Iranian Journal of Operations Research
Vol. 2, No.1, 2010, pp. 50-61

Characterization of efficient points of the production possibility set under variable returns to scale in DEA

F. Hosseinzadeh Lotfi[1], A.A. Noora[2], G.R. Jahanshahloo[3], J.Gerami[4] and M.R.Mozaffari[5]

We suggest a method for finding the non-dominated points of the production possibility set (PPS) with variable returns to scale (VRS) technology in data envelopment analysis (DEA). We present a multiobjective linear programming (MOLP) problem whose feasible region is the same as the PPS under variable returns to scale for generating non-dominated points. We demonstrate that Pareto solutions of the MOLP produce efficient units in DEA, and vice versa. We solve the MOLP problem by using a finite number of weights which are extreme rays of the cone generated by the efficient solutions. We obtain new efficient points by changing weights, and thus the efficient solutions set is produced.

Keywords: *Data envelopment analysis, Multi-objective linear programming, Production possibility set, Variable returns to scale.*

1. Introduction

Data Envelopment Analysis (DEA) was originally proposed by Charnes et al. [4] as a method for evaluating the relative efficiency of Decision Making Units (DMUs) performing essentially the same task. Units use similar multiple inputs to produce similar multiple outputs. DEA deals with the evaluation of the performance of DMU performing a transformation process of several inputs to several outputs. Relying on a technique based on linear programming (LP) and without having to introduce any subjective or economic parameters (weight, price, etc.), DEA provides a measure of efficiency of each DMU allowing, in particular, to separate efficient from non-efficient DMUs and to indicate for each non-efficient DMU its efficient peers. Charnes et al [5] have also had a significant impact on the development of multiple objective linear programming (MOLP) and DEA. However, researchers generally not have paid much attention to research performed in the other camp. DEA and MOLP address similar problems and are structurally very close to each other. In a broader picture, there have been various studies highlighting the similarities between DEA and multiple criteria decision making (MCDM) in general and MOLP in particular, though it is said that they retain their own distinctive traits, see Belton and Stewart. [2], Agrell and Tind [1], Joro et al. [8] and Stewart et al [13, 14]). Taking a step further Doyle and Green [6] suggested that DEA is an MCDM method itself. Belton and Vickers. [3] described the equivalence between the formulations of the basic DEA models and the classic linear multi-attribute value function of MCDM. More specifically, Belton and Stewart [2] pointed out that the mechanism of DEA involves comparison of DMUs on the basis of multiple criteria of both inputs and outputs, but the emphasis of DEA is put on evaluating DMUs against the best practice units and on setting targets

1 Corresponding author, Department of Mathematics, Science and Research Branch,Islamic Azad University,Tehran, Iran. farhad@hosseinzadeh.ir.
2 Department of Mathematics, Sistan & Baluchestan University, Zahedan, Iran
3 Department of Mathematics, Tarbiat Moallem University, Tehran, Iran.
4 Department of Mathematics, Science and Research Branch,Islamic Azad University,Tehran.
5 Department of Mathematics, Science and Research Branch,Islamic Azad University,Fars, Iran

to improve efficiency, while MCDM focuses on ranking and assessing alternatives. The MOLP model has been widely applied to many fields and has become a useful tool For decision making (for applications, see Leschine et al. [9], Gravel et al. [7] and Prabuddha et al. [10]). A requisite technique has already been developed for multiple objective linear programing models. Because DEA and MOLP models are structurally similar, we apply this technique to DEA problems as well. The approach recommended is based on the equivalence between DEA and MOLP; we obtain efficient units in DEA by solving an MOLP problem. Here we use an approach similar to the "combined constraint-space, objective space" approach together with the method of transferring a polyhedron from intersection form to sum form for building the efficient solution structure of an MOLP. It is well known that a polyhedron can be represented by a set of linear constraints, called "intersection form" or by a convex combination of finite extreme points and non-negative combination of finite extreme rays, called"sum form". A polyhedron can be transferred from intersection form to sum form (for the algorithms, see Charnes et al. [5], Wei and Yan [11, 12], or Yan et al. [15]). westart by studying the relation between DEA and MOLP problems. We show that by choosing weights properly and solving the weighted sum problem of MOLP associeted with these weights, we can obtain all the weak Pareto solutions and the Pareto solution of the MOLP problem that are efficient units in DEA.

The mainder of our work is organized as follows. Section 2 introduces the similar structure of MOLP and DEA problem and some of theair basic results for later use. Section 3 provides the efficient points structure of the DEA model. Section 4 gives a numerical example for illustrating our approach. Section 5 gives the conclusions.

2. Structural similarities between MOLP and DEA

Consider n decision making units, $DMU_j\ (j = 1, \dots, n)$, where each DMU consumes an m-vector input to produce an s-vector output. Suppose that $X^j = (x_1^j, x_2^j, \dots, x_m^j)^T$ and $Y^j = (y_1^j, y_2^j, \dots, y_s^j)^T$ are the vectors of inputs and outputs, respectively, for DMU_j, in where it is assumed that $X^j \geq 0, X^j \neq 0$ and $Y^j \geq 0,\ \ Y^j \neq 0$. We define the production possibility set of Data Envelopment Analysis under variable returns to scale as follows:

$$T_v = \{(X, Y) = (x_1, \dots, x_m, y_1, \dots, y_s) | \textstyle\sum_{j=1}^{n} \lambda_j y_r^j \geq y_r,\ r = 1, \dots, s,$$
$$\textstyle\sum_{j=1}^{n} \lambda_j x_i^j \leq x_i,\ i = 1, \dots, m, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, j = 1, \dots, n\}.$$

Definition 1. $DMU_o = (X^o, Y^o) \in T_v$ *is called an efficient unit if* and *only if there is not an* $(X, Y) \in T_v$ *such that* $(X, -Y) \leq (X^o, -Y^o)$ *and* $(X, -Y) \neq (X^o, -Y^o)$.

Definition 2. $DMU_o = (X^o, Y^o) \in T_v$ *is called a weak efficient unit if* and *only if there is not an* $(X, Y) \in T_v$ *such that* $(X, -Y) < (X^o, -Y^o)$.

Consider the following multiobjective linear programing (MOLP) problem,

$$\begin{aligned} \min \quad & C^T Z \\ s.t. \quad & Z \in R = \{Z | AZ \leq b, Z \geq 0\}, \end{aligned} \qquad (1)$$

where $C = (C_1^t, C_2^t, \dots, C_p^T)^{\mathrm{t}}$ is a $p \times n$ matrix, $c_i^{Tt} \in E^n$, $i = 1, \dots, p$, $Z = (\mathrm{z}_1, \mathrm{z}_2, \dots, \mathrm{z}_\mathrm{n}) \in E^n$, E^n is called the constraint space, A is an $m \times n$ matrix, $n \geq m$ and rank(A)=m, $b = (b_1, b_2, \dots, b_m) \in E^m$.

The Pareto solution and weak Pareto solution of (1) are defined as follows.

Definition 3. $\bar{Z} \in R$ *is called a Pareto solution of* (1) *if there does not exist* $Z \in R$ *such that* $C^T Z \leq C^T \bar{Z}, C^T Z \neq C^T \bar{Z}$.

Definition 4. $\bar{Z} \in R$ is called a weak Pareto solution of (1) if there does not exist $Z \in R$ so that $C^t Z < C^t \bar{Z}$.

Put $Z = (X, Y, \lambda) = (x_1, \dots, x_m, y_1, \dots, y_s, \lambda_1, \dots, \lambda_n)$, $X \in R^m$, $Y \in R^s$, $\lambda \in R^n$, $C_j^t = e_j^t, j = 1, \dots, m$, $C_j^t = -e_j^t, j = m+1, \dots, m+s$, $e_j^t \in R^{m+s+n}$ is a vector whose jth element is one and other elements are zero, $C = (C_1^T, C_2^T, \dots, C_{m+s}^T)^T$, and
$R = \{(x_1, \dots, x_m, y_1, \dots, y_s, \lambda_1, \dots, \lambda_n) | \sum_{j=1}^n \lambda_j y_r^j \geq y_r,\ r = 1, \dots, s,$
$\sum_{j=1}^n \lambda_j x_i^j \leq x_i,\ i = 1, \dots, m, \sum_{j=1}^n \lambda_j = 1, \lambda_j \geq 0, j = 1, \dots, n\}$.

Then problem (1) is converted to

$$\begin{array}{ll} \min & (x_1, \dots, x_m, -y_1, \dots, -y_s) \\ s.t. & \sum_{j=1}^n \lambda_j y_r^j \geq y_r, \quad r = 1, \dots, s, \\ & \sum_{j=1}^n \lambda_j x_i^j \leq x_i, \quad i = 1, \dots, m, \\ & \sum_{j=1}^n \lambda_j = 1, \\ & \lambda_j \geq 0, \quad j = 1, \dots, n, \\ & x_i \geq 0, \quad i = 1, \dots, m, \quad y_r \geq 0, \quad r = 1, \dots, s. \end{array} \tag{2}$$

Note (X, Y, λ) is a feasible solution of problem (2) while $(X, -Y)$ is a vector belonging to objective function space of problem (2).

By considering definition (2) (X^*, Y^*, λ^*) is called a weak pareto solution of (2),if there does not exist (X, Y, λ) such that $(X, -Y) < (X^*, -Y^*)$.

Theorem 1. *Let* $(X^*, Y^*) \in T_v$. *Then,*
(i) (X^*, Y^*, λ^*) *is a Pareto solution of* (2) *if and only if* (X^*, Y^*) *is an efficient unit in* T_v.
(ii) (X^*, Y^*, λ^*) *is a weak Pareto solution of* (2) *if and only if* (X^*, Y^*) *is a weak efficient unit in* T_v.

Proof: (i) Let (X^*, Y^*, λ^*). be a Pareto solution of (2). We show that (X^*, Y^*) is an efficient unit in T_v. By contradiction, suppose (X^*, Y^*) is not an efficient unit in T_v. Then there is an $(\bar{X}, \bar{Y}) \in T_v$ such that $(\bar{X}, -\bar{Y}) \leq (X^*, -Y^*)$ and $(\bar{X}, -\bar{Y}) \neq (X^*, -Y^*)$. Since $(\bar{X}, \bar{Y}) \in T_v$, there is a $\bar{\lambda} \in R^n$ such that $(\bar{X}, \bar{Y}, \bar{\lambda})$ is a feasible solution of (2) . Since $(\bar{X}, -\bar{Y}) \leq (X^*, -Y^*)$ and $(\bar{X}, -\bar{Y}) \neq (X^*, -Y^*)$ then we have a contradiction; therefore, (X^*, Y^*) is an efficient unit in T_v.

Now suppose (X^*, Y^*) is an efficient unit in T_v. Since $(X^*, Y^*) \in T_v$, there is a $\lambda^* \in R^n$ such that (X^*, Y^*, λ^*) is a feasible solution of (2). AS (X^*, Y^*) is an efficient unit in T_v, there is no $(\bar{X}, \bar{Y}) \in T_v$ such that $(\bar{X}, -\bar{Y}) \leq (X^*, -Y^*)$ and $(\bar{X}, -\bar{Y}) \neq (X^*, -Y^*)$. Since, there is a vector $\bar{\lambda}$ for each $(\bar{X}, \bar{Y}) \in T_v$ such that $(\bar{X}, \bar{Y}, \bar{\lambda})$ is a feasible solution of (2). Regarding the above relations there is no $(\bar{X}, \bar{Y}, \bar{\lambda})$ that is a feasible solution of (2) such that $(\bar{X}, -\bar{Y}) \leq$

$(X^*, -Y^*)$ and $(\bar{X}, -\bar{Y}) \neq (X^*, -Y^*)$. Therefore (X^*, Y^*, λ^*) is a Pareto solution of (2) and the proof is complete.

(*ii*) Proof is similar to (*i*). ■

Theorem 2. *The optimal- values of problem* (2) *are finite.*

Proof: Since $-\sum_{j=1}^{n} \lambda_j y_r^j \leq -y_r, r = 1, \dots, s,$ and $\sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, j = 1, \dots, n$, then $-y_r,\ r = 1, \dots, s$ are finite. Similarly $\sum_{j=1}^{n} \lambda_j x_i^j \geq x_i, i = 1, \dots, m$, and $\sum_{j=1}^{n} \lambda_j = 1$, $\lambda_j \geq 0, j = 1, \dots, n$. Then, $x_i,\ i = 1, \dots, m,$ are finite. Therefore, the optimal values of problem (2) are finite. ■

3. Constructing efficient points structure of DEA

Consider the following multiobjective linear programming problem,

$$\begin{aligned} \min \quad & (X, -Y) \\ \text{s.t.} \quad & (X, Y) \in T_v. \end{aligned} \tag{3}$$

Note that the above problem is the vector form of problem (2).

Using Theorem 1, we solve problem (3) to obtain the efficient units of T_v. Problem (3) being an MOLP problem, we can use a method of solving MOLP, as presented in [16].

Since, for every $(X, Y) \in T_v$ we have a corresponding point $(X, -Y)$ in the objective space of problem (3), there exists a injective correspondence between the units of T_v and the values of the objective function. Then, for finding the efficient units of T_v we solve problem (3) and obtain all the weak efficient points.

Here paper, we use the weighted sum problem method. So, we select vectors of positive weights $V \in E^m, U \in E^s$ $(V \geq 0, U \geq 0, V \neq 0, U \neq 0)$. The following linear programming problem is called a weighted sum problem of problem (2) associated with weight $W = (V, U)^T = (v_1,, \dots, v_m, u_1, \dots, u_s)^T \in E^{m+s}(V \geq 0, U \geq 0, V \neq 0, U \neq 0)$.

$$\begin{aligned} \min \quad & \sum_{i=1}^{m} v_i x_i - \sum_{r=1}^{s} u_r y_r \\ \text{s.t.} \quad & (x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_s) \in T_v. \end{aligned} \tag{4}$$

It is well known (from Theorem 3 below) that $(X, -Y)$ is an optimal solution for the linear programming problem (2) in objective function space if and only if $(X, -Y)$ is an optimal solution for the linear programming problem (4) for some $W \in E^{m+s}(W \geq 0, W \neq 0)$.

Theorem 3. *Let* $(X, Y) \in T_v$. *Then,*

(i)$(X, -Y)$ *is a weak Pareto solution of (2) in objective function space if and only if there exists a weight* $W = (w_1, w_2, \dots, w_{m+s}), W \in E^{m+s}, W \geq 0, W \neq 0,$ *such that* $(X, -Y)$ *is the optimal solution of the weighted sum problem (4).*

(ii) $(X, -Y)$ *is a Pareto solution of (2) in objective function space if and only if there exists a weight* $W = (w_1, w_2, \dots, w_{m+s}), W \in E^{m+s}, W > 0,$ *such that* $(X, -Y)$ *is the optimal solution of the weighted sum problem (4).*

Proof: See Zeleny [16] for proof. ■

Provided that we have obtained *k* weak Pareto solutions, $(\widehat{X},-\widehat{Y})^1,\dots,(\widehat{X},-\widehat{Y}),^k$ of (2), we hope to find new Pareto solutions of (2) by solving its weighted sum problem. In following, we show how we obtain these solutions. (see Theorems (3) and (4)). In order to do this, we should choose some weights to obtain the new efficient solutions.

Let $SE=\{(X,-Y)|(X,-Y)=\sum_{q=1}^{k}\lambda_q(\widehat{X},-\widehat{Y})^q,\sum_{q=1}^{k}\lambda_q=1,\lambda_q\geq 0,q=1,\dots,k\}+E_{+}^{m+s}$, where $E_{+}^{m+s}=\{(X,Y)|(X,Y)\in E^{m+s},(X,Y)\geq 0\}$. Let also Cone $\mathbf{C}$, is constructed as follows.

$$C=\{(V,U,u_0)|V^T\widehat{X}^q-U^T\widehat{Y}^q\geq u_0,(V,U)\geq 0,(V,U)\neq 0,q=1,\dots,k\}.$$

It is easy to see that SE is a closed convex set in E^{m+s} and $C\cup\{0\}$ is a polyhedral cone in E^{m+s}. C can be represented by a non-negative combination of its extreme rays. (For the algorithms, see Charnes et al. [5] or Wei and Yan. [11, 12.]. and Yan et al. [15]).

Denote the extreme rays of $\mathbf{C}$ by

$W^l=(\dot{w}_1,\dots\dot{w}_m,\ddot{w}_1,\dots,\ddot{w}_s,w_0)^l=(\dot{W},\ddot{W},w_0)^l\in E^{m+s+1},l=1,\dots,h$. Then,

$C=\{\sum_{l=1}^{h}\mu_l(\dot{W},\ddot{W},w^0)^l|\ \mu_l\geq 0,l=1,\dots,h\}$.

Denote $P=\{(X,-Y)|(\dot{W}^l)^TX-(\ddot{W}^l)^TY\geq w_0^l,l=1,\dots,h,(X,Y)\in T_v\}$.

Theorem 4. *Suppose SE and $\mathbf{P}$ are defined as above. Then $SE,=P$.*

Proof: First, we show that $SE\subseteq P$.

Let $(X^o,-Y^o)\in SE$. Then, there exists $\lambda^o=(\lambda_1^o,\lambda_2^o,\dots,\lambda_k^o)^T,\ \lambda^o\in E^k$, $\sum_{q=1}^{k}\lambda_q=1,\lambda_q\geq 0,q=1,\dots,k$, such that $(X^o,-Y^o)\geq\sum_{q=1}^{k}\lambda_q(\widehat{X}^q,-\widehat{Y}^q)$.

Therefore, $X^o\geq\sum_{q=1}^{k}\lambda_q\widehat{X}^q,-Y^o\geq-\sum_{q=1}^{k}\lambda_q\widehat{Y}^q$. Since $(\dot{W},\ddot{W},w_0)^l,l=1,\dots,h$, are the extreme rays of $\mathbf{C}$, we have, $\widehat{X}^T\dot{W}^l-\widehat{Y}^T\ddot{W}^l\geq w_0^l,\ l=1,\dots,h$.

Note that $(\dot{W},\ddot{W})^l\geq 0,l=1,\dots,h$. We have ${X^o}^T\dot{W}^l-{Y^o}^T\ddot{W}^l\geq\sum_{q=1}^{k}\lambda_q\left(\widehat{X}^q\right)^T\dot{W}^l-\sum_{q=1}^{k}\lambda_q\left(\widehat{Y}^q\right)^T\ddot{W}^l\geq\sum_{q=1}^{k}\lambda_q w_0^l=w_0^l,\ l=1,\dots,h,$, that is, $(X^o,-Y^o)\in P$.

On the other hand, assume that $P\subseteq SE$ is not true. Then, there exists $(X^o,-Y^o)\in P$, but $(X^o,-Y^o)$ is not in SE. Since SE is a closed convex set, by the separation theorem for convex sets, there exist $d\in E^{m+s},d\neq 0$, and $\alpha\in E^1$, such that for any $(X,-Y)\in SE$, we have, $d(X,-Y)^T\geq\alpha>d(X^o,-Y^o)^T$ Then $\sum_{i=1}^{m}d_ix_i-\sum_{r=1}^{s}d_{r+m}y_r\geq\alpha>\sum_{i=1}^{m}d_ix_i^o-\sum_{r=1}^{s}d_{r+m}y_r^o$.

Note that when all components of $(X,-Y)$ are very large, we still have $(X,-Y)\in SE$. Thus, $d\geq 0$. Since $(\widehat{X}^q,-\widehat{Y}^q)\in SE$, we have $\sum_{i=1}^{m}d_i\hat{x}_i^q-\sum_{r=1}^{s}d_{r+m}\hat{y}_r^q\geq\alpha,\ q=1,\dots,k$. Therefore, $(d,\alpha)^T\in C$, (by definition of $\mathbf{C}$).Tthat is, there are $\mu_l\geq 0,l=1,\dots,h$, such that $d=\sum_{l=1}^{h}\mu_l(\dot{W},\ddot{W})^l$, $\alpha=\sum_{l=1}^{h}\mu_l w_0^l$. Thus $\alpha\geq\sum_{i=1}^{m}d_ix_i^o-\sum_{r=1}^{s}d_{r+m}y_r^o=\sum_{l=1}^{h}\sum_{i=1}^{m}\mu_l\dot{w}_i^lx_i^o-\sum_{l=1}^{h}\sum_{i=1}^{m}\mu_l\ddot{w}_r^ly_r^o$. Note that $(X^o,-Y^o)\in P$. Then, $\alpha\geq\sum_{i=1}^{m}d_ix_i^o-\sum_{r=1}^{s}d_{r+m}y_r^o=\sum_{l=1}^{h}\sum_{i=1}^{m}\mu_l\dot{w}_i^lx_i^o-\sum_{l=1}^{h}\sum_{i=1}^{m}\mu_l\ddot{w}_r^ly_r^o>\sum_{l=1}^{h}\mu_l w_0^l.=\alpha$which is a contradiction. So, $SE\subseteq P$, and the proof is complete. ■

After all obtaining extreme rays of $\mathbf{C}$, we solve the following weighted sum problems associated with weight $W^l=(\dot{W},\ddot{W},w^0)^l,l=1,\dots,h$, (note that $W^l,l=1,\dots,h$ are the extreme rays of $\mathbf{C}$):

$$\begin{array}{ll} \min & \sum_{i=1}^{m} \dot{w}_i^l x_i - \sum_{r=1}^{s} \ddot{w}_r^l y_s \\ \text{s.t.} & (x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_s) \in T_v, \end{array} \qquad (5)$$

where $l = 1, \dots, h$.

Let $(\overline{X}, -\overline{Y})^l$ be an optimal solution of (5) corresponding to weights vector $W^l, l = 1, \dots, h$. From Theorem 3, $(\overline{X}, -\overline{Y})^l$ is a weak Pareto solution of (2)) in objective function space.

For any $(\overline{X}, -\overline{Y})^l, l = 1, \dots, h,$ we have two cases:

1. For any $l = 1, \dots, h$, we have $(\overline{X}, -\overline{Y})^l \in P$. Then, from Theorems 4, 7 and 8, (to be seen later) we have determined all weak Pareto solutions of problem (2)) in objective function space.There exists $l(0 \leq l \leq h)$ such that $(\overline{X}, -\overline{Y})^l$ is not in $\mathbf{P}$. Denote an index set
2. $I_0 = \{l | (\overline{X}, -\overline{Y})^l$ is not in $\mathbf{P}, l = 1, \dots, h\}$. Then, $(\overline{X}, -\overline{Y})^l, l \in I_0,$ are the new weak Pareto solutions of problem (2) in objective function space.

To determine the structure of all optimal solutions of an MOLP problem, we make use of extreme points and ray in MOLP. First, we obtain some of the optimal solutions in MOLP, which coincide with DEA efficient units. These solutions can be easily obtained in the first stage. Next, we obtain the structure of all optimal solutions in MOLP, which coincide with the DEA-efficient surfaces. This is the space made by the efficient surfaces. We construct cone C, which is made by the efficient surfaces; then we obtain the extreme rays of the cone and use them as the weights of the objective function in MOLP for finding new optimal points. If all the points produced lie in the space made by the efficient surfaces, then we have obtained the structure of all optimal solutions. Otherwise, we reconstruct cone C by adding the new optimal points and the surfaces containing them. The cone will then include the half-spaces made by the new and previous efficient surfaces. We pursue this process until we obtain the structure of all efficient surfaces. It should be noted that the union of all efficient surfaces will yield the set of all efficient points. As we have not obtained all efficient points in the second case, we make a new cone and obtain the new extremerays of this cone as the new weights of the weighted sum problem.

At First, we produce $m + s$ weak Pareto solutions of problem (2)) in objective function space for composing set SE, which will be efficient units in T_v, with regard to Theorems 5 and 6. We compose cone $\mathbf{C}$ using these points, and produce new efficient points by obtaining extreme rays of this cone as the new weights .

If all points belong to the corresponding set $\mathbf{P}$, we will have all efficient points. Otherwise we continue producing new weights and points. In addition, we present a procedure for studying the above conditions.

At first, we obtain m+s weak efficient unit $(X^*, Y^*)^q, q = 1, \dots, m + s$, by using $\mathbf{n}$ observed DMUs introduced in Section 2 ($DMU_j = (X^j, Y^j)$ such that

$X^j = (x_1^j, x_2^j, \dots, x_m^j)$ and $Y^j = (y_1^j, y_2^j, \dots, y_s^j), j = 1, \dots, n)$ as follows.

For $i = 1, \dots, m$, we put $(X^*, Y^*)^i = (x_1^l, x_2^l, \dots, x_m^l, y_1^l, y_2^l, \dots, y_s^l)$, where $\mathbf{l}$ is an index such that $x_i^l = \min\{x_i^j | j = 1, \dots, n\}$.

Similarly, for $r = 1, \dots, s$, we put $(X^*, Y^*)^{m+r} = (x_1^t, x_2^t, \dots, x_m^t, y_1^t, y_2^t, \dots, y_s^t)$, where $\mathbf{t}$ is an index such that $y_r^t = \max\{y_r^j | j = 1, \dots, n\}$.

Theorem 5. Suppose $x_i^l = \min\{x_i^j | j = 1, \dots, n\}$

(note that l *is corresponding to the index of the DMU that has the least input in* i*th component). Then,* $(X^*,Y^*)=(x_l^l,x_2^l,\dots,x_m^l,y_1^l,y_2^l,\dots,y_s^l)$ *will be a weak efficient unit in* T_v.

Proof: By contradiction, suppose (X^*,Y^*) is not a weak efficient unit in T_v. Therefore, there is $(X,Y)\in T_v$ such that $(X,-Y)<(X^*,-Y^*)$. Since $(X,Y)\in T_v$, there is a vector $\lambda\in E^n$ such that $\sum_{j=1}^n \lambda_j y_r^j \ge y_r, r=1,\dots,s$ and $\sum_{j=1}^n \lambda_j x_i^j \le x_i, i=1,\dots,m$, $\sum_{j=1}^n \lambda_j=1$, $\lambda_j\ge 0, j=1,\dots,n$.

Regarding above relations, we have $x_i^* > x_i \ge \sum_{j=1}^n \lambda_j x_i^j$, , which is a contradiction, because we have $x_i^l=\min\{x_i^j | j=1,\dots,n\}, x_i^l=x_i^*\le x_i^j, j=1,\dots,n$, then $x_i^*\le \sum_{j=1}^n \lambda_j x_i^j$, $\sum_{j=1}^n \lambda_j=1,\ \lambda_j\ge 0, j=1,\dots,n.$ Therefore, (X,Y) is a weak efficient unit in T_v, and the proof is complete. ■

Theorem 6. *Suppose* $y_r^t=\min\{y_r^j | j=1,\dots,n\}$ *(note that t is corresponding to the index of DMU that has the most output in* r*th component), then* $(X^*,Y^*)=(x_l^t,x_2^t,\dots,x_m^t,y_1^t,y_2^t,\dots,y_s^t)$ *will be a weak efficient unit in* T_v.

Proof. The Proof is similar to the proof of Theorem 5. ■

By attention to Theorem (3), since $(X^*,Y^*)^q,\ q=1,\dots,m+s$ are weak efficient units in T_v thus $(X^*,-Y^*)^q,\ q=1,\dots,m+s$ are weak pareto solutions in objective function space of problem (2). Denote $EF^1=\{(X^*,-Y^*)^q | q=1,\dots,m+s\}$ and $k=m+s$. Let

$C^1=\{(V,U,u_0) | V^T X^{*q}-U^T Y^{*q}\ge u_0, (V,U)\ge 0, (V,U)\ne 0, q=1,\dots,m+s\}$.

Obtain the extreme rays of C^1 and denote the rays by $W^l=(\dot{W},\ddot{W},w_0)^l, l=1,\dots,h_1$.

Now, denote $P^1=\{(X,-Y) | (\dot{W}^l)^T X-(\ddot{W}^l)^T Y\ge w_0^l, l=1,\dots,h_1, (X,Y)\in T_v\}$.

For $l=1,\dots,h_1$, solve the weighted sum problem (5).

Let $(\bar{X},-\bar{Y})^l$ be an optimal solution of (5). Denote an index set as follows.

$I_1=\{l | (\bar{X},-\bar{Y})^l$ is not in $P^1, l=1,\dots,h_1\}$.

If $I_1=\emptyset$ the stop, (denote $h=h_1, k=m+s$, we have case (1)), else denote

$EF^2=\{(\bar{X},-\bar{Y})^l | l\in I_1\}\cup EF^1$. Without loss of generality, denote

$EF^2=\{(\bar{X},-\bar{Y})^1,\dots,(\bar{X},-\bar{Y})^{m+s},(\bar{X},-\bar{Y})^{m+s+1},\dots,(\bar{X},-\bar{Y})^{k_2}\}, k_2>m+s$,

We provide an **algorithm** for finding all weak Pareto solutions of (2), in objective function space, as follows.

Step 0. Set $EF=EF^2$.

Step 1. We obtain cone C as following.

$C=\{(V,U,u_0) | V^T X-U^T Y\ge u_0, (V,U)\ge 0, (V,U)\ne 0, (X,-Y)\in EF\}$. Obtain the extreme rays of C and denote the rays by $W^l=(\dot{W},\ddot{W},w_0)^l, l=1,\dots,h$, and compose

$P=\{(X,-Y) | (\dot{W}^l)^T X-(\ddot{W}^l)^T Y\ge w_0^l, l=1,\dots,h, (X,Y)\in T_v\}$.

Step 2. For $l=1,\dots,h$, solve the weighted sum problem (5) associated with weight W^l. Let $(\bar{X},-\bar{Y})^l$ be an optimal solution of (5). Denote an index set $I=\{l | (\bar{X},-\bar{Y})^l$ does not belong to $P, l=1,\dots,h\}$.

Step 3. If $I=\emptyset$ then stop From theorems 4, 7and 8, we have determined all weak Pareto solutions in objective function space of problem (2). else go to step 4.

Step 4. Denote $\mathrm{EF} = \mathrm{EF} \cup \mathrm{I}$. Without loss of generality, denote

$\mathrm{EF}^2 = \{(\overline{\mathrm{X}}, -\overline{\mathrm{Y}})^1, \ldots, (\overline{\mathrm{X}}, -\overline{\mathrm{Y}})^k\}$, where $(\overline{\mathrm{X}}, -\overline{\mathrm{Y}})^j$, $j = 1, \ldots, k$, are the extreme points of outcome space of MOLP (2). Go to step 1.

Since the outcome space has a finite number of extreme points, this algorithm will finallyend after a finite number of steps.

According to the above algorithm, if we want to obtain the weak pareto solutions in objective function space of problem (2), in every step we will compose cone C and obtain its extreme rays we use them as the new weights of problem (5) to obtain the new weak pareto solutions.

Theorem 7. *Let* $(X, Y) \in T_v$ *and* $EF = \{(\bar{X}, -\bar{Y})^1, \ldots, (\bar{X}, -\bar{Y})^h\}$ *in the termination of the above algorithm. Then,* $P = \{(X, -Y) | (X, Y) \in T_v\} + E_+^{m+s}$.

Proof: Suppose $(X, Y) \in T_v$. At the end of the above algorithm, we have $(\bar{X}, -\bar{Y})^l \in P, l = 1, \ldots, h$. Regarding Theorem (4), $P = SE$, and therefore $(\bar{X}, -\bar{Y})^l \geq \sum_{q=1}^{k} \lambda_q (\hat{X}, -\hat{Y})^q$,

$\sum_{q=1}^{k} \lambda_q = 1, \lambda_q \geq 0, q = 1, \ldots, k$.

Since $(\widehat{\mathrm{X}}, -\widehat{\mathrm{Y}})^q \in \mathrm{SE}, q = 1, \ldots, k$, and $(\overline{\mathrm{X}}, -\overline{\mathrm{Y}})^l, l = 1, \ldots, h$, are the optimal solutions of problem (5) in objective function space corresponding to weight vectors $\mathrm{W}^l, l = 1, \ldots, h$, we have,

$(\mathrm{W}^l)^t(\mathrm{X}, -\mathrm{Y}) \geq (\mathrm{W}^l)^t(\overline{\mathrm{X}}, -\overline{\mathrm{Y}})^l \geq \sum_{q=1}^{k} \lambda_q (\mathrm{W}^l)^t(\widehat{\mathrm{X}}, -\widehat{\mathrm{Y}})^q \geq \sum_{q=1}^{k} \lambda_q \mathrm{w}_o^l = \mathrm{w}_o^l$. That is,

$(\mathrm{X}, -\mathrm{Y}) \in \mathrm{P}$. Now, suppose $(\mathrm{X}, -\mathrm{Y}) \in \mathrm{SE} = \mathrm{P}$. By the definition of SE, we have $(\mathrm{X}, -\mathrm{Y}) \geq \sum_{q=1}^{k} \lambda_q (\widehat{\mathrm{X}}, -\widehat{\mathrm{Y}})^q, \sum_{q=1}^{k} \lambda_q = 1, \lambda_q \geq 0, q = 1, \ldots, k$. Since $(\widehat{\mathrm{X}}, \widehat{\mathrm{Y}})^q \in \mathrm{T_v}$, $q = 1, \ldots, k$, and $\mathrm{T_v}$ is a convex set, $\sum_{q=1}^{k} \lambda_q (\widehat{\mathrm{X}}, \widehat{\mathrm{Y}})^q \in \mathrm{T_v}$ for $\sum_{q=1}^{k} \lambda_q = 1, \lambda_q \geq 0, q = 1, \ldots, k$. We put $\sum_{q=1}^{k} \lambda_q \hat{X}^q = \tilde{\mathrm{X}}$ and $\sum_{q=1}^{k} \lambda_q \hat{Y}^q = \tilde{\mathrm{Y}}$, for $\sum_{q=1}^{k} \lambda_q = 1, \lambda_q \geq 0, q = 1, \ldots, k$. Then $X \geq \tilde{X}$ and $Y \leq \tilde{Y}$ since $(\tilde{X}, \tilde{Y}) \in \mathrm{T_v}$, by definition of $\mathrm{T_v}$, we have $(X, Y) \in T_v$.

As for every $(\mathrm{X}, \mathrm{Y}) \in \mathrm{T_v}$ we have a corresponding point $(\mathrm{X}, -\mathrm{Y})$ in the objective space of problem (2), then there exists a injective correspondence between $(\mathrm{X}, \mathrm{Y}) \in \mathrm{T_v}$ and $(\mathrm{X}, -\mathrm{Y})$ of the objective function space of problem (3). Consequently, if $(\mathrm{X}, -\mathrm{Y}) \in \mathrm{P}$ then,

$(\mathrm{X}, -\mathrm{Y}) \in \{(\mathrm{X}, -\mathrm{Y}) | (\mathrm{X}, \mathrm{Y}) \in \mathrm{T_v}\} + \mathrm{E}_+^{m+s}$, and the proof is complete. ■

Theorem 8. Suppose we obtain all weights during the above algorithm as follows.With $\{W^1, \ldots, W^h\}$, we obtain the weak pareto solutions structure of problem (2) in objective function space as follows:

$$\overline{\mathrm{EF}} = \cup_{l=1}^{h} \{(\mathrm{X}, -\mathrm{Y}) | \sum_{i=1}^{m} \dot{\mathrm{w}}_i^l \mathrm{x}_i - \sum_{r=1}^{s} \ddot{\mathrm{w}}_r^l \mathrm{y}_r = \mathrm{w}_0^l,$$
$$(\mathrm{X}, \mathrm{Y}) = (\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_m, \mathrm{y}_1, \mathrm{y}_2, \ldots, \mathrm{y}_s) \in \mathrm{T_v}\}.$$

Proof: At first, suppose $(\mathrm{X}, \mathrm{Y}) \in \mathrm{T_v}$, Then, by Theorem 7, we have $(\mathrm{X}, -\mathrm{Y}) \in \mathrm{P} = \mathrm{SE}$ then $\sum_{i=1}^{m} \dot{\mathrm{w}}_i^l \mathrm{x}_i - \sum_{r=1}^{s} \ddot{\mathrm{w}}_r^l \mathrm{y}_r \geq \mathrm{w}_0^l$. Therefore, w_0^l is the minimum of the left-hand side of the above inequality.

If there is $(\overline{\mathrm{X}}, \overline{\mathrm{Y}}) \in \mathrm{T_v}$ such that $\sum_{i=1}^{m} \dot{\mathrm{w}}_i^l \bar{\mathrm{x}}_i - \sum_{r=1}^{s} \ddot{\mathrm{w}}_r^l \bar{\mathrm{y}}_r = \mathrm{w}_0^l$, for some $l = 1, \ldots, h$, Then we will have $(\overline{\mathrm{X}}, \overline{\mathrm{Y}})$ is an optimal solution of problem (4), if we select $(\mathrm{U}, \mathrm{V}) = (\dot{\mathrm{W}}, \ddot{\mathrm{W}})$. From Theorem 1 and 3, $(\overline{\mathrm{X}}, \overline{\mathrm{Y}})$ is an efficient unit in $\mathrm{T_v}$, thus $(\bar{X}, -\bar{Y})$ is a weak Pareto solution in objective function space of problem (2).

Now, suppose $(\overline{\mathrm{X}}, -\overline{\mathrm{Y}})$ is an efficient point in outcome space of MOLP (2). From Theorem 7, $(\overline{\mathrm{X}}, -\overline{\mathrm{Y}})$ is the optimal solution of the following problem,

$$\begin{aligned} \min \quad & (X, -Y) \\ \text{s.t.} \quad & (X, Y) \in P. \end{aligned} \qquad (6)$$

By contradiction, suppose $(\overline{X}, -\overline{Y})$ is not in
$\cup_{l=1}^{h} \{(X, -Y) | \sum_{i=1}^{m} \dot{w}_i^l x_i - \sum_{r=1}^{s} \ddot{w}_r^l y_r = w_0^l, (x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_r) \in T_v\}$.
Since $(\overline{X}, -\overline{Y}) \in P$ then $\sum_{i=1}^{m} \dot{w}_i^l \overline{x}_i - \sum_{r=1}^{s} \ddot{w}_r^l \overline{y}_r \geq w_0^l, l = 1, \dots, h$, since $(\overline{X}, -\overline{Y})$ dose no belong to
$\cup_{l=1}^{h} \{(X, -Y) | \sum_{i=1}^{m} \dot{w}_i^l x_i - \sum_{r=1}^{s} \ddot{w}_r^l y_r = w_0^l, (x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_r) \in T_v\}$,
Therefore $\sum_{i=1}^{m} \dot{w}_i^l \overline{x}_i - \sum_{r=1}^{s} \ddot{w}_r^l \overline{y}_r > w_0^l, l = 1, \dots, h$, which shows that $(\overline{X}, -\overline{Y})$ is not a weak Pareto solution of (6), and this is a contradiction. Therefore,
$(\overline{X}, -\overline{Y}) \in \cup_{l=1}^{h} \{(X, -Y) | \sum_{i=1}^{m} \dot{w}_i^l x_i - \sum_{r=1}^{s} \ddot{w}_r^l y_r = w_0^l,$
$(x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_s) \in T_v\}$ and Now, the proof is completed. ■

Suppose $(\overline{X}, -\overline{Y}) \in \overline{EF}$, Denote $J = \{l | \sum_{i=1}^{m} \dot{w}_i^l \overline{x}_i - \sum_{r=1}^{s} \ddot{w}_r^l \overline{y}_r = w_0^l, l = 1, \dots, h\}$.

Theorem 9. *Suppose $(\bar{X}, -\bar{Y}) \in \overline{EF}$ then $(\bar{X}, -\bar{Y})$ is a Paerto solution in objective function space of problem (2) if and only if $\sum_{l \in J} (\dot{w}^l, \ddot{w}^l) > 0$.*

Proof: For $l \in J$, we have $\sum_{i=1}^{m} \dot{w}_i^l \overline{x}_i - \sum_{r=1}^{s} \ddot{w}_r^l \overline{y}_r = w_0^l$ and for $l \in \{1, \dots, h\} - J$, $\sum_{i=1}^{m} \dot{w}_i^l \overline{x}_i - \sum_{r=1}^{s} \ddot{w}_r^l \overline{y}_r > w_0^l$. Assume $(\overline{X}, -\overline{Y})$ is a Paerto solution in objective function space of problem (2) but $\sum_{l \in J} (\dot{w}^l, \ddot{w}^l) > 0$ is not true. Since for $l \in \{1, \dots, h\}$, we have $(\dot{w}^l, \ddot{w}^l) \geq 0$, thus there exist j_0 such that the j_0th component of $\sum_{l \in J} (\dot{w}^l, \ddot{w}^l)$ is zero. Without losing generality, suppose, for each $l \in J$, we have $(\dot{w}_{j_0}^l) = 0$. If $J = \{1, \dots, h\}$, let δ be a positive number (for example $\delta = 1$), else let

$$\delta = \min\{\frac{\sum_{i=1}^{m} \dot{w}_i^l \overline{x}_i - \sum_{r=1}^{s} \ddot{w}_r^l \overline{y}_r - w_0^l}{\dot{w}_{j_0}^l} | \dot{w}_{j_0}^l > 0, l \in \{1, \dots, h\} - J\}.$$

Denote $(\widetilde{X}, -\widetilde{Y}) = (\overline{x}_1, \dots, \overline{x}_{j_0-1}, \overline{x}_{j_0} - \delta, , \overline{x}_{j_0+1}, \dots, \overline{x}_m, -\overline{y}_1, -\overline{y}_2, \dots, -\overline{y}_s)$, then
$\sum_{i=1}^{m} \dot{w}_i^l \tilde{x}_i - \sum_{r=1}^{s} \ddot{w}_r^l \tilde{y}_r \geq w_0^l, l = 1, \dots, h$ thus $(\widetilde{X}, -\widetilde{Y}) \in P$ from Theorem (7) $(\widetilde{X}, -\widetilde{Y})$ belongs to objective function space of problem (2).

By attention to above relation, we have $(\overline{X}, -\overline{Y}) > (\widetilde{X}, -\widetilde{Y})$, $(\overline{X}, -\overline{Y}) \neq (\widetilde{X}, -\widetilde{Y})$. This contradicts with the assumption that $(\overline{X}, -\overline{Y})$ is a Paerto solution in objective function space of problem (2), then $\sum_{l \in J} (\dot{w}^l, \ddot{w}^l) > 0$.

On the other hand, if $\sum_{l \in J} (\dot{w}^l, \ddot{w}^l) > 0$, for $l \in J$ we have $\sum_{i=1}^{m} \dot{w}_i^l \overline{x}_i - \sum_{r=1}^{s} \ddot{w}_r^l \overline{y}_r = w_0^l$ but, for each $(X, Y) \in T_v$, from Theorem (7) we have $\sum_{i=1}^{m} \dot{w}_i^l x_i - \sum_{r=1}^{s} \ddot{w}_r^l y_r \geq w_0^l$.

Therefore, for each $(X, -Y)$, that belongs belong to objective function space of problem (2) we have $(X, -Y)(\sum_{j \in J} (\dot{w}^l, \ddot{w}^l))^T \geq \sum_{l \in J} w_0^l = (\overline{X}, -\overline{Y})(\sum_{l \in J} (\dot{w}^l, \ddot{w}^l))^T$.

This indicates that $(\overline{X}, -\overline{Y})$ is an optimal solution of linear programing following.

$$\begin{aligned} \min \quad & (X, -Y)(\textstyle\sum_{l \in J} (\dot{w}^l, \ddot{w}^l))^T \\ \text{s.t.} \quad & (X, Y) \in T_v \end{aligned} \qquad (7)$$

Note, $(\sum_{l \in J} (\dot{w}^l, \ddot{w}^l)) > 0$, by Theorem (3) $(\overline{X}, -\overline{Y})$ is a Paerto solution in objective function space of problem (2). ■

4. Numerical example

In this section, we illustrate the problem by a numerical example. Consider the case where there are seven units with two input and one output, with details as given in Table1.

Table 1. The data of the eight DMUs.

DMU	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5	DMU_6	DMU_7	DMU_8
Input 1	4	7	8	4	2	10	12	10
Input 2	3	3	1	2	4	1	1	1.5
Output	2	4	7	5	2	5	8	7

The proposed model (2) for the data in Table (1) is summarized as follows:

$$\begin{aligned} \min\ & \{x_1, x_2, -y_1\} \\ \text{s.t}\ & 4\lambda_1 + 7\lambda_2 + 8\lambda_3 + 4\lambda_4 + 2\lambda_5 + 10\lambda_6 + 12\lambda_7 + 10\lambda_8 + s_1^- = x_1 \\ & 3\lambda_1 + 3\lambda_2 + \lambda_3 + 2\lambda_4 + 4\lambda_5 + \lambda_6 + \lambda_7 + 1.5\lambda_8 + s_2^- = x_2 \\ & 2\lambda_1 + 4\lambda_2 + 7\lambda_3 + 5\lambda_4 + 2\lambda_5 + 5\lambda_6 + 8\lambda_7 + 7\lambda_8 - s_1^+ = y_1 \\ & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 = 1 \\ & s_1^- \geq 0, s_2^- \geq 0, s^+ \geq 0, x_1 \geq 0, x_2 \geq 0, y_1 \geq 0, \lambda_j \geq 0, j = 1, \dots, 7. \quad (9) \end{aligned}$$

First, we obtain $\min\{x_{1j} | 1 \leq j \leq 7\} = x_{15} = 2$, and $\min\{x_{2j} | 1 \leq j \leq 7\} = x_{23} = 1$, $\max\{y_{1j} | 1 \leq j \leq 7\} = y_{17} = 8$.

Therefore, $(X^*, -Y^*)^1 = (2,4,-2)$, $(X^*, -Y^*)^2 = (8,1,-7)$, $(X^*, -Y^*)^3 = (12,1,-8)$ and $EF^1 = \{(2,4,-2), (8,1,-7), (12,1,-8)\}$. Let $C^1 = \{(v_1, v_2, u_1, u_0) | 2v_1 + 4v_2 - 2u_1 \geq u_0, 8v_1 + v_2 - 7u_1 \geq u_0, 12v_1 + v_2 - 8u_1 \geq u_0, (v_1, v_2, u_1) \geq 0, (v_1, v_2, u_1) \neq 0\}$

We obtain all extreme rays of C^1 as follows:

$W^1 = (\dot{W}_1, \dot{W}_2, \ddot{W}_1, w_0)^1 = (1,0,0,0)$, $W^2 = (\dot{W}_1, \dot{W}_2, \ddot{W}_1, w_0)^2 = (0,1,0,0)$

$W^3 = (\dot{W}_1, \dot{W}_2, \ddot{W}_1, w_0)^3 = (0,0,1,-8)$, $W^4 = (\dot{W}_1, \dot{W}_2, \ddot{W}_1, w_0)^4 = (1,1,1,2)$.

Using these weights to solve the weighted sum problem (5), the optimal solutions corresponding them are as follows, respectively:

$(\overline{X}, -\overline{Y})^1 = (2,4,-2)$, $(\overline{X}, -\overline{Y})^2 = (8,1,-7)$, $(\overline{X}, -\overline{Y})^3 = (12,1,-8)$, $(\overline{X}, -\overline{Y})^4 = (4,2,-5)$.

$P^1 = \{(x_1, x_2, -y_1) | x_1 \geq 0, x_2 \geq 0, -y_1 \geq -8, x_1 + x_2 - y_1 \geq 2, (x_1, x_2, y_1) \in T_v\}$

Since $(4,2,-5)$ is not in P^1 then $I_1 = \{4\} \neq \emptyset$.

Now, we have $EF^2 = \{(2,4,-2), (8,1,-7), (12,1,-8), (4,2,-5)\}$.

In step 1 of the algorithm, we put $EF = EF^2 = \{(2,4,-2), (8,1,-7), (12,1,-8), (4,2,-5)\}$.

$$C = \{(v_1, v_2, u_1, u_0) | 2v_1 + 4v_2 - 2u_1 \geq u_0, 8v_1 + v_2 - 7u_1 \geq u_0, 12v_1 + v_2 - 8u_1 \geq u_0, 4v_1 + v_2 - 5u_1 \geq u_0, (v_1, v_2, u_1) \geq 0, (v_1, v_2, u_1) \neq 0\}$$

We obtain all extreme rays of C as follows:

$W^1 = (\dot{W}_1, \dot{W}_2, \ddot{W}_1, w_0)^1 = (1,0,0,0)$, $W^2 = (\dot{W}_1, \dot{W}_2, \ddot{W}_1, w_0)^2 = (0,1,0,1)$,

$W^3 = (\dot{W}_1, \dot{W}_2, \ddot{W}_1, w_0)^3 = (2.5,0,5,-15)$, $W^4 = (\dot{W}_1, \dot{W}_2, \ddot{W}_1, w_0)^4 = (15,0,10,10)$, $W^5 = (\dot{W}_1, \dot{W}_2, \ddot{W}_1, w_0)^5 = (1.25,0,5,-25)$.

Using these weights to solve the weighted sum problem (5), the optimal solutions corresponding them are as follows, respectively:

$(\overline{X}, -\overline{Y})^1 = (2,4,-2)$, $(\overline{X}, -\overline{Y})^2 = (10,1,-5)$, $(\overline{X}, -\overline{Y})^3 = (8,1,-7)$,

$(\overline{X}, -\overline{Y})^4 = (2,4,-2)$, $(\overline{X}, -\overline{Y})^5 = (4,2,-5)$.

$$P^1 = \{(x_1, x_2, -y_1) | x_1 \geq 0, x_2 \geq 1, 2.5x_1 - 5y_1 \geq -15, 15x_1 - 10y_1 \geq 10, 1.25x_1 - 5y_1 \geq -25, (x_1, x_2, y_1) \in T_v\}$$

Since all points are in P, then $I = \emptyset$, and the algorithm terminates. So,

$EF = \{(2,4,-2), (8,1,-7), (12,1,-8), (4,2,-5)\}$.

$$\overline{\mathrm{EF}} = \{(x_1, x_2, -y_1) | x_2 = 1, (x_1, x_2, y_1) \in T_v\} \cup \{(x_1, x_2, -y_1) | 2.5x_1 - 5y_1 = 15, (x_1, x_2, y_1) \in T_v\} \cup \{(x_1, x_2, -y_1) | 15x_1 - 10y_1 = 10, (x_1, x_2, y_1) \in T_v\} \cup \{(x_1, x_2, -y_1) | 1.25x_1 - 5y_1 = -25, (x_1, x_2, y_1) \in T_v\}$$

Since $W^2 + W^4 > 0$, $W^2 + W^3 > 0$, $W^2 + W^4 + W^2 > 0$, $W^1 + W^1 + W^3 > 0$ According to Theorem (8) thus the corresponding points to these weights are pareto efficient. Therefore (2,4,-2),(8,1,-7),(4,2,-5), (12,1,-8) are pareto efficient in objective function space of problem (2).

We obtain the vertex set of T_v by converting $(x_1, x_2, -y_1)$ to (x_1, x_2, y_1) as follows: $\{(2,4,2), (8,1,7), (12,1,8), (4,2,5)\}$.

5. Conclusion

In data envelopment analysis, programming problems corresponding to DMU. Are applied. investigated the structure of weak Pareto solutions via solving an MOLP problem. We showed that by choosing weights properly and solving the weighted sum problems of the MOLP associated with these weights, all weak Pareto solutions and Pareto solutions of the MOLP problem were obtained. The method showed that weak Pareto solutions and Pareto solutions could be termined by solving only a finite number of linear weighted sum problems. If the number of inputs and outputs are smaller than the DMUs, the the method will be useful. If the weights are chosen suitably, it can help the convergence of the method. We can use the proposed method for obtaining benchmarks and other elements in DEA. Here we established a relation between DEA and multiobjective linear programming and showed how a DEA problem could be solved by an MOLP formulation. This provides a basis for applying techniques of MOLP to solve DEA problems.

References

[1] Agrell P.J., Tind J. (2001), A dual approach to nonconvex frontier models. *Productivity Analysis*; 16, 129-147.

[2] Belton V. T., Stewart J. (2001), Multiple Criteria Decision Analysis: An Integrated Approach. *Kluwer Academic Publishers*.

[3] Belton V.,Vickers S.P. (1993), Demystifying DEA – A visual interactive approach based on multiple criteria analysis. *Operational Research Society*; 44, 883-896.

[4] Charnes A, Cooper, W.W., Rhodes E. (1978), Measuring efficiency of decision making units. *European Operational Research;* 2, 429-444.

[5] Charnes A, Cooper W.W., Huang Z.M.and Sun D.B. (1991), Relations between half-space and finitely generated cones in polyhedral cone-rate DEA models. *International Journal Systems Science*; 22, 2057-2077.

[6] Doyle J., Green R. (1993), Data envelopment analysis and multiple criteria decision making; *Omega;* 21, 713-715.

[7] Gravel M, Martel J.M, Nadeau R, Price W., Tremblay R. (1992), A multicriterion view of optimal resource allocation in job–shop production. *European Journal Operational Research*; 61, 230-244.

[8] Joro R, Korhonen P.,Wallenius J. (1998), Structural comparison of data envelopment analysis and multiple objective linear programming. *Management Science*; 44, 962-970.

[9] Leschine T.M, Wallenius H., Verdini W.A. (1992), Interactive multiobjective analysis and assimilative capacitybased ocean disposal decision. *European Journal Operational Research*; 56, 278-289.

[10] Prabuddha D, Ghosh J.B., Wells C.E. (1992), On the minimization of completion time variance with a bicriteria extension. *Operational Research*; 40, 1148-1155.

[11] Wei Q.L.,Yan H. (1997), An algebra-based vertex identification process on polytopes. Beijing Math; 3, 40-48.

[12] Wei Q.L., Yan H. (2000), A method of transferring cones of intersection form to cones of sum form and its applications in data envelopment analysis models. *International Journal*

Systems Science; 31, 629-638.
[13] Stewart T.J. (1994), Data envelopment analysis and multiple criteria decision making: A response. *International Journal of Management Science*; 22, (2) 205-206.
[14] Stewart T.J. (1996), Relationships between data envelopment analysis and multicriteria decision analysis. *Operational Research Society*; 47, (5) 654-665.
[15] Yan H., Wei Q., Wang J. (2005), Constructing efficient solutions structure of multiobjective linear programming, *Journal of Mathematical Analysis and Applications*; 307, 504-523.
[16] Zeleny M. (1974), Linear Multiobjective Programming,Springer Verlag,Berlin New York.