different players.

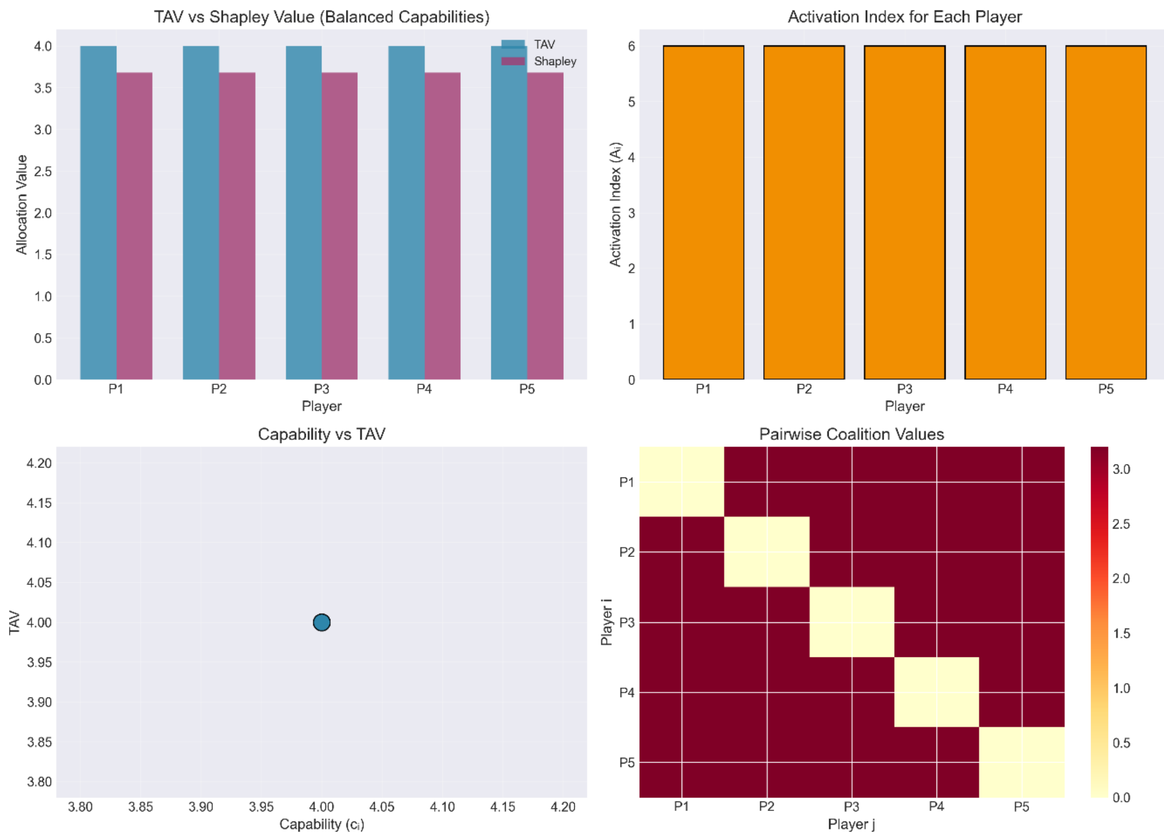

Figure 1. Comparison of TAV allocation and Shapley value in balanced capacity scenario

7.2. Scenario 2: Unequal Capacities with Key Player

In this scenario, one player with high capacity ($c_1 = 8$) and four other players with capacities ($c_2 = 3$, $c_3 = 3$, $c_4 = 2$, $c_5 = 2$) are considered. The cooperation threshold $\Theta = 10$ and the inefficiency coefficient $\lambda = 0.3$ are considered.

Table 2. Allocation results in the unequal capacities scenario

Player	Capability	A_i (Activation Index)	TAV	Shapley_Value	Difference (TAV - Shapley)	TAV Premium
P1	8	14	11.45454545	8.5	2.954545455	34.75935829
P2	3	2	1.636363636	2.4	-0.763636364	-31.81818182
P3	3	2	1.636363636	2.4	-0.763636364	-31.81818182
P4	2	2	1.636363636	1.81	-0.173636364	-9.593169262
P5	2	2	1.636363636	1.81	-0.173636364	-9.593169262

Analysis of Results:

The results of this scenario clearly demonstrate the model's ability to identify key players. The first player, with a capacity of 8 and an activation index of 14, receives a significant share of the total coalition value (v(N) = 18). The TAV for this player is 11.45, which is about 34.76% higher than the Shapley value (8.50).

This additional reward reflects the key role of the first player in activating subthreshold coalitions. For example, the first player alone converts 14 different coalitions from inactive to active, while the other players activate only 2 coalitions. This large difference in activation index leads to an unequal but fair allocation based on the structural role of the players in establishing cooperation.

The Gini coefficient of the TAV allocation in this scenario is calculated to be 0.436, indicating a significant inequality in the distribution of benefits. This inequality, although it may seem undesirable at first glance, precisely reflects the asymmetry in the capacities and structural roles of the players in the cooperation.

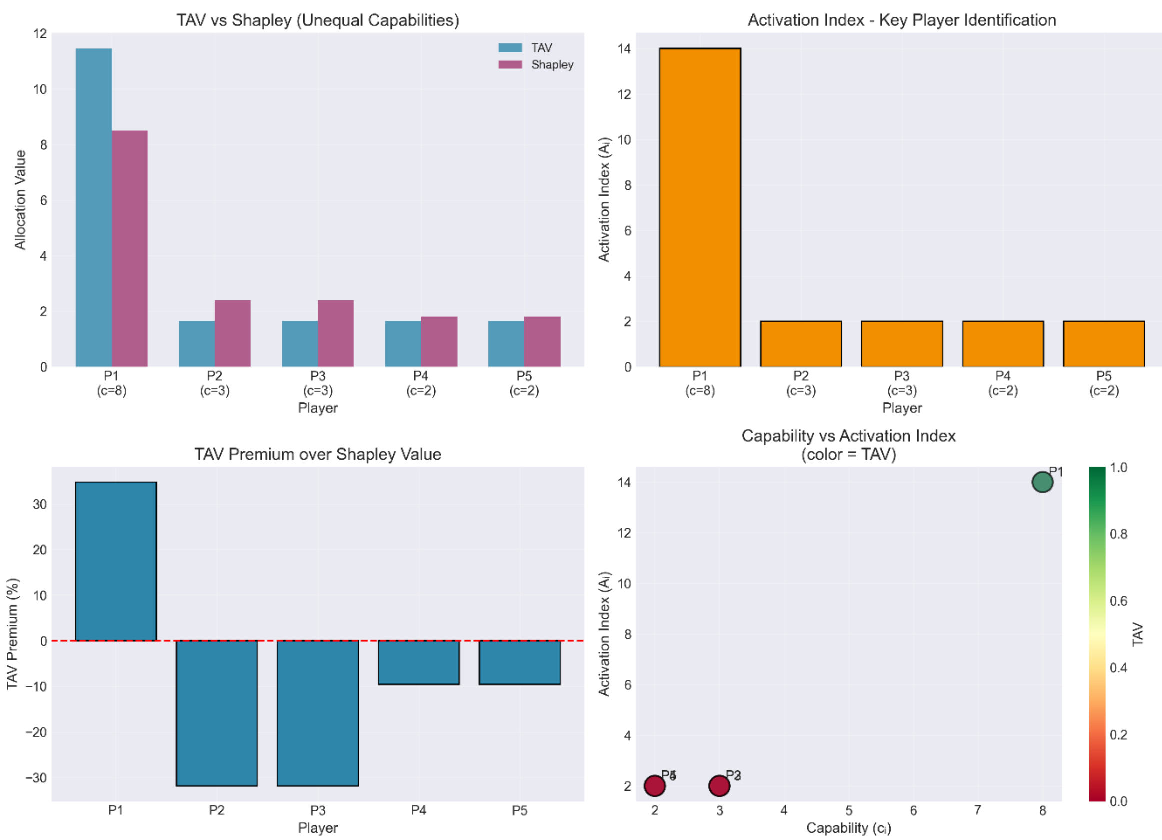

Figure 2. Comparison of TAV allocation and Shapley value in the unequal capacity scenario

7.3. Scenario Three: Sensitivity Analysis to Threshold (Θ)

In this scenario, by keeping capacities and the inefficiency coefficient constant, the effect of threshold changes on the activation index and payoff allocation is examined.

Table 3. Results of the sensitivity analysis to the threshold

Theta	Total_A	Grand_Value	Active_Players	TAV_1	TAV_2	TAV_3	TAV_4	TAV_5	Max_TAV	Min_TAV	TAV_Gini
8	24	20	5	6.666667	5	3.333333	3.333333	1.666667	6.666667	1.666667	0.233333
10	27	20	5	6.666667	5.185185	3.703704	2.222222	2.222222	6.666667	2.222222	0.237037
12	26	20	5	6.153846	4.615385	4.615385	3.076923	1.538462	6.153846	1.538462	0.215385
15	18	20	5	6.666667	4.444444	4.444444	2.222222	2.222222	6.666667	2.222222	0.222222
20	5	20	5	4	4	4	4	4	4	4	0

Analysis of Results:

Key observations from this sensitivity analysis are:

1. Non-uniform behavior of the activation index: As the threshold increases from 8 to 20, the total activation index first increases (from 24 to 27 at Θ=10), then decreases and reaches 5 at threshold 20. This behavior shows that the relationship between the threshold and the number of active coalitions is not linear and depends on the distribution of capacities.

2. Uniformity of allocation at high thresholds: At Θ=20, all players receive a TAV of (4) and the Gini coefficient approaches zero. This situation occurs when there are only 5 active coalitions and the role of all players in activation becomes equal.

3. Stability of the value of the large coalition: Despite the changes in the threshold, the value of the large coalition (v(N) = 20) remains constant. This is because the total capacity of the players (20) is always larger than all the thresholds examined.

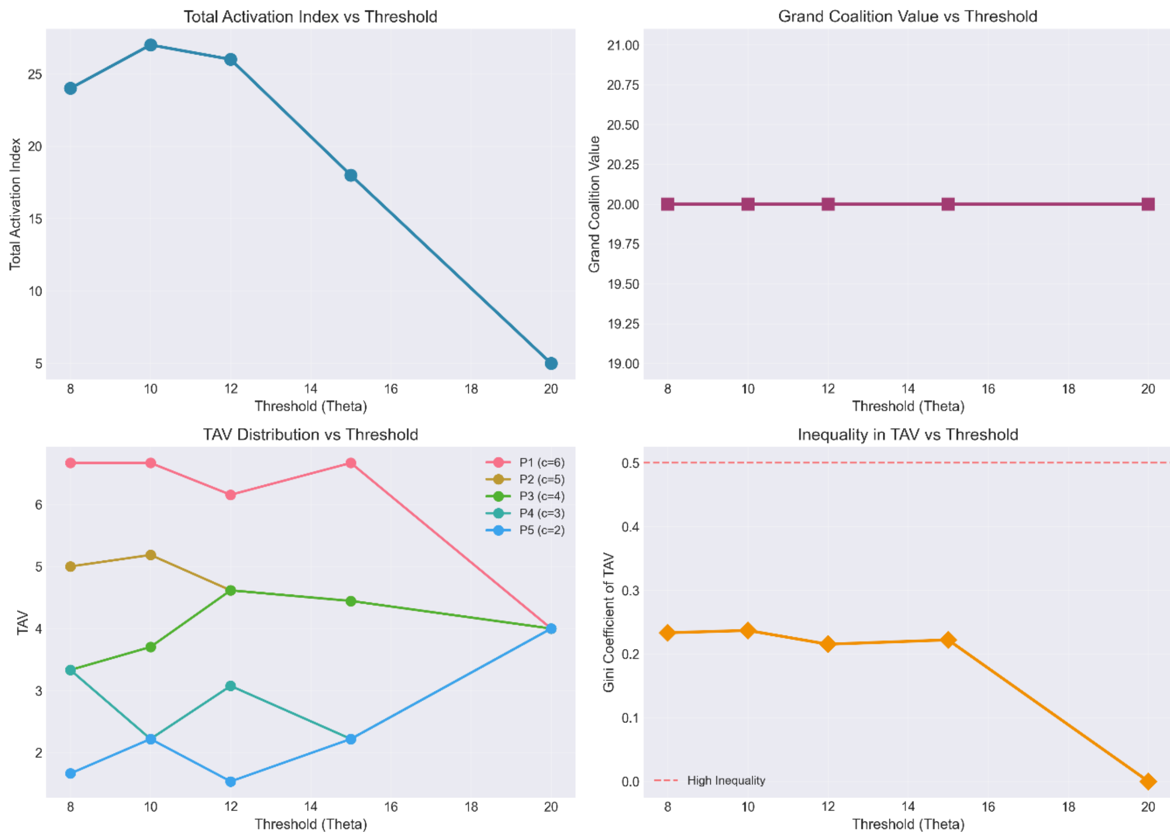

Figure 3. Sensitivity analysis to threshold changes

7.4. Scenario 4: Sensitivity Analysis to the Inefficiency Coefficient (λ)

In this scenario, the effect of changes in the inefficiency coefficient on the value of the alliance and the payoff allocation is examined.

Table 4. Results of the sensitivity analysis to the λ coefficient

Lambda	Grand_Value	TAV_1	TAV_2	TAV_3	TAV_4	TAV_5	Total_A
0	20	6.153846	4.615385	4.615385	3.076923	1.538462	26
0.1	20	6.153846	4.615385	4.615385	3.076923	1.538462	26
0.2	20	6.153846	4.615385	4.615385	3.076923	1.538462	26
0.3	20	6.153846	4.615385	4.615385	3.076923	1.538462	26
0.4	20	6.153846	4.615385	4.615385	3.076923	1.538462	26
0.5	20	6.153846	4.615385	4.615385	3.076923	1.538462	26
0.6	20	6.153846	4.615385	4.615385	3.076923	1.538462	26
0.7	20	6.153846	4.615385	4.615385	3.076923	1.538462	26
0.8	20	6.153846	4.615385	4.615385	3.076923	1.538462	26
0.9	20	6.153846	4.615385	4.615385	3.076923	1.538462	26

Analysis of Results:

The results of this scenario show that changes in the inefficiency coefficient (λ) do not affect the activation index and the TAV allocation ratio. This feature, which is also proven in the λ invariance theorem, is of great importance. In other words, although the value of λ affects the numerical value of subthreshold coalitions, it does not affect the structural role of players in activating coalitions.

Also, the value of the grand coalition (v(N) = 20) remains constant at all values of λ, because the grand coalition is above the threshold and its value is independent of λ.

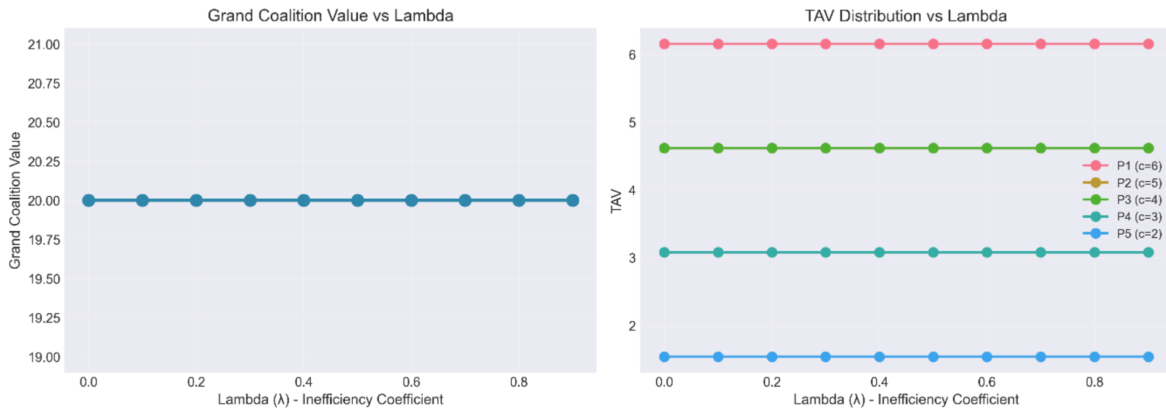

Figure 4. Sensitivity analysis to changes in the inefficiency coefficient

7.5. Scenario 5: Scale Invariance Test

In this scenario, the scale invariance property of the model is examined. For this purpose, the initial capacities [1, 2, 3, 4] and the threshold Θ = 6 are considered, and then both are multiplied simultaneously at different scales.

Table 5. Results of the scale invariance test

Scale	Capabilities	Theta	Activation_Indices	TAV	Grand_Value
0.5	[2.0, 1.5, 1.0, 0.5]	3	[5.0, 3.0, 3.0, 1.0]	[2.0833333333333335, 1.25, 1.25, 0.41666666666666663]	5
1	[4, 3, 2, 1]	6	[5.0, 3.0, 3.0, 1.0]	[4.166666666666667, 2.5, 2.5, 0.8333333333333333]	10
1.5	[6.0, 4.5, 3.0, 1.5]	9	[5.0, 3.0, 3.0, 1.0]	[6.25, 3.75, 3.75, 1.25]	15
2	[8, 6, 4, 2]	12	[5.0, 3.0, 3.0, 1.0]	[8.333333333333334, 5.0, 5.0, 1.6666666666666665]	20
3	[12, 9, 6, 3]	18	[5.0, 3.0, 3.0, 1.0]	[12.5, 7.5, 7.5, 2.5]	30

Analysis of the results:

The results of this test clearly confirm the invariance of the activation indices. At all scales, the activation indices remain constant at [5, 3, 3, 1]. Also, the ratio of the TAV of the players does not change. For example, at all scales, the ratio of the TAV of the first player to the fourth player is equal to 5 (2.08/0.42 = 4.17/0.83 = 6.25/1.25 = 8.33/1.67 = 12.50/2.50 = 5).

This feature shows that the model is robust to scale (unit of measurement) changes and its results are independent of the choice of the capacity unit. However, the value of the grand coalition increases linearly with the scale factor, which is a completely logical and expected behavior.

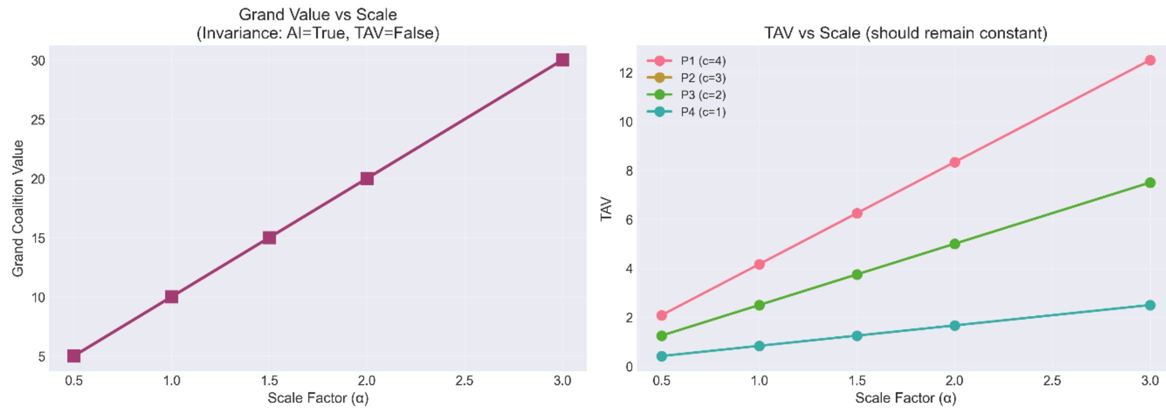

Figure 5. Scale invariance test

7.6. Scenario 6: Monte Carlo Simulation

In this scenario, in order to examine the behavior of the model under random conditions and ensure the stability of the results, 1000 simulations were performed with random distributions for capacities, threshold, and inefficiency coefficient.

Table 6. Descriptive statistics of Monte Carlo simulation results (1000 iterations)

Statistical Measure	Mean	Std. Dev.	Min	25%	Median	75%	Max
Total Activation Index	19.639	6.408	0	16	22	25	30
Grand Coalition Value	27.476	5.834	2.935	23.319	27.823	31.225	42.584
Maximum TAV	8.593	1.966	0	7.532	8.683	9.832	15.030
Minimum TAV	2.403	1.821	0	1.103	2.134	3.585	8.517
Gini Coefficient of TAV	0.237	0.117	≈0	0.178	0.218	0.320	0.600
Number of Active Players	4.754	0.694	0	5	5	5	5

Analysis of Results:

The Monte Carlo simulation results show that:

1. The average total activation index is 19.64, indicating that in a game with 5 players and random parameters, on average about 20 subthreshold coalitions are activated.

2. The average grand coalition value is 27.48, which, given the average total capacity of the players (about $27.5 = 5 \times 5.5$), indicates that in most cases the grand coalition is above the threshold.

3. The average Gini coefficient is 0.237, indicating that the TAV allocation has relatively low inequality on average. However, the range of Gini coefficient variations from 0 to 0.6 indicates that in some scenarios, the payoff allocation can be very unequal.

4. The number of active players was 5 in more than 75% of cases, meaning that in almost all simulations, all five players activated at least one coalition.

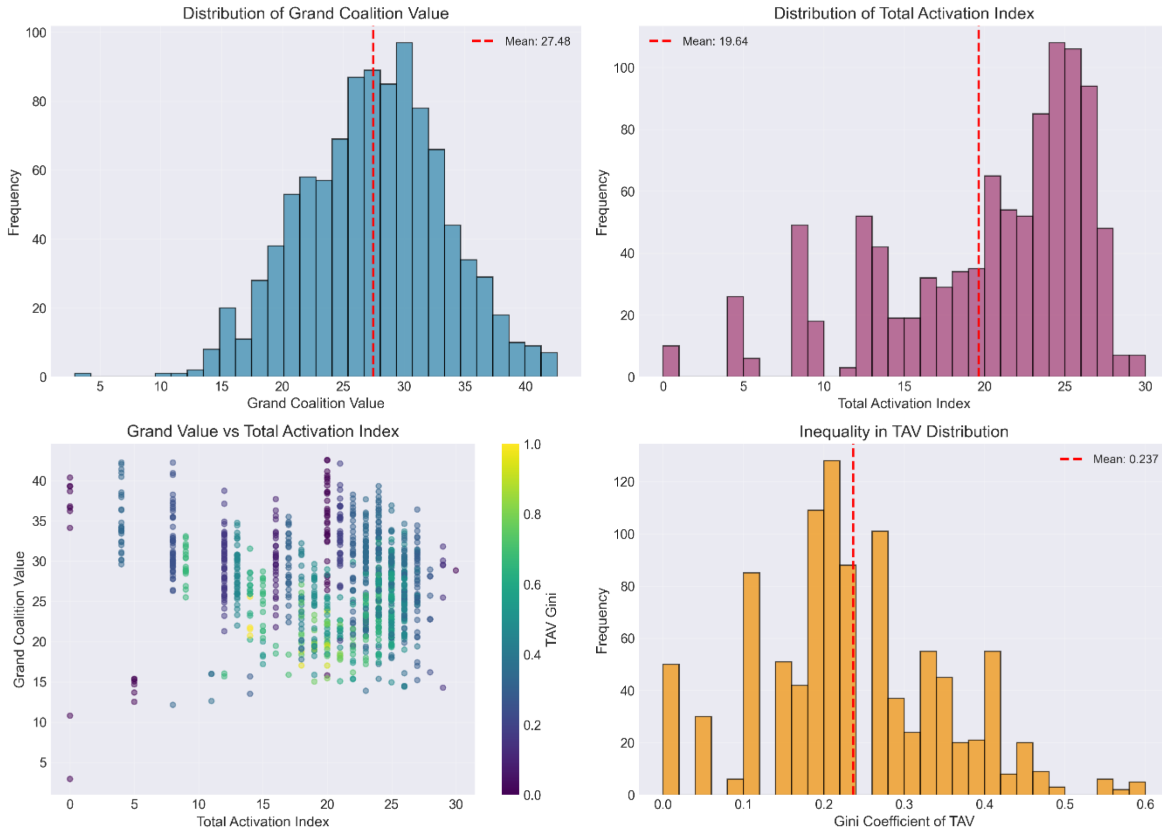

Figure 6. Monte Carlo simulation results - distribution of the grand coalition value, activation index, and Gini coefficient

8. Discussion

The model presented in this paper is developed to overcome one of the fundamental limitations of classical cooperative games. In many common models, the value of a coalition is considered as a function of its members, while in many real systems, cooperation takes an effective form only when the collective capacity or ability of the coalition exceeds a certain minimum level. The concept of threshold introduced in this paper explicitly incorporates this feature into the game structure. Theoretical results showed that threshold coalition games have desirable properties such as uniformity, conditional hyper-incrementality, and stability under certain conditions. In addition, the introduction of the coalition activation index allows the assessment of the structural role of each player in establishing cooperation, an issue that has not been considered in many classical allocation methods.

The results of numerical simulations performed in six different scenarios provide valuable insights into the behavior of the model. In the first scenario (balanced capacities), it was observed that the proposed allocation rule (TAV) provides an equal and fair allocation for all players, which is consistent with intuition and the symmetry property. Although the TAV value is slightly higher than the Shapley value, this difference is due to the greater emphasis of the proposed rule on the activation role of players. In the second scenario (unequal capacities with key players), the results clearly demonstrated the model's ability to identify key players and reward them commensurate with their structural role. The first player with a capacity of 8 and an activation index of 14 received about

34.76% more than the Shapley value. This finding emphasizes that in unequal conditions, the activation-based allocation rule can guide the payoff allocation in such a way that players who play a pivotal role in creating cooperation receive a fairer share of the total value. The Gini coefficient of 0.436 in this scenario indicates significant inequality in the distribution of payoffs, reflecting the inherent asymmetry in capacities and structural roles.

The sensitivity analysis to the threshold (scenario three) revealed the non-uniform behavior of the activation index. As the threshold increases, the number of active coalitions first increases and then decreases, indicating that the relationship between the threshold and the activation of coalitions is not linear and depends on the distribution of capacities. Also, at very high thresholds, the allocation of payoffs tends to be uniform, and the Gini coefficient approaches zero. This observation suggests that in a situation where there are only a limited number of active coalitions, the role of all players in activation becomes equal. The sensitivity analysis to the inefficiency coefficient (scenario four) revealed another important feature: changes in λ do not affect the activation index and the TAV allocation ratio. This feature, which is also proven in the invariance theorem with respect to λ, is of great practical importance, since it shows that although λ affects the numerical value of subthreshold coalitions, the structural role of players in the activation of coalitions is independent of this parameter.

The scale invariance test (scenario five) clearly confirmed the theoretical property of the model. The activation indices and the TAV ratio of players remained constant at all scales, indicating that the model is robust to changes in the unit of measurement of capacities. This property is very important for practical applications of the model, since the model results will be independent of the choice of scale. Monte Carlo simulation (scenario six) with 1000 iterations demonstrated the stability of the model well under random conditions. The average total activation index (19.64) and the average grand coalition value (27.48) indicate the predictable behavior of the model under random conditions. Also, the average Gini coefficient (0.237) indicates that, on average, the TAV allocation has relatively low inequality, although its range of variation (0 to 0.6) indicates that in some specific scenarios, inequality can be significant.

Based on theoretical and simulation findings, the proposed allocation rule not only fully distributes the total value of the coalition, but also determines the players' shares in proportion to their role in activating new coalitions. Although the model presented in this study is purely theoretical in nature, it can be generalized to various fields such as collaborative networks, research consortia, multi-agent systems, project management, digital economy, and supply chains. Also, future research can develop dynamic, fuzzy, probabilistic, or multi-threshold versions of this model and examine its behavior in more complex structures. Investigating the conditions for the existence of the core, studying the convexity of the game, developing alternative allocation rules, generalizing the model to dynamic and fuzzy games, and applying it to real problems are among the research directions that can be addressed in the continuation of this study.

9. Conclusion

In this study, a new class of cooperative games, called “threshold coalition games,” was introduced, in which the creation of value for coalitions depends on crossing a certain threshold of collective capacity. This framework allows modeling conditions where effective cooperation is formed only after reaching a minimum required power or resources. Subsequently, the fundamental properties of the model were investigated and proven from a mathematical perspective, and a new allocation rule based on the role of players in activating coalitions was proposed.

Theoretical results showed that the proposed rule has the properties of efficiency, symmetry, non-negativity, and uniqueness, and can reflect the role of players in terms of their ability to create cooperation better than criteria based solely on final contributions. Extensive numerical simulations in six different scenarios confirmed the validity of the model and provided valuable insights into its behavior under different conditions. In particular, the results showed that in the unequal scenario, the key player receives about 35% more payoff than the Shapley value, which indicates the accurate identification of the structural role of players in activating cooperation. Also, the sensitivity analysis showed that the model is robust to changes in the inefficiency coefficient and scale, and its behavior is stable under random conditions.

In addition to providing a new theoretical framework, the presented model provides a good basis for the development of future research. Investigating the conditions for the existence of the core, studying the convexity of the game, developing alternative allocation rules, generalizing the model to dynamic and fuzzy games, and applying it to real problems are among the research paths that can be addressed in the continuation of this study. According to the results obtained, this model has high potential for application in various fields such as collaborative networks, research consortia, project management, and the digital economy, and can be used as an efficient tool for analyzing and designing threshold-based cooperation structures.

The authors used large language models (e.g., ChatGPT) and online tools (e.g., Grammarly) to assist in improving the grammar and language quality of the manuscript.

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