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# A new machine replacement policy based on number of defective items and Markov chains

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*A novel optimal single machine replacement policy using a single as well as a two-stage decision making process is proposed based on the quality of items produced. In a stage of this policy, if the number of defective items in a sample of produced items is more than an upper threshold, the machine is replaced. However, the machine is not replaced if the number of defective items is less than a lower threshold. Nonetheless, when the number of defective item falls between the upper and the lower thresholds, the decision making process continues inspecting and possibly repairing the machine and the decision making process goes on to collect more samples. The primary objective of own work is to determine the optimal values of both the upper and the lower thresholds using a Markov process to minimize the total cost associated with a machine replacement policy.*

***Keywords:*** *Machine replacement, Inspection, Defective items, Markov process.*

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## 1. Introduction

As the optimal management of equipment replacement is continuously becoming more important nationally and locally (i.e., at a firm level), it is attracting a significant interest in operations research and management science. The traditional equipment replacement analysis focus on the evaluation of the keep or replace alternative for each period within the planning horizon for single, two, or multiple machines. These machines work in series or parallel, where several motivations such as machine deterioration, technological change, and capacity expansion have been focused. Solutions generally involve the use of dynamic programming under various assumptions concerning costs, planning horizon, number of machines and their types.

Two approaches have been applied to the machine replacement problem. In the first approach, modeling helps to understand how the optimal replacement depends on the intensity of technological change, the rate of capacity expansion, and the deteriorating rate. In the second approach, the maintenance policy that is being followed in practice is a combination of preventive and corrective maintenance.

Chand and Sethi [2] defined a deterministic approach improving technological environment in terms of a periodic process of technological improvements in every existing technology. Goldstein et al. [5] proposed a planning horizon for the first optimal replacement in an infinite-horizon problem when two types of machines were considered. The first type is based on existing technology; the second is based on a not-yet-achieved technological breakthrough to be available at some unknown future time. They employed a dynamic programming model to generate some conditional decision rules on optimal replacement. In their formulation, technological improvements were viewed in terms of purchasing and operating costs, assuming that the

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production capacity and quality characteristics of machines at different or identical technologies remained constant.

While the objective of a machine replacement problem is to minimize the cost, some researchers assumed the rate of machine breakdown and repair was known. They tried to minimize the cost of production, inventories, and backlogs. Sethi et al. [12] developed appropriate dynamic programming equations and established the existence of the solution using a verification theorem for optimality. Presman et al. [11] considered a production-planning problem in a two-machine flow-shop environment with breakdown and repair, subject to upper bound constraints on work-in-process. The main purpose in their research was to choose admissible input rates to minimize the average surplus and production costs over an infinite horizon. Niaki and Fallahnezhad [10] employed Bayesian inference and stochastic dynamic programming to design a decision-making framework in production environment. Further, Fallahnezhad et al. [3] determined the optimal policy for two-machine replacement problem using Bayesian inference in the context of the finite mixture model. They discussed the analysis of time-to-failure data and proposed an optimal decision-making procedure for machine replacement strategy. Moreover, Fallahnezhad and Niaki [4] proposed a dynamic programming model of the two-machine replacement problem.

Ivy and Nembhard [8] integrated statistical quality control (SQC) and partially observable Markov decision processes (POMDP) for maintenance decision making of deteriorating systems. In their work, they employed SQC to sample a real-world system and define the observation distribution for the POMDP modeling. Simulation methodology was used in their research to integrate SQC and POMDP to develop and evaluate maintenance policies as a function of process characteristics, system operating and maintenance costs.

Marsh and Nam [9] studied the equipment maintenance and replacement policies under the scenarios of increasing customer expectations (in terms of tighter product specifications), loss (in terms of process deviation), and process drift. The Taguchi loss function was first employed to estimate the loss due to target deviation. Then, a generalized Brownian motion-to-process was used to model the problem.

Hartman and Ban [7] developed a multiple machine replacement model that was characterized as a parallel flow shop environment. An integer programming formulation was developed in their work to determine the optimal purchase, salvage, utilization, and storage decisions for each machine over a finite horizon. They first showed the formulation was hard to solve and then came up with valid inequalities to improve the lower bound provided by the linear programming relaxation and a dynamic programming approach to provide initial lower bounds. Grosfeld-Nir [6] presented a two-state partially observable Markov decision process for machine replacement problem. He proved that the ‘‘dominance in expectation’’ (the expected profit is larger in the good state than in the bad state) suffices for the optimal policy to be of a control limit type (CLT). In other words, continue if and only if the good state probability exceeds the CLT.

Here, a new single-machine replacement policy with both single and two-stage decision-making process is proposed in which the optimal policy is derived based on the quality of items produced. The quality in each stage is compared with the optimal upper and lower thresholds obtained by a Markov process modeling. The rest of the paper is organized as follows. The problem is first stated in Section 2. Then, the notations are presented in Section 3. Next, the single-stage model together with an illustrative example is proposed in Section 4. Section 5 contains the two-stage modeling along with a numerical example. Finally, we conclude in Section 6.

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## 2. Problem Statement and Assumptions

Consider a single machine that produces a specific item. At each stage of the machine replacement policy, the states of the machine are defined in terms of the quality of items it produces. This quality is determined using the attribute acceptance sampling plans to accept or reject a production lot. The accept/reject decision is based on the count of the number of defectives items (see Taylor [13]). If this number is greater than an upper threshold, then the machine needs replacement. If it is less than a lower threshold, then the machine continues production. Nonetheless, if it is between the upper and the lower threshold, then an inspection and possibly repair is needed. After inspection, the machine returns to its initial state and the decision-making process starts over by collecting more samples. The objective is to find the optimal values of the thresholds that minimize the total cost associated with the machine replacement strategy. Furthermore, the following assumptions are used in the modeling process.

(1) The machine produces the items continuously.
(2) The quality of items produced by the machine is inspected periodically.
(3) The machine replacement strategy is employed in every period of item inspection.

## 3. Notations

To model the problem at hand, the following notations are used.

$E(TC)$ : The expected total system cost

$E(AC)$ : The expected total cost of accepting (not replacing) the machine

$E(RP)$ : The expected total cost of replacing the machine

$E(I)$ : The expected total cost of inspecting and repairing the machine

$n$ : The sample size of a single-stage machine replacement policy

$n_1$ : The sample size in the first stage of a two-stage machine replacement policy

$n_2$ : The sample size in the second stage of a two-stage machine replacement policy

$N$ : The number of produced items in a period

$p$ : The probability of producing a defective item

$c_1$ : The lower threshold for the number of defective items in the first stage

$c_2$ : The upper threshold for the number of defective items in the first stage

$c_3$ : The lower threshold for the number of defective items in the second stage

$c_4$ : The upper threshold for the number of defective items in the second stage

$I$ : The cost of inspecting and repairing a possible defect

$c$ : The cost of producing a defective item

$R$ : The cost of machine replacement

$\delta_1$ : The minimum acceptable level of the lot quality (accepted quality level (*AQL*)).

$\delta_2$ : The minimum rejectable level of the lot quality (lot tolerance proportion defective (*LTPD*))

$\varepsilon_1$ : The probability of type-one error in making a decision

$\varepsilon_2$ : The probability of type-two error in making a decision

$\mathbf{P}$ : The transition probability matrix

$\mathbf{Q}$ : The square matrix containing the transition probabilities of going from a non-absorbing state to another non-absorbing state

$\mathbf{R}$ : The matrix containing all probabilities of going from a non-absorbing state to an absorbing state

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(i.e., finished or scrapped product)
**A** : An identity matrix representing the probability of staying in a state
**O** : The matrix representing the probabilities of escaping an absorbing state (always zero)
**M** : The fundamental matrix containing the expected number of transitions from any non-absorbing state to another non-absorbing state before absorption occurs
**F** : The absorption probability matrix containing the long-run probabilities of the transition from any non-absorbing state to an absorbing state
$p_{ij}$ : The probability of going from state $i$ to state $j$ in a single stage
$m_{ij}$ : The expected long-run number of times that the transient state $j$ is occupied before absorption occurs, given that the initial state is $i$
$f_{ij}$ : The long-run probability of going from a non-absorbing state ($i$) to an absorbing state ($j$)

## 4. Model Development

Consider a machine that produces an item continuously. Depending on the number of decision-making stages (either one stage or two), a sample of proper size is first gathered for inspection. Then, the decision on whether or not replacing, or repairing the machine is made. The purpose of this research is to develop a Markovian model to determine the optimum values of the thresholds in each inspection stage. The paper starts by developing the model for a single-stage and then goes on to propose the two-stage replacement policy.

Based on the notations defined in Section 3, the expected total system cost can be expressed as:
$$E(TC) = E(AC) + E(RP) + E(I). \quad (1)$$

In what follows, the single-machine optimal replacement policy is first derived. Then, the modeling extends to the two-stage case in Section 5.

### 4.1. A Single-Stage Single-Machine Replacement Policy

The single-machine optimal replacement policy in a single stage is developed based on the following scenario. A lot containing $n$ items is first inspected. If the number of defective items in the lot is less than or equal to $c_1$, then the machine is in a good state and continues production. If the number of defective items is more than $c_2$, then the machine is in a bad state and needs to. Otherwise, if the number of defective items is greater than $c_1$ but less than or equal $c_2$, then the machine is inspected for malfunctioning, is adjusted or repaired accordingly, and the single-stage decision-making problem starts over.

A Markov chain process with the following states can model the above scenario.
State 1: The machine needs inspection and possibly adjustment or repair.
State 2: The machine continues production (no replacement is needed).
State 3: The machine needs replacement.

Note that in State 2, when the machine does not require replacement, it is classified as an acceptable machine that fulfills quality requirements. Hence, the decision-making process stops. Further more, in State 3, the performance of the machine is not satisfactory and causes to stop the decision-making process. In other words, states 2 and 3 are absorbing.

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Based on the definition given in Section 2 and the notations given in Section 3 we have.

Probability of inspecting the machine = $p_{11} = \sum_{i=c_1+1}^{c_2} \binom{n}{i} p^i (1-p)^{n-i} = F(c_2) - F(c_1)$,

probability of non-replacement = $p_{12} = \sum_{i=0}^{c_1} \binom{n}{i} p^i (1-p)^{n-i} = F(c_1)$,

probability of replacing the machine = $p_{13} = \sum_{i=c_2+1}^{n} \binom{n}{i} p^i (1-p)^{n-i} = 1 - F(c_2)$,

where $F(.)$ represents the cumulative binomial distribution function with parameters $n$ and $p$. Hence, the transition probability matrix of this process can be expressed as follows:

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}. \quad (2)$$

The $\mathbf{P}$ matrix is an absorbing Markov chain with states 2 and 3 being absorbing and state 1 being transient. Analyzing this absorbing Markov chain requires the rearrangement of the single-step probability matrix in the following form:

$$\mathbf{P} = \begin{bmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{R} & \mathbf{Q} \end{bmatrix}. \quad (3)$$

Rearranging the $\mathbf{P}$ matrix in the latter form yields the following matrix:

$$\mathbf{P} = \begin{array}{c} \\ 2 \\ 3 \\ 1 \end{array} \begin{array}{c} \begin{array}{ccc} 2 & 3 & 1 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ p_{12} & p_{13} & p_{11} \end{bmatrix} \end{array}. \quad (4)$$

Hence, the fundamental matrix $\mathbf{M}$, which is a one-by-one matrix in this case, can be obtained as follows, where $\mathbf{I}$ is the identity matrix (Bowling et al. [1]):

$$\mathbf{M} = m_{11} = (\mathbf{I} - \mathbf{Q})^{-1} = \frac{1}{1-p_{11}}. \quad (5)$$

The value $m_{11}$ represents the expected long-run number of times that the transient state 1 is occupied before absorption occurs (i.e., replaced or not replaced), given that the initial state is 1. Hence, the long-run absorption probability matrix, $\mathbf{F}$, can be calculated as follows (Bowling et al. [1]:

$$\mathbf{F} = \mathbf{M} \times \mathbf{R} = \begin{array}{c} \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} f_{12} = \frac{p_{12}}{1-p_{11}} & f_{13} = \frac{p_{13}}{1-p_{11}} \end{bmatrix} \end{array}, \quad (6)$$

where the elements of the $\mathbf{F}$ matrix, $f_{12}$ and $f_{13}$, represent the probabilities of a machine not being replaced and replaced, respectively.

Now, the expected total system cost given by Eq. (1) can be derived. The expected acceptance cost, $E(AC)$, is determined by the expected cost of the defective items, $c(Np)$, multiplied by the absorption probability of accepting the machine (i.e., $f_{12}$). The expected replacement cost, $E(RP)$, is obtained by the replacement cost, $R$, multiplied by the absorption probability of replacing machine (i.e., $f_{13}$). Also, once a machine goes into one of these two absorbing states (i.e., states 2 and 3), it

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cannot go back to state 1. Hence, the number of visits to the absorbing states is equal to 1, and the expected inspection cost is given by $I(m_{11}-1)$. Therefore, the expected cost for machine replacement policy can be expressed as a function of $f_{12}$, $f_{13}$, and $m_{11}$ as follows:

$$E(TC) = cNpf_{12} + Rf_{13} + I(m_{11}-1). \quad (7)$$

Substituting for $f_{12}$ and $m_{11}$, the expected cost Eq. (8) can be rewritten as:

$$E(TC) = cNp\frac{p_{12}}{1-p_{11}} + R\left(1-\frac{p_{12}}{1-p_{11}}\right) + I\left(\frac{p_{11}}{1-p_{11}}\right). \quad (8)$$

In terms of the cumulative binomial distribution, Eq. (8) becomes:

$$E(TC) = cNp\frac{F(c_1)}{1-F(c_2)+F(c_1)} + R\left(1-\frac{F(c_1)}{1-F(c_2)+F(c_1)}\right) + I\left(\frac{F(c_2)-F(c_1)}{1-F(c_2)+F(c_1)}\right), \quad (9)$$

where the terms $F(c_1)$ and $F(c_2)$ are functions of the probability of producing a defective item $p$.

The optimal replacement policy turns into determining the values of $c_1$ and $c_2$ that minimize the expected total system cost numerically. Alternatively, in order to determine the boundary limits of $c_1$ and $c_2$, the concepts of type-I and type-II error probabilities are utilized. The type-I error, $\varepsilon_1$, shows the probability of rejecting the machine when the defective production percent of the machine is acceptable and the type-II error, $\varepsilon_2$, is the probability of accepting the machine when the defective percentage is not acceptable. Then, by setting the parameters $p=\delta_1=AQL$ and $p=\delta_2=LTPD$, in case when $p=\delta_1$, the probability of rejecting the machine is less than $\varepsilon_1$ and if $p=\delta_2$, the probability of accepting the machine is less than $\varepsilon_2$. Hence, we have

$$\begin{aligned} p=\delta_1 &\rightarrow f_{13}\le\varepsilon_1 \rightarrow f_{12}\ge 1-\varepsilon_1, \\ p=\delta_2 &\rightarrow f_{12}\le\varepsilon_2 \rightarrow f_{13}\ge 1-\varepsilon_2. \end{aligned} \quad (10)$$

Based on the above inequalities, the feasible values of $c_1$ and $c_2$ among a set of alternative values are first determined. Then, the optimal replacement policy is derived using Eq. (9). To show this, a numerical example is given in the next subsection to illustrate the application of the proposed methodology.

### 4.2. An Illustrative Example

To demonstrate the application of the proposed methodology in a single-machine single-stage replacement strategy, a numerical example is solved. Consider a single-stage system with the number of produced items in a period, $N=1000$, the probability of producing a defective item $p=0.1$, the cost of producing a defective item, $c=6$, the cost of machine replacement, $R=600$, the cost of machine inspection and repair, $I=300$, the number of items in an inspected lot, $n=50$, $\delta_1=0.05$, $\delta_2=0.2$, $\varepsilon_1=0.05$, and $\varepsilon_2=0.1$.

The feasible values of $c_1$ and $c_2$ among existing alternatives are first obtained using Eq. (10) as follows:

$$\begin{aligned} p=0.05 &\rightarrow f_{13}\le 0.05 \rightarrow f_{12}\ge 0.95, \\ p=0.2 &\rightarrow f_{12}\le 0.1 \rightarrow f_{13}\ge 0.9. \end{aligned}$$

In other words, the probability of accepting the machine when $\delta_1=0.05$ should be more than 0.95 and the probability of rejecting the machine when $\delta_2=0.2$ should be more than 0.9. Table 1 shows 12 different alternative combination values of $c_1$ and $c_2$ together with their probability of rejecting or

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accepting the machine, of which the ones in bold are feasible. Then, Eq. (9) is numerically solved for all feasible sets of Table 1. The results are given in Table 2. Based on the results, the best combination value is $c_1 = 4$ and $c_2 = 6$ with the minimum value of the expected total cost being 753.88.

**Table1.** The feasible values of $c_1$ and $c_2$ among alternatives

| Probability of rejecting the machine when $\delta_2 = 0.2$ | Probability of accepting the machine when $\delta_1 = 0.05$ | $c_2$ | $c_1$ |
|---|---|---|---|
| 0.99981 | 0.53838 | 3 | 1 |
| 0.99980 | 0.8809 | 5 | 1 |
| 0.99976 | 0.98872 | **7** | **1** |
| 0.99870 | 0.83914 | 4 | 2 |
| 0.99857 | 0.97866 | **6** | **2** |
| 0.99815 | 0.99860 | **8** | **2** |
| 0.97979 | 0.98702 | **6** | **4** |
| 0.97399 | 0.99916 | **8** | **4** |
| 0.95747 | 0.99997 | **10** | **4** |
| 0.87011 | 0.99924 | 8 | 6 |
| 0.80110 | 0.99997 | 10 | 6 |
| 0.642783 | 0.99999 | 12 | 6 |

**Table 2.** The expected total cost for different feasible combinations of $c_1$ and $c_2$

| $c_1$ | $c_2$ | $p_{11}$ | $p_{12}$ | $p_{13}$ | $E(TC)$ |
|---|---|---|---|---|---|
| 1 | 7 | 0.84407 | 0.03379 | 0.12214 | 2223.93 |
| 2 | 6 | 0.65850 | 0.11173 | 0.22977 | 1178.47 |
| 2 | 8 | 0.83040 | 0.11173 | 0.05787 | 2068.91 |
| 4 | 6 | 0.33903 | 0.43120 | 0.22977 | 753.88 |
| 4 | 8 | 0.51093 | 0.43120 | 0.05787 | 913.41 |
| 4 | 10 | 0.55945 | 0.43120 | 0.00935 | 980.96 |

## 5. A Two-Stage Machine Replacement Policy

In a first inspected two-stage single-machine replacement strategy, assume a lot containing $n_1$ items is inspected first. If the number of defective items in the lot is less than or equal to $c_1$, then the machine is in a good state and is accepted. If the number of defective items is greater than $c_1$, but less than or equal $c_2$, the machine is inspected and possibly adjusted or repaired. Then the decision-making process starts over. Otherwise, if the number of defective items is more than $c_2$, then the machine is evaluated in the second stage. In the second stage, a lot containing an additional $n_2$ items is inspected. If the total number of defective items is less than or equal $c_3$, then the machine is accepted. If the total number of defective items is greater than $c_3$, but less than or equal $c_4$, the machine is inspected and possibly adjusted or repaired. Then the two-stage decision-making process starts over. Otherwise, if the total number of defective items is more than $c_4$, then the machine is rejected and should be replaced.

Similar to the single-stage replacement strategy, a Markov chain process with the following states can

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model the above scenario.
State 1: The machine should be inspected.
State 2: The second stage replacement policy should be applied to the machine.
State 3: The machine should be accepted.
State 4: The machine should be replaced.
Then, based on the notations given in Section 3, we have:

the first stage probability of accepting the machine = $p_{13} = \sum_{i=0}^{c_1} \binom{n_1}{i} p^i (1-p)^{n_1-i} = F_1(c_1)$,

probability of deciding in the second stage = $p_{12} = \sum_{i=c_2+1}^{n_1} \binom{n_1}{i} p^i (1-p)^{n_1-i} = 1-F_1(c_2)$,

the first stage probability of inspecting the machine = $p_{11} = \sum_{i=c_1+1}^{c_2} \binom{n_1}{i} p^i (1-p)^{n_1-i} = F_1(c_2)-F_1(c_1)$,

the second stage probability of accepting the machine = $p_{23} = \sum_{i=0}^{c_3} \binom{n_2}{i} p^i (1-p)^{n_2-i} = F_2(c_3)$,

the second stage probability of inspecting the machine = $p_{21}$

$= \sum_{i=c_3+1}^{c_4} \binom{n_2}{i} p^i (1-p)^{n_2-i} = F_2(c_4)-F_2(c_3)$,

the second stage probability of rejecting the machine = $p_{24} = \sum_{i=c_4+1}^{n_2} \binom{n_2}{i} p^i (1-p)^{n_2-i} = 1-F_2(c_4)$,

where $F_1(.)$ and $F_2(.)$ denote the cumulative binomial distribution functions of stage 1 and stage 2 of the decision making process. In this case, the transition probability matrix $\mathbf{P}$ can be expressed as follows:

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} p_{11} & p_{12} & p_{13} & 0 \\ p_{21} & 0 & p_{23} & p_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}. \tag{11}$$

Again, analyzing this absorbing Markov chain requires the rearrangement of the single-step probability matrix in the following form:

$$\mathbf{P} = \begin{bmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{R} & \mathbf{Q} \end{bmatrix}. \tag{12}$$

Rearranging the $\mathbf{P}$ matrix in the latter form yields the following matrix:

$$\mathbf{P} = \begin{array}{c} \\ 3 \\ 4 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ p_{13} & 0 & p_{11} & p_{12} \\ p_{23} & p_{24} & p_{21} & 0 \end{bmatrix} \end{array}. \tag{13}$$

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The fundamental matrix $\mathbf{M}$ can be obtained as follows (Bowling et al. [1]):

$$\mathbf{M} = \left(\mathbf{I} - \mathbf{Q}\right)^{-1} = \begin{bmatrix} m_{11} = 1 - p_{11} & m_{12} = -p_{12} \\ m_{21} = -p_{21} & m_{22} = 1 \end{bmatrix}^{-1} = \begin{matrix} 1 \\ 2 \end{matrix} \overset{\begin{matrix} 1 & 2 \end{matrix}}{\begin{bmatrix} \dfrac{1}{1 - p_{11} - p_{12}p_{21}} & \dfrac{p_{12}}{1 - p_{11} - p_{12}p_{21}} \\ \dfrac{p_{22}}{1 - p_{11} - p_{12}p_{21}} & \dfrac{1 - p_{11}}{1 - p_{11} - p_{12}p_{21}} \end{bmatrix}}, \quad (14)$$

where $m_{ii}$ represents the expected number of times in the long-run that the transient state $i$, $(i = 1,2)$, is occupied before absorption occurs (i.e., accepted or rejected), given that the initial state is 1.

The long-run absorption probability matrix, $\mathbf{F}$, can then be determined as follows (Bowling et al. [1]):

$$\mathbf{F} = \mathbf{M} \times \mathbf{R} = \begin{bmatrix} \dfrac{1}{1 - p_{11} - p_{12}p_{21}} & \dfrac{p_{12}}{1 - p_{11} - p_{12}p_{21}} \\ \dfrac{p_{22}}{1 - p_{11} - p_{12}p_{21}} & \dfrac{1 - p_{11}}{1 - p_{11} - p_{12}p_{21}} \end{bmatrix} \begin{bmatrix} p_{13} & 0 \\ p_{23} & p_{24} \end{bmatrix}$$

$$= \begin{matrix} 1 \\ 2 \end{matrix} \overset{\begin{matrix} 3 & 4 \end{matrix}}{\begin{bmatrix} \dfrac{p_{13} + p_{12}p_{23}}{1 - p_{11} - p_{12}p_{21}} & \dfrac{p_{12}p_{24}}{1 - p_{11} - p_{12}p_{21}} \\ \dfrac{p_{22}p_{13} + (1 - p_{11})p_{23}}{1 - p_{11} - p_{12}p_{21}} & \dfrac{(1 - p_{11})p_{24}}{1 - p_{11} - p_{12}p_{21}} \end{bmatrix}}, \quad (15)$$

where $f_{14}$ is the probability of machine replacement.

Now, the expected total system cost can be obtained using Eq. (1). Assuming that at the start of the process the machine is in state one, the expected acceptance cost is simply the acceptance cost $(cNp)$ multiplied by the probability of accepting the machine (i.e., $f_{13}$). The expected replacement cost is the replacement cost ($R$) multiplied by the probability of machine replacement $(f_{14} = 1 - f_{13})$. The expected inspection cost is the inspection cost ($I$) multiplied by the number of machine inspections at stage 1 (i.e., $m_{11} - 1$) plus inspection cost ($I$) multiplied by the number of machine inspections at stage 2 (i.e., $m_{22} - 1$) multiplied by the probability of continuing to the second stage $(p_{12})$. Therefore, the expected cost for a two-stage machine replacement policy can be expressed as follows:

$$E(TC) = cNpf_{13} + R(1 - f_{13}) + I\left((m_{11} - 1) + (m_{22} - 1)p_{12}\right), \quad (16)$$

or

$$E(TC) = cNp\left(\frac{p_{13} + p_{12}p_{23}}{1 - p_{11} - p_{12}p_{21}}\right) + R\left(1 - \frac{p_{13} + p_{12}p_{23}}{1 - p_{11} - p_{12}p_{21}}\right) + I\left(\left(\frac{1}{1 - p_{11} - p_{12}p_{21}} - 1\right) + \left(\frac{1 - p_{11}}{1 - p_{11} - p_{12}p_{21}} - 1\right)p_{12}\right), \quad (17)$$

which leads to:

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$$E(TC)=$$
$$cNp\left(\frac{F_1(c_1)+(1-F_1(c_2))F_2(c_3)}{1-F_1(c_2)+F_1(c_1)-(1-F_1(c_2))(F_2(c_4)-F_2(c_3))}\right)+$$
$$R\left(1-\frac{F_1(c_1)+(1-F_1(c_2))F_2(c_3)}{1-F_1(c_2)+F_1(c_1)-(1-F_1(c_2))(F_2(c_4)-F_2(c_3))}\right)+$$
$$I\left(\begin{array}{l}\left(\frac{1}{1-F_1(c_2)+F_1(c_1)-(1-F_1(c_2))(F_2(c_4)-F_2(c_3))}-1\right)+\\ \left(\frac{1-F_1(c_2)+F_1(c_1)}{1-F_1(c_2)+F_1(c_1)-(1-F_1(c_2))(F_2(c_4)-F_2(c_3))}-1\right)(1-F_1(c_2))\end{array}\right). \quad (18)$$

Then, similar to what was derived in a single stage case, we have

$$\begin{aligned} p=\delta_1 &\rightarrow f_{13}\geq 1-\varepsilon_1 \\ p=\delta_2 &\rightarrow 1-f_{13}\geq 1-\varepsilon_2. \end{aligned} \quad (19)$$

In the next subsection, a numerical example is given to illustrate the application of the proposed methodology.

### 5.1. An Illustrative Example

Consider a two-stage system with the number of produced items in a period, $N=1000$, the probability of producing a defective item, $p=0.15$, the cost of producing a defective item, $c=5$, the cost of machine replacement, $R=600$, the cost of machine inspection and repair, $I=200$, the first sample size in an inspected lot, $n_1=50$, the second sample size, $n_2=40$, $\delta_1=0.1, \delta_2=0.2, \varepsilon_1=0.01,$ and $\varepsilon_2=0.02$.

In order to determine the feasible values of $c_1, c_2, c_3,$ and $c_4$ among existing alternatives, using Eq. (19), we have

$$\begin{aligned} p=0.1 &\rightarrow f_{13}\geq 0.99, \\ p=0.2 &\rightarrow 1-f_{13}\geq 0.98. \end{aligned}$$

In other words, the probability of accepting the machine when $\delta_1=0.1$ should be more than 0.99 and the probability of rejecting the machine when $\delta_2=0.2$ should be more than 0.98. Table 3 shows 16 different alternative combination values of $c_1, c_2, c_3,$ and $c_4$ together with their probabilities of rejecting or accepting the machine, of which the ones in bold are feasible. Then, Eq. (18) is numerically solved for all feasible sets of Table 3. The results are given in Table 4. Based on the results, the best combination value is $c_1=2, c_2=5, c_3=1,$ and $c_4=10$ with the objective function value being 7215.411.

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**Table 3.** The feasible values of $c_1, c_2, c_3,$ and $c_4$

| Probability of rejecting the machine when $\delta_2 = 0.2$ | Probability of accepting the machine when $\delta_1 = 0.1$ | $c_4$ | $c_3$ | $c_2$ | $c_1$ |
|---|---|---|---|---|---|
| 0.998019 | 0.449585 | 5 | 1 | 5 | 1 |
| 0.99771 | 0.947088 | 5 | 1 | 10 | 1 |
| 0.996658 | 0.643003 | 5 | 1 | 5 | 2 |
| 0.994605 | 0.983135 | 5 | 1 | 10 | 2 |
| 0.989752 | 0.991352 | 10 | 1 | 5 | 1 |
| 0.988169 | 0.999602 | 10 | 1 | 10 | 1 |
| 0.982807 | 0.99606 | 10 | 1 | 5 | 2 |
| 0.972484 | 0.999878 | 10 | 1 | 10 | 2 |
| 0.990382 | 0.601092 | 5 | 2 | 5 | 1 |
| 0.990078 | 0.948952 | 5 | 2 | 10 | 1 |
| 0.989042 | 0.713564 | 5 | 2 | 5 | 2 |
| 0.98702 | 0.983329 | 5 | 2 | 10 | 2 |
| 0.951783 | 0.995294 | 10 | 2 | 5 | 1 |
| 0.950318 | 0.999617 | 10 | 2 | 10 | 1 |
| 0.945359 | 0.997148 | 10 | 2 | 5 | 2 |
| 0.935804 | 0.999879 | 10 | 2 | 10 | 2 |

**Table 4.** The expected total cost for different combination values of $c_1, c_2, c_3,$ and $c_4$

| $c_1$ | $c_2$ | $c_3$ | $c_4$ | $E(TC)$ |
|---|---|---|---|---|
| 1 | 5 | 1 | 10 | 9321.193 |
| 1 | 10 | 1 | 10 | 26001.17 |
| 2 | 5 | 1 | 10 | 7215.411 |

## 6. Conclusion

The optimum value of control thresholds in an acceptance sampling design for a machine replacement problem was determined numerically using a Markovian approach. We developed a general model for the expected cost by considering acceptance, replacement, and inspection costs. The model was then used to determine the optimum value of control thresholds for a single-stage and a two-stage replacement strategy. Numerical examples were solved to demonstrate the application of the proposed approach.

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