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A Path-Following Infeasible Interior-Point Algorithm for Semidefinite Programming

H. Mansouri[1,*], T. Siyavash[2], M. Zangiabadi[3]

We present a new algorithm obtained by changing the search directions in the algorithm given in [8]. *This algorithm is based on a new technique for finding the search direction and the strategy of the central path. At each iteration, we use only the full Nesterov-Todd (NT)step. Moreover, we obtain the currently best known iteration bound for the infeasible interior-point algorithms with full NT steps, namely* $O\left(n\log\frac{n}{\varepsilon}\right)$*, which is as good as the linear analogue.*

Keywords: *Infeasible interior-point algorithm, Central path, Semidefinte programming, Polynomial complexity.*

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1. Introduction

We are concerned with the semidefinite programming (SDP) problem given in the following standard form:

$$\min_{X\succeq 0}\left\{Tr(CX): Tr(A_iX)=b_i,\ i=1,\cdots,m\right\}, \qquad (P)$$

and its associated dual problem:

$$\max_{S\succeq 0}\left\{b^Ty: \sum_{i=1}^m y_iA_i+S=C\right\}, \qquad (D)$$

where $b,y\in R^m$ and each A_i , $i=1,\cdots,m$ and C are symmetric matrices, i.e., $A_i ,C\in S^n$. Furthermore, $X\succeq(X\succ 0)$ means that X is a positive semidefinite (positive definite) matrix. Without loss of generality, we assume that matrices A_i are linearly independent. The SDP problem has wide applications in continuous and combinatorial optimization [1, 14]. In the past decade, SDP has become a popular research area in mathematical programming when it became clear that the algorithm for linear optimization (LO) can often be extended to the more general SDP case. Several interior-point methods (IPMs) designed for (LO) have been successfully extended to SDP [11, 15]. Many researchers have studied SDP and obtained substantiona results. For an overview of these results, see [9, 14].

Here, we present a full-Newton step infeasible interior-point algorithm for solving the SDP problem and prove that the complexity of the algorithm coincides with the best known iteration bound for infeasible IPMs.

*Corressponding Author.
[1] Department of Applied Mathematics, Shahrekord University, P.O. Box 115, Shahrekord, Iran. E-mail: mansouri@sci.sku.ac.ir
[2] Department of Applied Mathematics, Shahrekord University, P.O. Box 115, Shahrekord, Iran.
[3] Department of Applied Mathematics, Shahrekord University, P.O. Box 115, Shahrekord, Iran. E-mail: zangiabadi_m@sci.sku.ac.ir

The following notations are used throughout the paper. R^n, R_+^n and R_{++}^n denote the set of vectors with n components, the set of nonnegative vectors and set of positive vectors, respectively, $\|.\|$ denotes the Frobenius norm for matrices, and the 2-norm for vectors, $R^{m\times n}$ is the space of all $m\times n$ matrices, S^n, S_+^n and S_{++}^n denote the cone of symmetric, symmetric positive semidefinite and symmetric positive definite $n\times n$ matrices, respectively, I denotes the $n\times n$ identity matrix, P and D denote the feasible sets of the primal and dual problems, respectively. We use the classical Löwner partial order $\succeq (\succ)$ for symmetric matrices meaning that $A-B$ is positive semidefinite (positive definite). The matrix inner product is defined by $A \bullet B = Tr(A^T B)$. For any symmetric positive definite matrix $Q \in S_{++}^n$, the expression $Q^{1/2}$ denotes the symmetric square root of Q. We denote the diagonal matrix Λ with entries λ_i by $diag(\lambda_i)$. For any $V \in S_{++}^n$, we denote $\lambda(V)$ to bethe vector of eigenvalues of V arranged in non-increasing order, that is, $\lambda_1(V) \ge \lambda_2(V) \ge \cdots \ge \lambda_n(V)$. For any matrix M, we denote $\sigma_1(M) \ge \sigma_2(M) \ge \cdots \sigma_n(M)$, the singular values of M. Specially, if M is symmetric, then one has $\sigma_i(M) = |\lambda_i(M)|, i = 1,\cdots,n$. Finally, if $g(x) \ge 0$ is a real valued function of a real nonnegative variable, then the notation $g(x) = O(x)$ means that $g(x) \le \bar{c}x$ for some positive constant $\bar{c}$ and $g(x) = \Theta(x)$ means that $c_1 x \le g(x) \le c_2 x$ for two positive constants c_1 and c_1.

2. Preliminaries on Matrices and Matrix Functions

We recall some concepts from linear algebra. For more details, see [5].

Lemma 2.1.

1) $Tr(A) = \sum_{i=1}^{n} \lambda_i(A)$, where $\lambda_i(A)$ is the i th eigenvalue of the matrix A ,
2) $Tr(A) = Tr(A^T)$,
3) $Tr(AB) = Tr(BA)$,
4) $Tr(A+B) = Tr(A) + Tr(B)$.

Theorem 2.2. (Spectral Theorem for Symmetric Matrix [14]).

The real $n\times n$ matrix A is symmetric if and only if there exists a matrix $Q \in R^{n\times n}$ such that $QQ^T = 1$ and $Q^T A Q = \Lambda$ where Λ is a diagonal matrix.

Throughout the paper, we assume that $\psi(t)$ is a real valued function on $[0,\infty)$ and differentiable on $(0,\infty)$ such that $y(t) > 0$ for all $t > 0$. Now, we are ready to show a matrix function obtained from $y(t)$.

Definition 1 Let $V \in S_+^n$ and

$$v = Q^T diag\left(l_1(V), L, l_n(V)\right) Q,$$

where Q is any orthogonal matrix $Q^T = Q^{-1}$ that diagonalizes V . The matrix valued function

$\psi(\mathrm{v})$ is defined by

$$y(v) = Q^T diag\left(y(l_1(V)), L, y(l_n(V))\right)Q. \tag{1}$$

It should be noted that the matrix Q is not unique, but $\psi(\mathrm{v})$ is well defined on the eigenvalues of V [11]. Furthermore, replacing $\psi(\lambda_i(V))$ in (1) by $\psi'(\lambda_i(V))$, we obtain the matrix function $y'(V)$ as follws:

$$y'(V) = Q^T diag\left(y'(l_1(V)), L, y'(l_n(V))\right)Q. \tag{2}$$

Two matrices A and B are called similar (abbreviated $A \sim B$) if $\mathrm{A} = \mathrm{PBP}^{-1}$ for some invertible matrix P and, moreover, if A and B are symmetric then this happen if and only if A and B have the same eigenvalues [5].

Lemma 2.3. (Lemma 2.6 in [13])

Let $A, B \in S^n$, and $AB = BA$. Then

$$\lambda_i(A+B) = \lambda_i(A) + \lambda_i(B),\ i = 1, \cdots, n.$$

Furthermore, if $|\lambda_i(B)|$ is small enough, then we have

$$\psi(A+B) \approx \psi(A) + \psi'(A)B.$$

3. Perturbed Problems and Central Path

As usual for infeasible interior-point methods (IIPMs), we assume that the initial iterates (X^0, y^0, S^0) are as follows:

$$X^0 = S^0 = \zeta I, y^0 = 0,\ \mu^0 = \zeta^2, \tag{3}$$

where I is the $n \times n$ identity matrix, μ^0 is the initial duality gap and a positive scaler ζ such that $X^* + S^* \leq \zeta I$, for some optimal solution (X^*, y^*, S^*) of (P) and (D) such that $Tr(X^* S^*) = 0$. It is generally agreed that the total number of inner iterations required by the algorithm is an appropriate measure for its efficiency and this number is referred to as the iteration complexity of the algorithm. Using $Tr(X^0 S^0) = n\zeta^2$, the total number of iterations in the algorithm of Mansouri and Roos [8] is bounded above by

$$O\left(n \log \frac{\max\left\{n\zeta^2, \left\| r_b^0 \right\|, \left\| R_c^0 \right\|\right\}}{\varepsilon}\right), \tag{4}$$

where r_b^0 and R_c^0 are initial values of the primal and dual residuals, respectively:

$$(r_b^0)_i = b_i - Tr(A_i X^0),\ i = 1, \cdots, m, \tag{5}$$

$$R_c^0 = c - \sum_{i=1}^{m} y_i^0 A_i - S^0. \tag{6}$$

Up to a constant factor, the iteration bound (4) was first obtained by Kojima et al. [7] and Potra and Sheng [12] and it is still the best-known iteration bound for infeasible IPMs. Now, we recall the main idea underlying the algorithm in [8]. For any ν with $0 < \nu \leq 1$, we consider the perturbed problem (P_ν), defined by

$$\min\left\{\mathrm{Tr}\left(CX - \nu R_c^0 X\right) : Tr(A_i X) = b_i - \nu(r_b^0)_i,\ i = 1,\cdots,m\ \ X \succeq 0\right\}, \qquad (P_\nu)$$

and its dual problem (D_ν) which is given by

$$\max\left\{\sum_{i=1}^{m}\left(b_i - \nu(r_b^0)_i\right)y_i : \sum_{i=1}^{m} y_i A_i + S = \mathrm{C} - \nu R_c^0,\ S \succeq 0\right\}. \qquad (D_\nu)$$

Note that if $\nu = 1$, then $X = X^0$ yields a strictly feasible solution of (P_ν) and (y, S) a strictly feasible solution of (D_ν). Due to the choice of initial iterates, we may conclude that if $\nu = 1$, then (P_ν) and (D_ν) each have a strictly feasible solution, which then means that both perturbed problems satisfy the well-known interior-point condition (IPC).

Lemma 3.1. (Lemma 4.1 in [8])

Let the original problems (P) and (D) be feasible. Then, for each ν, $0 < \nu \le 1$, the perturbed problems (P_ν) and (D_ν) satisfy the IPC.

Here we assume that the problems (*P*) and (*D*) are feasible, It follows from lemma 3.1 that the problems (P_ν) and (D_ν) satisfy the IPC, for each $\nu \in (0,1]$. Hence, their central path exists. This means that the system

$$\begin{aligned} b_i - Tr(A_i X) &= \nu(r_b^0)_i,\ i = 1,\cdots,m,\quad X \succeq 0, \\ C - \sum_{i=1}^{m} y_i A_i - S &= \nu R_c^0,\ \ S \succeq 0, \\ XS &= \mu I, \end{aligned} \qquad (7)$$

has a unique solution, for every $\mu > 0$.

In the sequel, this unique solution is denoted by $(\mathrm{X}(\mu,\nu), \mathrm{y}(\mu,\nu), \mathrm{S}(\mu,\nu))$. These are the μ - centers of the perturbed problems (P_ν) and (D_ν).

Note that since $X^0 S^0 = \mu^0 I$, X^0 is μ^0 -center of the perturbed problem (P_1) and (y^0, S^0) is the μ^0 -center of (D_1).

On the other words, $(X(\mu^0,1), y(\mu^0,1), S(\mu^0,1)) = (X^0, y^0, S^0)$. In the sequel, we will always have $\mu = \nu\mu^0$ and we will accordingly denote $(\mathrm{X}(\mu,\nu), \mathrm{y}(\mu,\nu), \mathrm{S}(\mu,\nu))$ simply as

$$(\mathrm{X}(\nu), \mathrm{y}(\nu), \mathrm{S}(\nu)).$$

The set of μ centers (with μ running all positive real numbers) gives a homotopy path, which is called the central path of (P) and (D). If $\mu \to 0$, then the limit of the central path exists and since the limit points satisfy the complementarity condition, the limit yields an ε- approximate solution for (P) and (D) [6, 14].

In [4], Darvay presented a new technique for finding a class of search direction for LO. He replaced the standard centering equation $XS = \mu e$ by $\psi(\frac{xs}{\mu}) = \psi(e)$, where $\psi(.)$ is the vector function induced by the function $\psi(t)$, and then applies Newton's method to obtain the search directions. Similar to the LO case, we replace the standard centering equation $\mathrm{XS} = \mu \mathrm{I}$ by $\psi(\frac{XS}{\mu}) = \psi(I)$. Then, the system (7) can be written as:

$$\begin{gathered} b_i - Tr(A_i X) = \nu (r_b^0)_i,\ i = 1, \cdots, m, \quad X \succeq 0, \\ C - \sum_{i=1}^{m} y_i A_i - S = \nu R_c^0, \quad S \succeq 0, \\ \psi\left(\frac{XS}{\mu}\right) = \psi(I). \end{gathered} \tag{8}$$

We, we use the above system to get the central path.

4. Centering Steps

Let (X, y, S) be a strictly feasible solution for (P_ν) and (D_ν). Applying Newton's method to system (8), we obtain the following system for the search directions ΔX , Δy and ΔS :

$$\begin{gathered} Tr(A_i(X + \Delta X)) = b_i - \nu (r_b^0)_i,\ i = 1, \cdots, m, \ X \succeq 0, \\ \sum_{i=1}^{m} (y_i + \Delta y_i) A_i + (S + \Delta S) = C - \nu R_c^0,\ S \succeq 0, \\ \psi\left(\frac{(X + \Delta X)(S + \Delta S)}{\mu}\right) = \psi(I). \end{gathered} \tag{9}$$

The third equation of the system (9) is equivalent to:

$$\psi\left(\frac{XS}{\mu} + \frac{X\,\Delta S + \Delta X S + \Delta S\,\Delta X}{\mu}\right) = \psi(I). \tag{10}$$

Applying Lemma 2.3 and neglecting the term $\Delta S\,\Delta X$, the equation (10) can be rewritten as

$$\psi\left(\frac{XS}{\mu}\right) + \psi'\left(\frac{XS}{\mu}\right)\left(\frac{X\,\Delta S + \Delta X S}{\mu}\right) = \psi(I). \tag{11}$$

Then we consider the following system:

$$\begin{gathered} Tr(A_i \Delta X) = 0,\ i = 1, \cdots, m, \\ \sum_{i=1}^{m} \Delta y_i A_i + \Delta S = 0, \\ \Delta X + X \Delta S S^{-1} = \mu \left(\psi'\left(\frac{XS}{\mu}\right)\right)^{-1} \left(\psi(I) - \psi\left(\frac{XS}{\mu}\right)\right) S^{-1}. \end{gathered} \tag{12}$$

It is obvious that ΔS is symmetric due to the second equation in (12). However, a crucial observation is that ΔX is not necessarily symmetric because $X\,\Delta S S^{-1}$ may not be symmetric [14]. Several ways exist for symmetrizing the third equation in the Newton system such that the resulting new system has a unique symmetric solution [6, 9, 11]. Here, we consider the Nesterov-Todd (NT)-symmetrization scheme in [9]. Let us define

$$P := X^{\frac{1}{2}}\left(X^{\frac{1}{2}}SX^{\frac{1}{2}}\right)^{\frac{-1}{2}}X^{\frac{1}{2}} = S^{\frac{-1}{2}}\left(S^{\frac{1}{2}}XS^{\frac{1}{2}}\right)^{\frac{1}{2}}S^{\frac{-1}{2}}, \tag{13}$$

which is a symmetric nonsingular matrix. We replace the term $X\,\Delta SS^{-1}$ in the third equation of (12) by $P\Delta SP^{T}$. The system (12) becomes

$$\begin{aligned} &Tr(A_i\Delta X)=0,\ i=1,\cdots,m,\\ &\sum_{i=1}^{m}\Delta y_i A_i+\Delta S=0,\\ &\Delta X+P\Delta SP^{T}=\mu\left(\psi'\left(\frac{XS}{\mu}\right)\right)^{-1}\left(\psi(I)-\psi\left(\frac{XS}{\mu}\right)\right)S^{-1}. \end{aligned} \tag{14}$$

Furthermore, we define $D=P^{\frac{-1}{2}}$. The matrix D can be used to scale X and S to the same matrix V, because

$$V := \frac{1}{\sqrt{\mu}}D^{-1}XD^{-1} = \frac{1}{\sqrt{\mu}}DSD. \tag{15}$$

Note that the matrices D and V are symmetric and positive definite. By using (15), we have

$$\mathrm{V}^2 = \frac{1}{\mu}D^{-1}XSD. \tag{16}$$

From Definition 1, we obtain

$$\psi\left(\frac{XS}{\mu}\right) = D\psi\left(V^2\right)D^{-1} \quad \text{and} \quad \psi'\left(\frac{XS}{\mu}\right) = D\psi'\left(V^2\right)D^{-1}. \tag{17}$$

Let us further define

$$\bar{A}_i := \frac{1}{\sqrt{\mu}}DA_iD,\ i=1,\cdots,m,\ \ D_X := \frac{1}{\mu}D^{-1}\Delta XD^{-1},\ \ D_S := \frac{1}{\mu}D\Delta SD. \tag{18}$$

Then it follows from (18) that the scaled NT search direction $(D_X, D_{\mathrm{y}}, D_S)$ is defined by the following system

$$\begin{aligned} &Tr(\bar{A}_i D_X)=0,\ i=1,\cdots,m,\\ &\sum_{i=1}^{m}\Delta y_i\bar{A}_i+D_S=0,\\ &D_X+D_S=P_V. \end{aligned} \tag{19}$$

Where,

$$P_V = \mu D^{-1}\left(D\psi'\left(V^2\right)D^{-1}\right)^{-1}\left(\psi(I)-D\psi\left(V^2\right)D^{-1}\right)S^{-1}D^{-1}. \tag{20}$$

Recently, Peng et al. [10] introduced a class of search directions based on self-regular kernel functions and Bai and Roos. [2] also defined a class of new search direction by using the so-called eligible kernel functions. The general approach in this paper can be particularized in such a way as to obtain the directions defined in [3] only by a constant multiplier, such as $\psi(t)=t$ yielding $P_V = V^{-1} - V$ which gives the classical search direction. The classical search direction has been studied by many researchers [1, 6, 9]. Here we restrict the analysis to the case where $\psi(t)=\sqrt{t}$, This yields

$$P_V = 2(I - V). \tag{21}$$

Furthermore, we have

$$V^2 + V P_V = V^2 + 2V(I-V) = I-(I-V)^2 = I - \frac{P_V^2}{4}. \tag{22}$$

For the analysis of the algorithm, we define a norm-based proximity measure $\delta(X,S;\mu)$ as follows:

$$\delta(V) := \delta(X,S;\mu) = \frac{\|P_V\|}{2} = \|I-V\|. \tag{23}$$

Due to the first and second equations of the system (19), D_X and D_S are orthogonal. One can easily verify that

$$\delta(V)=0 \Leftrightarrow V=I \Leftrightarrow D_X = D_S = 0 \Leftrightarrow XS = \mu I. \tag{24}$$

Hence, the value of $\delta(V)$ can be considered as a measure for the distance between the given pair (X,y,S) and the μ- center $(X(\mu), y(\mu), S(\mu))$.

The new search directions D_X and D_S are obtained by solving (19) with $P_V = 2(I - V)$ such that ΔX and ΔS are computed via (18). If $(X,y,S) \neq (X(\mu), y(\mu), S(\mu))$ then ΔX , Δy , ΔS are nonzero. One can construct a new full-Newton triple according to

$$X^+ = X + \Delta X,\ y^+ = y + \Delta y,\ S^+ = S + \Delta S, \tag{25}$$

where ΔX , Δy, and ΔS are called the centering steps. For the analysis of the algorithm, we introduce the notation

$$Q_V = D_X - D_S. \tag{26}$$

We cite two useful lemmas in [6], which will be used in the proof of Theorem 7.1 in Section 7.

Lemma 4.1. (Lemma 6.1 in [6])

Suppose that $X \succ 0$ and $S \succ 0$. If one has

$$\det(X(\alpha)S(\alpha)) > 0, \quad \forall\ 0 \le \alpha \le \bar{\alpha},$$

then $X(\bar{\alpha}) \succ 0$ and $S(\bar{\alpha}) \succ 0$.

Lemma 4.2. (Lemma 6.3 in [6])

Suppose that $Q \in S_{++}^n$ and $M \in \mathbb{R}^{n\times n}$ be skew-symmetric, i.e. $M = -M^T$. One has $\det(Q+M) > 0$. Moreover, if $\lambda_i(Q+M) \in R(i=1,\cdots,n)$, then

$$0 < \lambda_{\min}(Q) \le \lambda_{\min}(Q+M) \le \lambda_{\max}(Q+M) \le \lambda_{\max}(Q),$$

Lemma 4.3.

Suppose that $X \succ 0$ and $S \succ 0$, Then

$$\det(XS) \geq 0, \ \mathrm{Tr}(XS) \geq 0.$$

The following lemma shows the strict feasibility of the centering iterations under the condition $\delta(X, S; \mu) < 1$.

Lemma 4.4. (Lemma 6.3 in [13])

Let $\delta := \delta(X, S; \mu) < 1$. Then, the centering iterations are strictly feasible.

We proceed to prove the local quadratic convergence of full NT step to the target point $(X(\mu), y(\mu), S(\mu))$.

Lemma 4.5. (Lemma 6.4 in [13])

Let $\delta := \delta(X, S; \mu) < 1$. Then

$$\delta(X^{+}, S^{+}; \mu) \leq \frac{\delta^2}{1+\sqrt{1-\delta^2}}.$$

The following lemma gives an upper bound of the duality gap after a centering step (a full NT step).

Lemma 4.6. (Lemma 6.5 in [13])

After a centering step, we have

$$\mathrm{Tr}(X^{+}S^{+}) = n\mu.$$

5. A main Iteration of the Algorithm

Now, we describe a main iteration of the algorithm. Suppose that for some $\nu \in (0, 1]$ we have X, y and S satisfying the feasibility conditions (7) such that

$$\mathrm{Tr}(XS) = n\mu, \quad \text{and} \quad \delta(X, S; \mu) \leq \tau,$$

where $\mu = \nu\zeta^2$ and τ is a positive value.

First, we find new iterates X^{f} and (y^{f}, S^{f}) that the satisfy feasibility conditions of (P_ν) and (D_ν) with ν replaced by $\nu^{+} = (1+\theta)\nu$. As we will see, by taking θ small enough, this can be realized by one feasibility step. For the feasibility step, we use search directions $\Delta^f X$, $\Delta^f y$ and $\Delta^f S$ that are defined by the system

$$\mathrm{Tr}(A_i \Delta^f X) = \theta\nu(r_b^0)_i, \quad i = 1, \cdots, m,$$

$$\sum_{i=1}^{m} \Delta^f y_i A_i + \Delta^f S = \theta\nu R_c^0, \tag{27}$$

Therefore the following system is used to define $\Delta^f X, \Delta^f y$ and $\Delta^f S$:

$$\mathrm{Tr}(A_i \Delta^f X) = \theta\nu(r_b^0)_i, \quad i = 1, \cdots, m,$$

$$\sum_{i=1}^{m} \Delta^f y_i A_i + \Delta^f S = \theta \nu R_c^0, \tag{28}$$

$$\Delta X + X \Delta S S^{-1} = \mu \left(\psi' \left(\frac{XS}{\mu} \right) \right)^{-1} \left(\psi(I) - \psi \left(\frac{XS}{\mu} \right) \right) S^{-1}.$$

After the feasibility step, the iterates are given by

$$X^f = X + D^f X,$$
$$y^f = y + \Delta^f y,$$
$$S^f = S + \Delta^f S.$$

By definition, after the feasibility step, the iterates satisfy the affine equations (7), with $\nu = \nu^+$. The hard part in the analysis will be to guarantee that (X^f, y^f, S^f) are positive definite and satisfy $\delta(X^f, S^f; \mu^+) \le \eta$. In other words, we want that X^f and S^f belong to the region of quadratic convergence of μ^+-centers. Proving this is the crucial part in the analysis of the algorithm.

After the feasibility step, assume that we have the iterates (X^f, y^f, S^f) such that $\delta(X^f, S^f; \mu^+)$

can be obtained and denoting this bound by η. We perform few centering steps in order to get iterates (X^+, y^+, S^+) that satisfy

$$\mathrm{Tr}(X^+ S^+) = n\mu^+ \quad \text{and} \quad \delta(X^+, S^+; \mu^+) \le \tau.$$

By using Lemma 4.5, the required number of centering steps can easily be obtained. Indeed, assuming $\delta = \delta(X^f, S^f; \mu^+) \le \eta$, after k centering steps we will have iterates (X^+, y^+, S^+) that are still feasible for (P_{ν^+}) and (D_{ν^+}) and satisfy

$$\delta(X^+, y^+, S^+) \le \eta^{2^k}.$$

Just as in the linear case [10] this implies that after at most

$$\log_2 \left(\frac{\log_2 \tau}{\log_2 \eta} \right) \tag{29}$$

centering steps, we have $\delta(X^+, y^+, S^+) \le \tau$.

6. A Generic Primal-Dual IIPM for SDP

A formal description of the algorithm is given in Figure 1, where r_b and R_c denote the primal and dual residuals, respectively. One may easily verify that after each iteration the residuals and the duality gap are reduced by a factor $1-\theta$. The algorithm stops if the norms of residual vectors and the duality gap are less than the accuracy parameter ε.

7. Feasibility Steps

In section 5, we proved that the feasibility step generates the new iterates (X^f, y^f, S^f) that satisfy the feasibility conditions of (P_{ν^+}) and (D_{ν^+}), except for the nonnegativity constrains, Another element in the analysis is to show that after the feasibility step we have $\delta(X^f, S^f; m^+) \leq \eta$, where $0 < \eta < 1$.

7.1. Effect of the Feasibility Step

According to Section 6, $\Delta^f X$, $\Delta^f y$ and $\Delta^f S$ satisfy the system (28). Hence, the matrix $\Delta^f X$ is not necessarily symmetric, because $X \Delta^f SS^{-1}$ may not be symmetric [14] we use the NT-trick to

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Input:
An accuracy parameter ε **;**
a fix barrier update parameter θ**,** $0 < \theta < 1$**;**
a threshold parameter ζ, *and* $\tau > 0$;
begin
$X^0, S^0 := \zeta I, y^0 := 0, \mu^0 = \zeta^2$;
While $\max\left\{Tr(XS), \left\|r_b^0\right\|, \left\|R_c^0\right\|\right\} \geq \varepsilon$ **do**
begin
feasibility step:
$(X,y,S) = (X,y,S) + (\Delta^f X, \Delta^f y, \Delta^f S)$;
μ **and** ν **update:**
$\mu = (1-\theta)\mu$;
centering steps:
While $\delta(X,S;\mu) > \tau$ **do**
begin
$(X,y,S) = (X,y,S) + (\Delta X, \Delta y, \Delta S)$;
end
end
end

Figure 1. Infeasible-point full Newton step algorithm

symmetrize $\Delta^f X$ with P as defined in (13). Therefore, the system (28) can be rewritten as:

$$\mathrm{Tr}(A_i \Delta^f X) = \theta\nu (r_b^0)_i, \quad i = 1, \cdots, m,$$

$$\sum_{i=1}^{m} \Delta^f y_i A_i + \Delta^f S = \theta\nu R_c^0, \tag{30}$$

$$\Delta X + P\Delta^f SP^T = \mu\left(\psi'\left(\frac{XS}{\mu}\right)\right)^{-1}\left(\psi(I)-\psi\left(\frac{XS}{\mu}\right)\right)S^{-1}.$$

Let X , y and S be the iterates at the start of an iteration. Define

$$D_X^f := \frac{1}{\sqrt{\mu}} D^{-1}\Delta^f XD^{-1}, \quad D_S^f = \frac{1}{\sqrt{\mu}} D\Delta^f SD ,$$

where $D = P^{-\frac{1}{2}}$. We can now rewrite (30) as follows : (31)

$$\mathrm{Tr}(DA_i DD_X^f) = \frac{1}{\sqrt{\mu}}\theta\nu(r_b^0)_i, \quad i=1,\cdots,m,$$

$$\frac{1}{\sqrt{\mu}}\sum_{i=1}^{m}\Delta^f y_i DA_i D + D_S^f = \frac{1}{\mu}\theta\nu DR_c^0 D, \tag{32}$$

$$D_X^f + D_S^f = P_v,$$

where (P_v) is as defined in (20) .

Using (31), we may write

$$X^f = X + \Delta^f X = \sqrt{\mu}D(V + D_X^f)D,$$

$$S^f = S + \Delta^f S = \sqrt{\mu}D^{-1}(V + D_S^f)D^{-1}.$$

Therefore,

$$X^f S^f = \mu D(V + D_X^f)(V + D_S^f)D^{-1}.$$

The last equality shows that the matrix $X^f S^f$ is similar to

$$\mu(V + D_X^f)(V + D_S^f).$$

This means that we have

$$X^f S^f \sim (V + D_X^f)(V + D_S^f). \tag{33}$$

According to Section 4, we limit the analysis to $\psi(t) = \sqrt{t}$, This yields $P_V = 2(I - V)$. Putting

$$Q_V := D_X^f - D_S^f. \tag{34}$$

we have

$$D_X^f = \frac{P_V + Q_V}{2}, \quad D_S^f = \frac{P_V - Q_V}{2}, \quad D_X^f D_S^f + D_S^f D_X^f = \frac{P_V^2 - Q_V^2}{2}. \tag{35}$$

In the sequel, we denote

$$w(V) := \frac{1}{2}\sqrt{\left\|D_X^f\right\|^2 + \left\|\Delta^f S\right\|^2}. \tag{36}$$

This implies

$$\left\|D_X^f\right\| \le 2w(V), \quad \left\|D_X^f\right\| \le 2\,w(V). \tag{37}$$

Theorem 7.1.

Let $X \succ 0$ and $S \succ 0$. Then, the iterates (X^f, y^f, S^f) are strictly feasible if $2\,w(V) \le \delta < 1$.

Proof. The proof of this lemma is similar to the proof of Lemma 5.4 in [8]. □

Using (33) and (34), we have

$$X^f S^f \sim \mu\left(I - \frac{Q_V^2}{4} + M\right), \tag{38}$$

where,

$$M = \frac{\mu}{2}\left(\left(D_X^f V + V D_S^f - V D_X^f - D_S^f V\right) + \left(D_X^f D_S^f - D_S^f D_X^f\right)\right).$$

dividing (38) by μ^+, we obtain:

$$\frac{X^f S^f}{\mu^+} \sim \frac{I - \frac{Q_V^2}{4} + M}{1-\theta}, \tag{39}$$

which yields:

$$(V^f)^2 = \frac{D X^f S^f D^{-1}}{\mu^+} \sim \frac{I - \frac{Q_V^2}{4} + M}{1-\theta}. \tag{40}$$

Assuming $2w(V) \leq \delta < 1$, which guarantees strict feasibility of the iterates (X^f, y^f, S^f) we proceed by deriving an upper bound for $\delta(X^f, S^f; \mu^+)$.
Recall from definition δ that

$$\delta(X^f, S^f; \mu^+) = \left\| I - V^f \right\|, \tag{41}$$

with V^f as defined in (40). In the sequel we denote $\delta(\mathrm{X}^f, \mathrm{S}^f; \mu^+)$ shortly by $\delta(\mathrm{V}^f)$. we proceed to find an upper bound for $\delta(V^f)$ in terms of $w(V)$. To this end we need some technical results given information on the eigenvalues and the norm of V^f.

Lemma 7.2.

One has

$$\lambda_n\left((V^f)^2\right) \geq \frac{1}{1-\theta}\left(1 - 4w^2(v)\right).$$

Proof. Using (40) and Lemma 4.2, we have

$$\lambda_n\left((V^f)^2\right) = \lambda_n\left(\frac{I - \frac{Q_V^2}{4} + M}{1-\theta}\right) = \frac{\lambda_n\left(I - \frac{Q_V^2}{4} + M\right)}{1-\theta}$$

$$\geq \frac{\lambda\left(I - \frac{Q_V^2}{4}\right)}{1-\theta} = \frac{1 - \lambda_n\left(\frac{Q_V^2}{4}\right)}{1-\theta} = \frac{1 - \frac{\lambda_n^2(Q_V)}{4}}{1-\theta}.$$

According to the definition of $w(V)$ and (34) and the properties of the Frobenius matrix norm, we obtain $|\lambda n(Q_V)| \leq 4w(V)$. Substituting this yields

$$\lambda_n\left((V^f)^2\right) \geq \frac{1}{1-\theta}\left(1-4w^2(V)\right),$$

which completes the proof. □

Lemma 7.3.

One has

$$\| I-(V^f)^2 \| \leq \frac{1}{1-\theta}\left(\sqrt{n}\theta + 4w^2(V)\right).$$

Proof. Using (40) and properties of the Frobenius norm, we have

$$\| I-(V^f)^2 \|^2 = Tr\left(I-(V^f)^2\right)^2 = \sum_{i=1}^{n}\left(\lambda_i\left(I-(V^f)^2\right)\right)^2$$

$$= \sum_{i=1}^{n}\left(\lambda_i\left(\frac{I-\frac{Q_V^2}{4}+M}{1-\theta}\right)-1\right)^2$$

$$= \frac{1}{(1-\theta)^2}\sum_{i=1}^{n}\left(\lambda_i\left(I-\frac{Q_V^2}{4}+M\right)-(1-\theta)\right)^2$$

$$= \frac{1}{(1-\theta)^2}\sum_{i=1}^{n}\left(\lambda_i\left(-\frac{Q_V^2}{4}+M\right)+\theta\right)^2$$

$$= \frac{1}{(1-\theta)^2}\left(n\theta^2 + \sum_{i=1}^{n}\left(\lambda_i\left(-\frac{Q_V^2}{4}+M\right)\right)^2 + 2\theta\sum_{i=1}^{n}\lambda_i\left(-\frac{Q_V^2}{4}+M\right)\right)$$

$$\leq \frac{1}{(1-\theta)^2}\left(\left\|-\frac{Q_V^2}{4}\right\| - 2\theta Tr\left(\frac{Q_V^2}{4}\right)+\theta^2 n\right).$$

In the last inequality, we use property of skew-symmetric matrices (M is a skew-symmetric matrix). Now, let $\lambda\left(\frac{Q_V^2}{4}\right)$ be the vector consisting of the eigenvalues of $\frac{Q_V^2}{4}$. Using the Cauchy-Schwartz inequality, we get

$$Tr\left(\left(\frac{Q_V^2}{4}\right)\right) = \sum_{i=1}^{n}\lambda_i\left(\frac{Q_V^2}{4}\right) = e^T\lambda\left(\frac{Q_V^2}{4}\right)$$

$$\leq \|e\|\left\|\frac{Q_V^2}{4}\right\| = \sqrt{n}\left\|\frac{Q_V^2}{4}\right\|.$$

Thus, we have

$$\| \mathrm{I}-(V^f)^2 \|^2 = \frac{1}{(1-\theta)^2}\left(\left\|\frac{Q_V^2}{4}\right\|^2 - 2\theta Tr\left(\frac{Q_V^2}{4}\right)+\theta^2 n\right)$$

$$\leq \frac{1}{(1-\theta)^2}\left(\left\|\frac{Q_V^2}{4}\right\| - \sqrt{n}\theta\right)^2.$$

On the other hand, according to the definition of $w(V)$ and (34) we get

$$\left\| I - (V^f)^2 \right\|^2 \leq \frac{1}{(1-\theta)}\left(\left\|\frac{Q_V^2}{4}\right\|^2 - \sqrt{n}\theta\right) \leq \frac{1}{(1-\theta)}\left(4w^2(V) + \sqrt{n}\theta\right),$$

which completes the proof. □

Lemma 7.4.

Let $2w(V) \leq \delta < 1$. Then we have

$$\delta(X^f, S^f; \mu) \leq \frac{\sqrt{n}\theta + 4w^2(V)}{\sqrt{(1-\theta)\left(1-\left(4w^2(V)\right)\right)}}.$$

Proof. We may write, using definition δ

$$\begin{aligned}\delta(V^f) &= \left\| I - V^f \right\| = \left\|(I - V^f)(I + V^f)(I + V^f)^{-1}\right\| \\ &\leq \lambda_1\left((I + V^f)^{-1}\right)\left\|(I - (V^f)^2)\right\| \\ &= \frac{1}{1+\lambda(V^f)}\left\|\text{I-}(\text{V}^\text{f})^2\right\|.\end{aligned}$$

Using the bound in lemmas 7.2 and 7.3 the result follows. □

We wish the new iterates (X^f, y^f, S^f) are within the neighborhood where the Newton process targeting the μ^+-centers of (P_{ν^+}) and (D_{ν^+}) is quadratically convergent, i.e., $\delta(\text{V}^\text{f}) \leq \eta$. According to Lemma 7.4, it suffices to have

$$\frac{\sqrt{n}\theta + 4w^2(V)}{\sqrt{(1-\theta)\left(1-\left(4w^2(V)\right)\right)}} \leq \eta. \tag{42}$$

putting

$$\eta := \frac{1}{\sqrt{2}}.$$

Lemma 7.5.

Let $w(V) = \dfrac{1}{4\sqrt{2}}$ and $\quad 0 \leq \theta < \dfrac{1}{2\sqrt{2}n}$.

Then, the iterates (X^f, y^f, S^f) are strictly feasible for (P_ν) and (D_ν), respectively, and $\delta(\text{V}^\text{f}) \leq \eta = \dfrac{1}{\sqrt{2}}$.

Proof. Due to Theorem 7.1 and $w(V)=\frac{1}{4\sqrt{2}}$, the iterates (X^f, y^f, S^f) are strictly feasible. We just established that if inequality (42) with $\eta=\frac{1}{\sqrt{2}}$ is satisfied then $\delta(V^f)\le\eta=\frac{1}{\sqrt{2}}$. The left hand side in (42) is monotonically increasing in $w(V)$. By substituting $w(V)=\frac{1}{4\sqrt{2}}$, the inequality (42) reduces to:

$$\frac{\sqrt{n}\theta+\frac{1}{8}}{\sqrt{(1-\theta)\left(1-\frac{1}{8}\right)}}\le\frac{1}{\sqrt{2}}, \tag{43}$$

which is equivalent to

$$n\theta^2+4\left(\sqrt{n}+\frac{7}{16}\right)\theta-\frac{27}{64}\le 0. \tag{44}$$

Thus, if $0\le\theta<\frac{1}{2\sqrt{2n}}$, then the above inequality is satisfied. Therefore, the proof is complete. □

8. An upper Bound for w(V)

Consider the linear space L as follows:

$$L=\left\{\xi\in \mathrm{S}^n : DA_iD\bullet\xi=0,\ \ i=1,\cdots,m\right\}.$$

Using the linear space L , it is clear from the first equation in system (30) that the affine space

$$\left\{\xi\in \mathrm{S}^n : DA_iD\bullet\xi=\frac{1}{\sqrt{\mu}}\theta\nu(r_b^0)_i,\ \ i=1,\cdots,m,\right\}$$

equals D_X^f+L. By the second equation in the system (30), we have $D_S^f\in\frac{1}{\sqrt{\mu}}\theta\nu DR_c^0D+L^\perp$.
Since $L\cap L^\perp=\{0\}$, the spaces D_X^f+L and $D_S^f+L^\perp$ meet in a unique matrix. This matrix is denoted by Q . We can get a similar result of Mansouri and Roos [8].

Lemma 8.1.

Let Q be the (unique) matrix in the intersection of the affine spaces D_X^f+L and $D_S^f+L^\perp$. Then,

$$2w(V)\le\sqrt{\|Q\|^2+\left(\|Q\|+2d(V)\right)^2} \tag{45}$$

Proof. The proof of this Lemma is similar to the proof of Lemma 5.6 in [10], and is therefore omitted . □

Recall from Lemma 7.5 that in order to guarantee that $\delta(V^f)\leq\frac{1}{\sqrt{2}}$, we want to have $w(V\ )\leq\frac{1}{4\sqrt{2}}$. Due to Lemma 8.1 this will certainly hold if $\|Q\|$ satisfies

$$\|Q\|^2+\left(\|Q\|+2\delta(V)\right)^2\leq\frac{1}{8}. \tag{46}$$

8.1. An Upper Bound for $\|\mathrm{Q}\|$

Recall from Lemma 8.1 that Q is the unique solution of the system

$$Tr(DA_iDQ)=\frac{1}{\mu}\theta\nu(r_b^0)_i,\quad i=1,\cdots,m,$$

$$\sum_{i=1}^{m}\frac{\xi_i}{\mu}DA_iD+Q=\frac{1}{\mu}\theta\nu DR_c^0D. \tag{47}$$

We proceed to find an upper bound for $\|Q\|$. As will become clear below, specially in the proofs of Lemma 8.2 and Lemma 8.4, it will be convenient to choose the initial iterates (X^0,y^0,S^0) as follows:

$$X^0=S^0=\zeta I,\ y^0=0,\ \ \mu^0=\zeta^2, \tag{48}$$

where $\zeta>0$ is such that

$$\mathrm{X}^*+\mathrm{S}^*\preceq\zeta\mathrm{I}, \tag{49}$$

for some $(X^*,y^*,S^*)\in F^*=\{(X,y,S)\in P\times D\colon Tr(XS)=0\}$. It may be noted that this choice of the initial iterates has become usual for infeasible IPMs for SDP [8].

Lemma 8.2. (Lemma 5.13 in [8])

With (X^0,y^0,S^0) as defined by (48), we have

$$\|Q\|\leq\frac{\theta}{\zeta\lambda_{\min}(V\ )}Tr(X\ +S\ ). \tag{50}$$

In the following lemma we get a bound for $\lambda_i(V\ )$ where V is defined in (15).

Lemma 8.3.

Let $\delta=\delta(\mathrm{V})$ be given by (23). Then

$$1-\delta\leq\lambda_i(V\ )\leq1+\delta,\ \ i=1,\cdots,n. \tag{51}$$

Proof. We can rewrite $\delta(\mathrm{V})$ in (23) as follows

$$\delta(V)=\|I-V\|=\|V-I\|=\sqrt{Tr((V-I)^2)}$$

$$= \sqrt{\sum_{i=1}^{n} \lambda_i^2 (V - I)} = \sqrt{\sum_{i=1}^{n} (\lambda_i (V) - 1)^2}.$$

It implies that

$$\left| \lambda_i (V) - I \right| \leq \delta.$$

The proof is complete.

□

Lemma 8.4.

Let X and (y, S) be feasible for the perturbed problems (P_ν) and (D_ν), respectively, and let (X^0, y^0, S^0) and $(X^*, y^*, S^*) \in F^*$ be as defined in (48) and (49). Then, we have

$$nTr(S^0 X + X^0 S) = Tr(XS) + \nu^2 Tr(S^0 X^0)$$
$$+ \nu(1-\nu) Tr(S^0 X^* + X^0 S^*) - (1-\nu) Tr(SX^* + S^* X).$$

Proof. See [8]. □

Lemma 8.5.

Using the same notations as in Lemma 8.4, we have

$$Tr(X + S) \leq \left((1 + d)^2 + 1\right) \mathrm{n}\zeta. \tag{52}$$

Proof. Since X, S, X^* and S^* are positive definite, $Tr(SX^*)$ and $Tr(XS^*)$ are nonnegative. Dividing both sides of the inequality in Lemma 8.4 by $0 < \nu \leq 1$, we get

$$Tr(S^0 X + X^0 S) \leq \frac{Tr(XS)}{\nu} + \nu Tr(S^0 X^0) + (1-\nu) Tr(S^0 X^* + X^0 S^*).$$

Since $X^0 = S^0 = \zeta \mathrm{I}$ and $X^* + S^* \preceq \zeta \mathrm{I}$, we have

$$Tr(S^0 X^* + X^0 S^*) = \zeta Tr(X^* + S^*) \leq \zeta^2 Tr(I) = n\zeta^2,$$

and

$$Tr(S^0 X + X^0 S) = \frac{Tr(XS)}{\nu} + n\zeta^2 = \frac{\mu Tr(V^2)}{\nu} + n\zeta^2 = \zeta^2 Tr(V^2) + n\zeta^2.$$

The last equality is true because of $\nu = \dfrac{\mu}{\mu^0}$ and $\mu^0 = \zeta^2$. According to (51), we get

$$Tr(V^2) = \sum_{\mathrm{i}=1}^{\mathrm{n}} \lambda_{\mathrm{i}}(V^2) = \sum_{\mathrm{i}=1}^{\mathrm{n}} \lambda_i^2(V) \leq n(1 + \delta)^2.$$

Thus,

$$Tr(S^0 X + X^0 S) \geq \left((1 + \delta)^2 + 1\right) n\zeta^2.$$

On the other hand, we have $X^0 = S^0 = \zeta I$. Therefore,

$$Tr(S^0 X + X^0 S) = \zeta Tr(X + S),$$

and the proof is complete.

□

By substituting (51) and (52) into (50), we get

$$\|Q\| \le \frac{n\theta}{1-\delta}\left((1+\delta)^2+1\right). \tag{53}$$

At this stage let

$$\tau = \frac{1}{8}. \tag{54}$$

Since $\delta \le \tau = \frac{1}{8}$, we have

$$\|\mathrm{Q}\| \le \frac{145n\theta}{56}. \tag{55}$$

In (46), we found that, in order to have $\delta(V^f) \le \frac{1}{\sqrt{2}}$, we should have $\|Q\|^2 + \left(\|Q\| + 2\delta(V)\right)^2 \le \frac{1}{8}$. Therefore, since $\delta(V) \le \tau = \frac{1}{8}$, it suffices to have $\|Q\|$ satisfy $\|Q\|^2 + \left(\|Q\| + \frac{1}{4}\right)^2 \le \frac{1}{8}$. The latter holds if $\|Q\| \le \frac{5}{32}$. Hence, using (55) we obtain $\delta(V^f) \le \frac{1}{\sqrt{2}}$ if

$$\frac{145n\theta}{56} \le \frac{5}{32}.$$

We deduce that by taking

$$\theta = \frac{1}{17n}. \tag{56}$$

We have $\delta(V^f) \le \frac{1}{\sqrt{2}}$

9. Iteration Bound

In the pervious sections, we have found out that if at the start of an iteration the iterates satisfy $\delta(X,S;\mu) \le \tau$ with τ as defined in (54), and θ as in (56), then after the feasibility step, the iterates satisfy $\delta(\mathrm{X}^{\mathrm{f}}, \mathrm{S}^{\mathrm{f}}; \mu) \le \frac{1}{\sqrt{2}}$.

According to (29), at most

$$\log_2\left(\log_2 \frac{1}{\tau^2}\right) \le \log_2\left(\log_2 64\right)$$

centering steps suffice to get the iterates that satisfy $\delta(X,S;\mu^+) \le \tau$. So, each main iteration (57) consists of one feasibility step and at most 3 centering steps. In each main iteration, both the duality gap and the norm of residual vectors are reduced by the factor $(1-\theta)$. Hence, using $Tr(X^0S^0) = n\zeta^2$ the total number of iterations is bounded above by

$$\frac{1}{\theta}\log\frac{\max\left\{n\zeta^2,\left\|r_b^0\right\|,\left\|R_c^0\right\|\right\}}{\varepsilon} \tag{58}$$

Due to θ as in (56), the total number of iterations is bounded above by

$$68n\log\frac{\max\left\{n\zeta^2,\left\|r_b^0\right\|,\left\|R_c^0\right\|\right\}}{\varepsilon}. \tag{59}$$

Theorem 1. If (P) and (D) have optimal solutions $(X^*, y^*, S^*)\in F$ such that $X^*+S^*\preceq\zeta I$, then after at most

$$68n\log\frac{\max\left\{n\zeta^2,\left\|r_b^0\right\|,\left\|R_c^0\right\|\right\}}{\varepsilon}$$

Iterations, the algorithm finds an ε-solution of (P) and (D).

10. Concluding Remarks

We have extended an infeasible primal-dual path-following interior-point algorithm for LO to SDP with full NT step and derived the currently best known iteration bound for the algorithm with full Newton step, namely, $\left(n\log\frac{n}{\varepsilon}\right)$, which is the same iteration bound as in the LO case. Some interesting topics remain for further research. First, the search directions used have are all based on the NT-symmetrization scheme. It may be possible to design similar algorithms using other symmetrization schemes to obtain polynomial-time iteration bounds. Second, the extensions to SOCO and the general convex optimization deserve to be investigated. Furthermore, numerical test are needed to investigate the behavior of the algorithm so as to be with other approaches.

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