

## Location problems in regions with $l_p$ and block norms

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We consider the two well known minimax and minisum single facility location problems in the plane  $R^2$  which has been divided into two regions,  $S_1$  and  $S_2$  by a straight line. The two regions are measured by various norms. We focus on three special cases in which the regions  $S_1$  and  $S_2$  are measured by  $l_1$  and  $l_p$  norms,  $l_1$  and block norms, two distinct block norms. Based on the properties of block norms then we use linear or almost linear problems in different cases to achieve the optimal solution.

**Keywords:** Continuous location, Single facility, Block norms, Optimization.

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### 1. Introduction

The minisum and minimax facility location problems are two important problems playing major roles in location theory. In the minisum and minimax single facility problems, we want to find a point  $x$  such that respectively is the sum of the weighted distances and the maximum weighted distances from  $x$  to all given points is minimized, Let  $a_i = (a_{i1}, \dots, a_{in})^T$ , for  $i = 1, \dots, m$ , be the demand points in the plane  $R^n$  and  $w_i$  be a positive weight corresponding to each point  $a_i$ . The minisum and minimax single facility location problems under a considered given norm  $\| \cdot \|$  can respectively be written as follows:

$$\min z = \sum_{i=1}^m w_i \|x - a_i\| \quad (1)$$

and

$$\min z = \max_{i=1, \dots, m} w_i \|x - a_i\|. \quad (2)$$

Here, we focus on a special case that the points are given in the plane  $R^2$  being divided into two regions  $S_1$  and  $S_2$  by a straight line  $L: x = \alpha$ , using different norms. The distance measures in the regions are considered to be  $l_1$ ,  $l_p$  and block norms.

Parlar [7] studied the minisum problem where the dividing line of the plane  $R^2$  was considered to be  $L: y = \beta x$ , with different regions using  $l_2$  and  $l_1$  norms distinctly. He formulated the minisum problem as a mixed integer programming problem and proposed a modified Weiszfeld procedure to solve it. Brimberg et al. [1] extended this problem to a more general case having more than two regions with different norms in the plane  $R^2$  and the dividing lines parallel to the  $y$  axis. They

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showed that the shortest path between two points  $S$  and  $t$  where  $S$  is in the region with  $l_1$  norm and  $t$ , is in the region with  $l_2$  norm, passed through the projection of  $t$  onto the line  $L$ . Zaferanieh et al. [10] considered this case when the dividing line was  $L: y = \beta x$ . They provided a characterization of the crossing points and showed that the optimal solution was the rectangular hull of the existing facilities. They also presented an efficient solution procedure using the big square small square (BSSS) method.

When there exists a bounded region in the plane  $R^2$  which is measured by one norm and the rest of the plane is measured by a different norm, the problem becomes considerably more difficult. This case was considered by Brimberg et al. [2].

Ward and Wendell [8, 9] used block norms to solve the location problems. Using some properties of block norms, they presented linear programming models for the minisum and minimax single facility location problems. For the case with some barrier in the plane, Hamacher and Klamroth [6] and Dearing et al. [4] presented polynomial algorithms to solve the planar location problems.

We investigate the problem of finding the shortest path between two points situated in two distinct regions in Section 2. In Section 3, the minisum and minimax problems are studied extensively, and based on the characterization of block norms as linear models, some solution procedures are proposed. Then, some illustrating examples are given to clarify the proposed methods. Conclusions are given in Section 4.

## 2. The Shortest Path Between Two Points

Here we state a property of the shortest path between two points that are situated in two distinct regions  $S_1$  and  $S_2$  being the partitions of the plane  $R^n$  and are created by a separating hyperplane  $e_i^T x = \alpha$ , where  $e_i$  is the  $i$ th unit vector.

We can see a distinct difference between the  $l_1$  and other norms. In a regular urban region, the street patterns are usually designed grid by perpendicular vertical and horizontal lines in order to simplify the motion and reducing traffic. Moreover,  $l_1$  is an  $e_2$ -block norm to be defined later. These norms have special property that the distance along the vertical and horizontal lines are the same. Also,  $l_1$  is contained in both the  $l_p$  and block norm sets. These properties justify the use of the  $l_1$  norm favorably.

Assume the distance measures in the regions  $S_1$  and  $S_2$  are  $l_1$  and  $l_p$  norms, respectively. In Lemma 2.1, we state a property as an extension of the special case provided by Brimberg et al. [1].

**Lemma 2.1.** *Let  $A = (x_1, \dots, x_n) \in S_1$  and  $D = (y_1, \dots, y_n) \in S_2$  be two arbitrary points. The shortest path from  $A$  to  $D$  passes through the point  $B = (x_1, \dots, x_{i-1}, \alpha, x_{i+1}, \dots, x_n)$ , which is the projection of  $A$  onto the hyperplane  $H = \{x : e_i^T x = \alpha\}$ .*

**Proof.** Let  $C = (x'_1, \dots, x'_{i-1}, \alpha, x'_{i+1}, \dots, x'_n)$ ,  $C \neq B$ , be an arbitrary point on the hyperplane  $H$ . We would show

$$\|A - B\|_1 + \|B - D\|_p \leq \|A - C\|_1 + \|C - D\|_p. \quad (3)$$

By definition of the  $l_p$  norm, we have,

$$\begin{aligned} \|A - B\|_1 + \|B - D\|_p &= |x_i - \alpha| + (|x_1 - y_1|^p + \dots + |x_{i-1} - y_{i-1}|^p \\ &\quad + |\alpha - y_i|^p + |x_{i+1} - y_{i+1}|^p + \dots + |x_n - y_n|^p)^{\frac{1}{p}} \\ &= |x_i - \alpha| + (|x_1 - x'_1 + x'_1 - y_1|^p + \dots + |x_{i-1} - x'_{i-1} + x'_{i-1} - y_{i-1}|^p \\ &\quad + |\alpha - y_i|^p + |x_{i+1} - x'_{i+1} + x'_{i+1} - y_{i+1}|^p + \dots + |x_n - x'_n + x'_n - y_n|^p)^{\frac{1}{p}}. \end{aligned}$$

Employing the triangle inequality, we get

$$\begin{aligned} \|A - B\|_1 + \|B - D\|_p &\leq |x_i - \alpha| + (|x_1 - x'_1|^p + \dots + |x_{i-1} - x'_{i-1}|^p \\ &\quad + |x_{i+1} - x'_{i+1}|^p + \dots + |x_n - x'_n|^p)^{\frac{1}{p}} \\ &\quad + (|x'_1 - y_1|^p + \dots + |x'_{i-1} - y_{i-1}|^p + |\alpha - y_i|^p + |x'_{i+1} - y_{i+1}|^p + \dots + |x'_n - y_n|^p)^{\frac{1}{p}} \\ &\leq |x_i - \alpha| + (|x_1 - x'_1| + \dots + |x_{i-1} - x'_{i-1}| + |x_{i+1} - x'_{i+1}| + \dots + |x_n - x'_n|) \\ &\quad + (|x'_1 - y_1|^p + \dots + |x'_{i-1} - y_{i-1}|^p + |\alpha - y_i|^p + |x'_{i+1} - y_{i+1}|^p + \dots + |x'_n - y_n|^p)^{\frac{1}{p}} \\ &= \|A - C\|_1 + \|C - D\|_p, \end{aligned}$$

completing the proof.  $\square$

Now, consider a different case in which the points are given at the plane  $R^2$  and the distance measures in the regions  $S_1$  and  $S_2$  are  $l_1$  and a block norm,  $l_B$ , respectively. The block norms are those whose contours are polytopes. Ward and Wendell [8, 9] demonstrated that a block norm could be characterized as follows:

$$\|x\|_B = \min \left\{ \sum_{g=1}^r |\lambda_g| : x = \sum_{g=1}^r \lambda_g b_g \right\}, \quad (4)$$

where the vector points  $b_g$  and  $-b_g$ ,  $g = 1, \dots, r$ , constitute the extreme points of the polytope

corresponding to the unit contour. They also presented another characterization of the block norms based on the concept of polar sets as follows:

$$\|x\|_B = \max\{|xb_g^0| : g = 1, \dots, r^0\} \tag{5}$$

where  $b_g^0$  and  $-b_g^0$ ,  $g = 1, \dots, r^0$ , are the extreme vector points of the polar set

$$B^0 = \{v : b_g v \leq 1, g = \pm 1, \pm 2, \dots, \pm r\}.$$

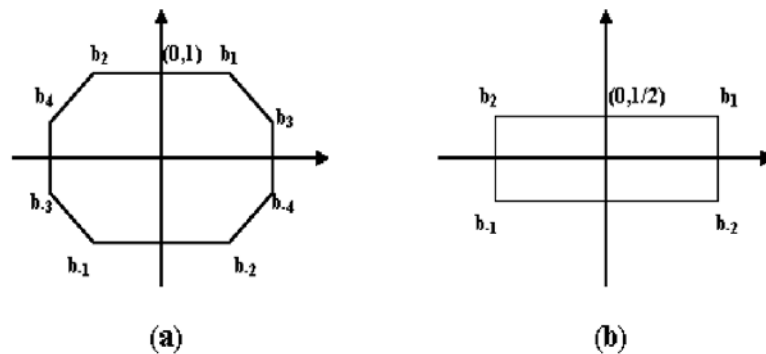
**Definition 2.2.** Let  $v$  be a vector in the plane  $R^2$  and  $l_B$  be a block norm satisfying in the following conditions:

$$(1) \sum_{g=1}^r \lambda_g b_g = v,$$

$$(2) \sum_{g=1}^r \lambda_g = 1.$$

Then,  $l_B$  is called a  $v$ -block norm.

Specially, if  $v$  is  $e_2 = (0,1)$ , then the point  $(0,1)$  is on the contour of any considered  $e_2$ -block norm (see Figure 1). Furthermore, the distance between each pair of points on the vertical axis using any  $e_2$ -block norm is the same. This is a key point that we use for the remainder of our Work.



**Figure 1.** (a) Contour of an  $e_2$ -block norm with 8 extreme points, and (b) contour of a non  $e_2$ -block norm

**Lemma 2.3.** Let  $R^2$  be divided into two regions  $S_1$  and  $S_2$  by a straight line  $L = \{x : x = \alpha\}$ , and the distance measure in  $S_1$  and  $S_2$  are  $l_1$  and  $e_2$ -block norm  $l_B$ , respectively. Also, let  $A \in S_1$  and  $D \in S_2$  be two arbitrary points. Then, the shortest path from  $A$  to  $D$  passes through the point  $E$  which is the projection of  $A$  onto the line  $L$ .

**Proof.** Similar to the proof of Lemma 2.1, let  $C$  be an arbitrary distinct point from  $E$  onto the line  $L$ . We want to show that

$$\|A - E\|_1 + \|E - D\|_B \leq \|A - C\|_1 + \|C - D\|_B. \tag{6}$$

Using the triangle inequality, the following relationship is obtained from (4):

$$\|A - E\|_1 + \|E - D\|_B \leq \|A - E\|_1 + \|E - C\|_B + \|C - D\|_B.$$

Since  $l_1$  is an  $e_2$ -block norm, and for all  $e_2$ -block norms, the distance between each pair of points on the vertical line  $L$  is the same then the following equality is attained:

$$\|A - E\|_1 + \|E - C\|_1 + \|C - D\|_B = \|A - C\|_1 + \|C - D\|_B,$$

completing the proof.  $\square$

### 3. Minisum and Minimax Problems

Here, we consider the single facility minisum and minimax location problems on the plane  $R^2$  being divided into two regions  $S_1$  and  $S_2$  by a straight line  $L = \{x : x = \alpha\}$ . In all of the following subsections, consider  $m$  points  $a_1, \dots, a_m$  with weights given two separate sets  $J_1 = \{i | a_i \in S_1\}$  and  $J_2 = \{i | a_i \in S_2\}$  respectively.

At first, the minisum and minimax problems in three special different cases are studied in detail and some relaxation methods are given to effectively solve the problems. Then, in the next section some explanatory examples are given to clarify the discussions.

#### 3.1 $l_1$ and $l_p$ Norms

Let  $S_1$  and  $S_2$  be measured by  $l_1$  and  $l_p$  norms, respectively. We consider two different cases in which the optimal solution  $x^*$  belongs to the regions  $S_1$  and  $S_2$  respectively (see Figure 2).

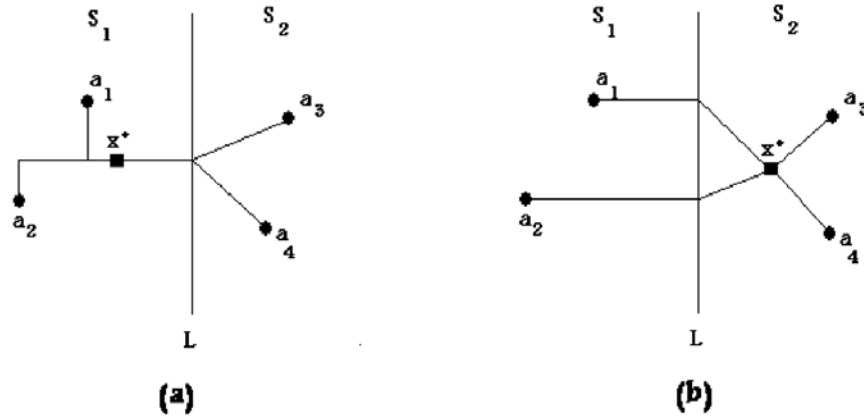


Figure 2. (a)  $x^*$  is in  $S_1$ , and (b)  $x^*$  is in  $S_2$

First, consider the optimal solution  $x^*$  being in the region  $S_2$ . By Lemma 2.1, the minisum problem can be written as:

$$\min z = \sum_{i \in S_2} w_i (|x_1 - a_{i1}|^p + |x_2 - a_{i2}|^p)^{\frac{1}{p}} + \sum_{i \in S_1} w_i |\alpha - a_{i1}| + \sum_{i \in S_1} w_i (|\alpha - a_{i1}|^p + |x_2 - a_{i2}|^p)^{\frac{1}{p}}$$

or

$$\min z = \sum_{i \in S_1 \cup S_2} w_i (|x_1 - c_{i1}|^p + |x_2 - a_{i2}|^p)^{\frac{1}{p}} + K, \tag{7}$$

where  $c_{i1}$  and  $K$  are constants. The solution of this problem is obtained by using a modified Weiszfeld procedure given in [3]. Similar to the minisum problem the minimax problem, can be formulated as:

$$\min z = \max_{i \in S_2} \{ \max_{i \in S_2} w_i [ |x_1 - a_{i1}|^p + |x_2 - a_{i2}|^p ]^{\frac{1}{p}}, \max_{i \in S_1} w_i ( |\alpha - a_{i1}| + [ |x_1 - \alpha|^p + |x_2 - a_{i2}|^p ]^{\frac{1}{p}} ) \}.$$

The solution of this problem is obtained with regard to the following minimax problem which is concerned with only one type of  $l_p$  norm and basically is a nonlinear problem.

$$\min z = \max_{i \in S_1 \cup S_2} w_i [ |x_1 - c_{i1}|^p + |x_2 - a_{i2}|^p ]^{\frac{1}{p}} + K_i, \tag{8}$$

where  $c_{i1}$  and  $K_i$  are appropriately defined based on the previous relation. Now, consider the optimal solution,  $x^*$ , lying in the region  $S_1$ . Then, the minisum problem is given by:

$$\min z = \sum_{i \in S_1} w_i (|x_1 - a_{i1}| + |x_2 - a_{i2}|) + \sum_{i \in S_2} w_i |x_1 - \alpha| + \sum_{i \in S_2} w_i (|\alpha - a_{i1}|^p + |x_2 - a_{i2}|^p)^{\frac{1}{p}}. \tag{9}$$

The problem (9) is separable with respect to the variables  $x_1$  and  $x_2$ . Therefore, the optimal solutions of the two following one variable problems give the optimal solution of the problem (9):

$$\min z_1 = \sum_{i \in S_1} w_i |x_1 - a_{i1}| + \sum_{i \in S_2} w_i |x_1 - \alpha| \tag{10}$$

and

$$\min z_2 = \sum_{i \in S_1} w_i |x_2 - a_{i2}| + \sum_{i \in S_2} w_i [ |\alpha - a_{i1}|^p + |x_2 - a_{i2}|^p ]^{\frac{1}{p}}. \tag{11}$$

After considering the minisum problem, we turn our attention back to the minimax problem:

$$\min z = \max \{ \max_{i \in S_1} w_i [ |x_1 - a_{i1}| + |x_2 - a_{i2}| ], \max_{i \in S_2} w_i ( |x_1 - \alpha| + [ |\alpha - a_{i1}|^p + |x_2 - a_{i2}|^p ]^{\frac{1}{p}} ) \}.$$

This problem can be clearly reformulated as an almost linear programming problem with only one type of nonlinear constraints.

$$\begin{aligned}
& \min z && (12) \\
& \text{s.t.} \\
& w_i(y_1 + y_2) \leq z, \quad i \in S_1 \\
& x_1 - a_{i1} \leq y_1, \quad i \in S_1 \\
& -x_1 + a_{i1} \leq y_1, \quad i \in S_1 \\
& x_2 - a_{i2} \leq y_2, \quad i \in S_1 \\
& -x_2 + a_{i2} \leq y_2, \quad i \in S_1 \\
& x_1 - \alpha \leq y_3, \quad i \in S_2 \\
& -x_1 + \alpha \leq y_3, \quad i \in S_2 \\
& w_i(|\alpha - a_{i1}|^p + |x_2 - a_{i2}|^p)^{\frac{1}{p}} + y_3 \leq z, \quad i \in S_2.
\end{aligned}$$

Since this problem is a convex optimization problem, then any local solution is also the global solution and convex optimization methods can be used to solve this problem.

### 3.2 $l_1$ and $e_2$ -Block Norms

Let the regions  $S_1$  and  $S_2$  be measured with  $l_1$  and  $e_2$ -block norm  $l_B$ , and  $a'_i$  be the projection of  $a_i$  onto the line  $L$ . If the optimal solution lies in the region  $S_2$  then the minisum and minimax problems can be respectively stated as follows:

$$\min z = \sum_{i \in J_1} w_i |\alpha - a_{i1}| + \sum_{i \in S_1} w_i \|x - a'_i\|_B + \sum_{i \in S_2} w_i \|x - a_i\|_B \quad (13)$$

and

$$\min z = \max\left\{\max_{i \in S_1} w_i (|\alpha - a_{i1}| + \|x - a'_i\|_B), \max_{i \in S_2} w_i \|x - a_i\|_B\right\}. \quad (14)$$

Note that  $|\alpha - a_{i1}|$  is a constant, and value therefore both problems (13) and (14) can be rewritten as linear programming problems by using the relations (4) and (5) (see [9]).

Now, consider the case that the optimal solution lies in region  $S_1$ . Let  $x'$  be the projection of  $x$  onto the line  $L: x = \alpha$ . Then, the minisum and minimax formulas are devised as follows respectively:

$$\min z = \sum_{i \in S_1} w_i [|x_1 - a_{i1}| + |x_2 - a_{i2}|] + \sum_{i \in S_2} w_i (\|x' - a_i\|_B + |x_1 - \alpha|) \quad (15)$$

and

$$\min z = \max\left\{\max_{i \in S_1} w_i (|x_1 - a_{i1}| + |x_2 - a_{i2}|), \max_{i \in S_2} w_i (\|x' - a_i\|_B + |x_1 - \alpha|)\right\}. \quad (16)$$

Since  $x'_1 = \alpha$ , then the solution of the problem (15) can be obtained by the investigation of problems (10) and (17):

$$\min z = \sum_{i \in S_1} w_i |x_2 - a_{i2}| + \sum_{i \in S_2} w_i \|x' - a_i\|_B. \tag{17}$$

The minimax problem (16) can be rewritten in the following convenient form:

$$\min z \tag{18}$$

*s.t.*

$$\begin{aligned} w_i(y_1 + y_2) &\leq z, \quad i \in S_1 \\ x_1 - a_{i1} &\leq y_1, \quad i \in S_1 \\ -x_1 + a_{i1} &\leq y_1, \quad i \in S_1 \\ x_2 - a_{i2} &\leq y_2, \quad i \in S_1 \\ -x_2 + a_{i2} &\leq y_2, \quad i \in S_1 \\ x_1 - \alpha &\leq y_3, \quad i \in S_2 \\ -x_1 + \alpha &\leq y_3, \quad i \in S_2 \\ w_i(\|x' - a_i\|_B + y_3) &\leq z, \quad i \in S_2. \end{aligned}$$

Problems (17) and (18) can be easily changed to linear and can readily be solved (see Section 3.3).

### 3.3. Two Block Norms

Let  $S_1$  and  $S_2$ , the regions beside line  $L$ , be measured by two block norms,  $l_{B_1}$  and  $l_{B_2}$ , respectively. Without loss of generality assume that the optimal solution  $x^*$  be included in the region  $S_1$ . Let  $u_i = (\alpha, u_{i2})$  be a point on the line  $L$  such that the shortest path from  $x^*$  to  $a_i$  passes through  $z_i$ . The minimum and minimax problems are given as follows:

$$\min z = \sum_{i \in S_1} w_i \|x - a_i\|_{B_1} + \sum_{i \in S_2} w_i (\|u_i - a_i\|_{B_2} + \|x - u_i\|_{B_1}) \tag{19}$$

and

$$\min z = \max \left\{ \max_{i \in S_1} w_i \|x - a_i\|_{B_1}, \max_{i \in S_2} w_i (\|u_i - a_i\|_{B_2} + \|x - u_i\|_{B_1}) \right\}. \tag{20}$$

The block norms within the problems (19) and (20) can be replaced by the following equalities with respect to the relation (5):

$$\begin{aligned} y_i &= \|x - a_i\|_{B_1} = \max \{ |(x - a_i) b_{g_1}^0| : g_1 = 1, \dots, r_1^0 \} \\ y'_i &= \|u_i - a_i\|_{B_2} = \max \{ |(u_i - a_i) b_{g_2}^0| : g_2 = 1, \dots, r_2^0 \} \end{aligned}$$

$$y_i'' = \|x - u_i\|_{B_1} = \max\{(x - u_i)b_{g_1}^0 \mid g_1 = 1, \dots, r_1^0\}.$$

So, the two following minisum and minimax linear programming problems, (21) and (22), are respectively derived:

$$\min z = \sum_{i \in S_1} w_i y_i + \sum_{i \in S_2} w_i (y'_i + y''_i) \quad (21)$$

*s t.*

$$\begin{aligned} (x - a_i)b_{g_1}^0 &\leq y_i, \quad i \in J_1, g_1 = 1, \dots, r_1^0 \\ -(x - a_i)b_{g_1}^0 &\leq y_i, \quad i \in J_1, g_1 = 1, \dots, r_1^0 \\ (u_i - a_i)b_{g_2}^0 &\leq y'_i, \quad i \in J_2, g_2 = 1, \dots, r_2^0 \\ -(u_i - a_i)b_{g_2}^0 &\leq y'_i, \quad i \in J_2, g_2 = 1, \dots, r_2^0 \\ (u_i - x)b_{g_1}^0 &\leq y''_i, \quad i \in J_2, g_1 = 1, \dots, r_1^0 \\ -(u_i - x)b_{g_1}^0 &\leq y''_i, \quad i \in J_2, g_1 = 1, \dots, r_1^0 \end{aligned}$$

and

$$\min z \quad (22)$$

*s t.*

$$\begin{aligned} w_i(x - a_i)b_{g_1}^0 &\leq z, \quad i \in J_1, g_1 = 1, \dots, r_1^0 \\ -w_i(x - a_i)b_{g_1}^0 &\leq z, \quad i \in J_1, g_1 = 1, \dots, r_1^0 \\ w_i(y'_i - y''_i) &\leq z, \quad i \in J_2 \\ (u_i - a_i)b_{g_2}^0 &\leq y'_i, \quad i \in J_2, g_2 = 1, \dots, r_2^0 \\ -(u_i - a_i)b_{g_2}^0 &\leq y'_i, \quad i \in J_2, g_2 = 1, \dots, r_2^0 \\ (u_i - x)b_{g_1}^0 &\leq y''_i, \quad i \in J_2, g_1 = 1, \dots, r_1^0 \\ -(u_i - x)b_{g_1}^0 &\leq y''_i, \quad i \in J_2, g_1 = 1, \dots, r_1^0. \end{aligned}$$

Problems (21) and (22) can be readily solved using linear programming.

### 3.4. Sufficient Conditions

Note that for all of the considered problems, in order to check whether the optimal solution lies in the region  $S_1$  or  $S_2$ , we need to solve two problems, one for each region, and compare the obtained results to get the optimal solution. Fortunately, in a special case discussed in Lemma 3.1 below, sufficient conditions exist, under which solution of only one problem gives the optimal solution.

**Lemma 3.1.** *Let the occupying norm in the region  $S_1$  be  $l_1$  and in  $S_2$  be  $l_p$  or  $e_2$ -block norm,  $W_1 = \sum_{i \in S_1} w_i$  and  $W_2 = \sum_{i \in S_2} w_i$ . If  $W_2 \geq W_1$ , then  $x^*$ , the optimal solution of the single facility minisum problem, is included in  $S_2 \cup L$ .*

**Proof.** If  $x^* \in S_1$ , then the solution of (10) implies  $x_1^* = \alpha$ . Thus,  $x^* \in S_2 \cup L$ .  $\square$

### 3.5. Examples

Now, we give some illustrating examples to clarify the above discussion. First, a small example with 3 points is given to specify the problem formulation. Then, an example with 18 points is considered. For these two examples, we assume that the given points are distributed in the plane  $R^2$  and the dividing line,  $L: x = 0$ , belongs to  $S_1$  which is measured with the  $l_1$  norm. For both examples, two cases of the region  $S_2$  being measured with  $l_p$  and  $l_B$  norms are considered. The considered  $l_B$  norm is an  $e_2$ -block norm with the extreme points of its contour and its polar set as shown in Table 1.

**Table 1.** The extreme points of  $l_B$  and corresponding polar set

extreme points of $l_B$	extreme points of polar set
$b_1 = (0,1)$	$b_1^0 = (\frac{\sqrt{3}}{3}, 1)$
$b_2 = (\frac{\sqrt{3}}{2}, \frac{1}{2})$	$b_2^0 = (1, 2 - \sqrt{3})$
$b_3 = (1,0)$	$b_3^0 = (1, -2 + \sqrt{3})$
$b_4 = (\frac{\sqrt{3}}{2}, -\frac{1}{2})$	$b_4^0 = (\frac{\sqrt{3}}{3}, -1)$
$b_{-1} = (0,-1)$	$b_{-1}^0 = (-\frac{\sqrt{3}}{3}, -1)$
$b_{-2} = (-\frac{\sqrt{3}}{2}, -\frac{1}{2})$	$b_{-2}^0 = (-1, -2 + \sqrt{3})$
$b_{-3} = (-1,0)$	$b_{-3}^0 = (-1, 2 - \sqrt{3})$
$b_{-4} = (-\frac{\sqrt{3}}{2}, \frac{1}{2})$	$b_{-4}^0 = (-\frac{\sqrt{3}}{3}, 1)$

**Example 1.** Let  $a_1 = (-1, 1) \in S_1$ ,  $a_2 = (-2, 0) \in S_1$  and  $a_3 = (1, 1) \in S_2$  with relevant weights  $w_1 = w_2 = 1$  be given. We solve the minisum problem in three different cases with  $w_3 = 1, 1.5, 2$ .

In the two cases of  $w_3 = 1$  and  $w_3 = 1.5$ , the sum of the weights,  $W = \sum_{i=1}^3 w_i$ , does not satisfy the condition of Lemma 3.1, and so obtaining the optimal solution requires solving the two problems and getting the best solution of the two regions  $S_1$  and  $S_2$ . But for the case  $w_3 = 2$ , the condition of Lemma 3.1 is satisfied, and therefore an optimal solution lies in the region  $S_2$ . Consider the following two cases.

(1) Let the region  $S_2$  be measured by an  $l_p$  norm. It is necessary to solve the problem in the two foregoing cases to get the optimal solution.

(a) To obtain the best solution  $x \in S_1$ , it is needed to solve the following problems:

$$\min z_{11} = |x_1 + 1| + |x_1 + 2| + w_3 |x_1|$$

and

$$\min z_{12} = |x_2 - 1| + |x_2| + w_3 (1 + |x_2 - 1|^p)^{\frac{1}{p}}.$$

We assume the cases, corresponding to  $p = 2, 3, 10, 100, \infty$  and  $w_3 = 1, 1.5, 2$ . In all cases the best solution in the region  $S_1$  is  $x = (-1, 1)$  with the objective function values as presented in Table 2.

**Table 2.** The results of Example 1 in the case  $x \in S_1$

$w_3$	$p = 2, 3, 10, 100, \infty$		
	$z_{11}$	$z_{12}$	$z_1 = z_{11} + z_{12}$
1	2	2	4
1.5	2.5	2.5	5
2	3	3	6

(b) To obtain the best solution  $x \in S_2$ , the following problem needs to be solved:

$$\min z_2 = (x_1^p + x_2^p)^{\frac{1}{p}} + (x_1^p + |x_2 - 1|^p)^{\frac{1}{p}} + w_3 (|x_1 - 1|^p + |x_2 - 1|^p)^{\frac{1}{p}} + 3.$$

Table 3 contains the results for this problem using different values of  $p$  and  $w_3$ . In the three cases, where  $w_3 = 1, 1.5, 2$  and  $p = \infty$ , the two points  $(0.5, 0.5)$  and  $(1, 1)$  are optimal solutions lying in the region  $S_2$ .

**Table 3.** Results of the  $l_1$  and  $l_p$  for the case  $x \in S_2$  in Example 1

p	$w_3 = 1$			$w_3 = 1.5$			$w_3 = 2$		
	$x_1$	$x_2$	$z_2$	$x_1$	$x_2$	$z_2$	$x_1$	$x_2$	$z_2$
2	0.21	0.79	4.93	0.46	0.81	5.29	1	1	5.41
3	0.35	0.65	4.81	0.51	0.70	5.11	1	1	5.26
10	0.47	0.53	4.59	0.51	0.55	4.86	1	1	5.07
100	0.49	0.50	4.51	0.50	0.50	4.76	1	1	5
$\infty$	0.50	0.50	4.50	0.50	0.50	4.75	1	1	5

Table 4 contains the optimal solutions of the main problem, i.e.,  $z^* = \min\{z_1, z_2\}$ . In the case  $w_3 = 2$ , the optimal solution lies in the region  $S_2$ , as expected. For the values  $p = 10, 100, \infty$  and  $w_3 = 1.5$ , in spite of the fact that the sum of the weights in the region  $S_1$  is more than the weights in the region  $S_2$ , the optimal solutions are included in the region  $S_2$ . This result illustrates that the condition of Lemma 3.1 is sufficient but not necessary.

**Table 4.** Optimal solutions of Example 1 with  $l_1$  and  $l_p$  norms

p	$w_3 = 1$			$w_3 = 1.5$			$w_3 = 2$		
	$x_1^*$	$x_2^*$	$z^*$	$x_1^*$	$x_2^*$	$z^*$	$x_1^*$	$x_2^*$	$z^*$
2	-1	1	4	-1	1	5	1	1	5.41
3	-1	1	4	-1	1	5	1	1	5.26
10	-1	1	4	0.51	0.55	4.86	1	1	5.07
100	-1	1	4	0.50	0.50	4.76	1	1	5.00
$\infty$	-1	1	4	0.50	0.50	4.75	1	1	5

(2) Let the measuring norm in the region  $S_2$  be the foregoing mentioned  $l_B$  norm. The two following cases must be traced.

(a) Find the best point  $x \in S_1$ , obtained by solving the following two problems:

$$\min z_{11} = |x_1 + 1| + |x_1 + 2| + w_3 |x_1|$$

and

$$\min z_{12} = |x_2 - 1| + |x_2| + w_3 y$$

s.t.:

$$-\frac{\sqrt{3}}{3} - 1 + x_2 \leq y$$

$$\frac{\sqrt{3}}{3} + 1 - x_2 \leq y$$

$$\frac{\sqrt{3}}{3} - 1 + x_2 \leq y$$

$$\begin{aligned}
-\frac{\sqrt{3}}{3} + 1 - x_2 &\leq y \\
-1 + (x_2 - 1)(2 - \sqrt{3}) &\leq y \\
1 - (x_2 - 1)(2 - \sqrt{3}) &\leq y \\
-1 + (x_2 - 1)(-2 + \sqrt{3}) &\leq y \\
1 + (x_2 - 1)(-2 + \sqrt{3}) &\leq y \\
y &\geq 0.
\end{aligned}$$

(b) Finding the best point  $x \in S_2$ , obtained by solving the following problem:

$$\min z_2 = y_1 + y_2 + w_3 y_3 + 3$$

s.t.:

$$\begin{aligned}
\pm (x_1, x_2 - 1) \begin{pmatrix} \frac{\sqrt{3}}{3} \\ 1 \end{pmatrix} &\leq y_1 \\
\pm (x_1, x_2 - 1) \begin{pmatrix} \frac{\sqrt{3}}{-1} \\ 1 \end{pmatrix} &\leq y_1 \\
\pm (x_1, x_2 - 1) \begin{pmatrix} 1 \\ 2 - \sqrt{3} \end{pmatrix} &\leq y_1 \\
\pm (x_1, x_2 - 1) \begin{pmatrix} -1 \\ 2 - \sqrt{3} \end{pmatrix} &\leq y_1 \\
\pm (x_1, x_2) \begin{pmatrix} \frac{\sqrt{3}}{3} \\ 1 \end{pmatrix} &\leq y_2 \\
\pm (x_1, x_2) \begin{pmatrix} \frac{\sqrt{3}}{-1} \\ 1 \end{pmatrix} &\leq y_2 \\
\pm (x_1, x_2) \begin{pmatrix} 1 \\ 2 - \sqrt{3} \end{pmatrix} &\leq y_2 \\
\pm (x_1, x_2) \begin{pmatrix} -1 \\ 2 - \sqrt{3} \end{pmatrix} &\leq y_2 \\
\pm (x_1 - 1, x_2 - 1) \begin{pmatrix} \frac{\sqrt{3}}{3} \\ 1 \end{pmatrix} &\leq y_3 \\
\pm (x_1 - 1, x_2 - 1) \begin{pmatrix} \frac{\sqrt{3}}{-1} \\ 1 \end{pmatrix} &\leq y_3 \\
\pm (x_1 - 1, x_2 - 1) \begin{pmatrix} 1 \\ 2 - \sqrt{3} \end{pmatrix} &\leq y_3 \\
\pm (x_1 - 1, x_2 - 1) \begin{pmatrix} -1 \\ 2 - \sqrt{3} \end{pmatrix} &\leq y_3 \\
y_1, y_2, y_3 &\geq 0.
\end{aligned}$$

Table 5 shows the obtained results for the different regions  $S_1$  and  $S_2$ . In the case  $w_3 = 2$ , the weights of the points satisfy in the condition in Lemma 3.1, and therefore the optimal solution lies in the region  $S_2$ , as shown in the table.

**Table 5.** The results for Example 1

$w_3$	$x \in S_1$				$x \in S_2$			$x_1^*$	$x_2^*$	$z^*$
	$x_1$	$z_{11}$	$x_2$	$z_{12}$	$x_1$	$x_2$	$z_2$			
1	-1	2	1	2	0	1	5	-1	1	4
1.50	-1	2.50	1	2.50	0.50	0.71	5.43	-1	1	5
2	-1	3	1	3	1	1	5.57	1	1	5.57

**Example 2.** consider 18 points as shown in Table 6.

**Table 6.** The points for Example 2

points in $S_1$	(-3,3)	(-3,0)	(-3,-2)	(-2,2)	(-2,-1)	(-1,4)	(-1,1)	(-1,0)	(-1,-2)
points in $S_2$	(1,3)	(1,1)	(1,-1)	(2,2)	(2,0)	(2,-2)	(3,4)	(3,-1)	(4,0)

Two cases are considered: (1) the weights of all points equal to one, (2) the weights of all points equal to one except for the point  $a_1 = (-3,3)$  with  $w_1 = 5$ . In the first case, the sum of the weights satisfies the condition of Lemma 3.1, and so the optimal solution  $x^*$  lies in the region  $S_2$ . But in the second case, the condition of Lemma 3.1 is not satisfied, and therefore it is not known where the optimal solution may lie.

(1) Let the region  $S_2$  be measured by the norm  $l_p$  with  $p = 2,3,10,100$ . The problem needs to be solved in two cases to get the optimal solution.

To find the best solution  $x \in S_1$ , we have to consider the models (10) and (11). The model (10) is independent of  $p$  and for all cases of  $p$  the solution is the same. To find the best point  $x \in S_2$ , it is required to consider model (7). The results for  $w_1 = 1$  and  $w_1 = 5$  and different values of  $p$  are presented in Tables 7 and 8.

**Table 7.** The results corresponding to the  $l_1$  and  $l_p$  norms for Example 2 with  $w_1 = 1$

$p$	$x \in S_1$					$x \in S_2$		
	$x_1$	$z_{11}$	$x_2$	$z_{12}$	$z_1 = z_{11} + z_{12}$	$x_1$	$x_2$	$z_2$
2	-0.9967	17	0.1961	40.7674	57.7674	0.8444	0.5192	55.2776
3	-0.9967	17	0.2790	39.0373	56.0373	0.9135	0.66398	53.1641
10	-0.9967	17	0.1427	37.3880	54.3880	0.9930	0.9149	51.4627
100	-0.9967	17	0.01517	37.0346	54.0346	0.9979	0.9817	51.0323

**Table 8.** The results corresponding to the  $l_1$  and  $l_p$  norms for Example 2 with  $w_1 = 5$ .

$P$	$x \in S_1$					$x \in S_2$		
	$x_1$	$z_{11}$	$x_2$	$z_{12}$	$z_1 = z_{11} + z_{12}$	$x_1$	$x_2$	$z_2$
2	-1	25	1.03	49.52	74.03	0.71	1.13	76.43
3	-1	25	1.18	47.63	72.63	0.89	1.12	73.68
10	-1	25	1.44	46.10	71.10	1.01	1.07	71.48
100	-1	25	1.46	46.00	71.00	1.00	1.02	71.03

(2) Let the norm in the region  $S_2$  be the previously mentioned  $l_B$  norm. To obtain the optimal point, the two following cases are considered. First, the best solution belongs to region  $S_1$  is found by employing the two problems (10) and (17). Then, the best solution of the region  $S_2$  is obtained from the problem (13). Table 9 give the results.

**Table 9.** The results corresponding to  $l_1$  and  $l_B$  for the Example 2

$w_1$	$x \in S_1$					$x \in S_2$		
	$x_1$	$z_{11}$	$x_2$	$z_{12}$	$z_1 = z_{11} + z_{12}$	$x_1$	$x_2$	$z_2$
1	-0.10	17	0.42	42.96	59.96	0.86	0.50	59.04
5	-1.00	25	1.00	51.58	76.58	0.00	1.00	80.57

The results show that for the case  $w_1 = 1$ , the optimal solutions lie in the region  $S_2$ , satisfying the condition of Lemma 3.1, and for the case  $w_1 = 5$ , the optimal solutions lies in the region  $S_1$ .

#### 4. Summary and Conclusion

We considered the single facility minisum and minimax location problems on the plane  $R^2$  being divided into two regions  $S_1$  and  $S_2$  by a straight line  $L: x = \alpha$ . The regions  $S_1$  and  $S_2$  were considered to be equipped with different norms. First, a special case with the distance measure in one region being the  $l_1$  norm and in the other region being the  $l_p$  norm was discussed. Then other cases considering the  $l_1$  norm and a block norm or two block norms measuring the, two regions were studied. Our formulation policy was to replace the  $l_1$  norm and the block norms by linear formulations. Illustrating examples were given to clarify the solution methods.

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