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Location problems in regions with l_p and block norms

J. Fathali[1,*], M. Zaferanieh[2]

We consider the two well knownminimax and minisum single facility location problems in the plane R^2 which has been divided into two regions, S_1 and S_2 by a straight line. The two regions are measured by various norms. We focus on three special cases in which the regions S_1 and S_2 are measured by l_1 and l_p norms, l_1 and block norms, two distinct block norms. Based on the properties of block norms then we use linear or almost linear problems in different cases to achieve the optimal solution.

Keywords: *Continuous location, Single facility, Block norms, Optimization*.

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1. Introduction

The minisum and minimax facility location problems are two important problems playing major roles in location theory. In the minisum and minimax single facility problems, we want to find a point x such thatrespectively isthe sum of the weighted distances and the maximum weighted distances from x to all given points is minimized, Let $a_i = (a_{i1},...,a_{in})$T, for $i = 1,...,m$, be the demand points in the plane R^n and w_i be a positive weight corresponding to each point a_i. The minisum and minimax single facility location problems under a considered given norm $\|\,.\|$ can respectively be written as follows:

$$\min \quad z = \sum_{i=1}^{m} w_i \| x - a_i \| \tag{1}$$

and

$$\min \quad z = \max_{i=1,...m} w_i \| x - a_i \|. \tag{2}$$

Here, we focus on a special case that the points are given in the plane R^2 being divided into two regions S_1 and S_2 by a straight line $L : x = \alpha$,using different norms. The distance measures in the regions are considered to be l_1, l_p and block norms.

Parlar [7] studied the minisum problem where the dividing line of the plane R^2 wasconsidered to be $L : y = \beta x$, with different regions using l_2 and l_1 norms distinctly. He formulated the minisum problem as a mixed integer programming problem and proposed a modified Weiszfeld procedure to solve it. Brimberg et al. [1] extended this problem to a more general case having more than two regions with different norms in the plane R^2 and the dividing lines parallel to the y axis. They

* Corresponding author.
[1] Department of Mathematics, Shahrood University of Technology, University Blvd., Shahrood, Iran. Email: fathali@shahroodut.ac.ir
[2] Department of Mathematics, SabzevarTarbiatMoallem University, Tovhid town, Sabzevar, Iran. Email: zaferanieh@localhost.sttu.ac.ir

showed that the shortest path between two points s and t where s is in the region with l_1 norm and t, is in the region with l_2 norm, passed through the projection of t onto the line L. Zaferanieh et al. [10] considered this case when the dividing line was $L: y = \beta x$. They provided a characterization of the crossing points and showed that the optimal solution was the rectangular hull of the existing facilities. They also presented an efficient solution procedure using the big square small square (BSSS) method.

When there exists a bounded region in the plane R^2 which is measured by one norm and the rest of the plane is measured by a different norm, the problem becomes considerably more difficult. This case was considered by Brimberg et al. [2].

Ward and Wendell [8, 9] used block norms to solve the location problems Using some properties of block norms, they presented linear programming models for the minisum and minimax single facility location problems. For the case with some barrier in the plane, Hamacher and Klamroth [6] and Dearing et al. [4] presented polynomial algorithms to solve the planar location problems.

We investigate the problem of finding the shortest path between two points situated in two distinct regions in Section 2. In Section 3, the minisum and minimax problems are studied extensinely, and based on the characterization of block norms as linear models, some solution procedures are proposed. Then, some illustratingexamples are given to clarify the proposed methods. Conclusions are given in Section 4.

2.The Shortest Path Between Two Points

Here we state a property of the shortest path between two points that are situated in two distinct regions S_1 and S_2 being the partitions of the plane R^n and are created by a separating hyperplane $e_i^T x = \alpha$, where e_i is the*i*thunit vector.

We can see a distinct difference between the l_1 and other norms. In a regular urban region, the street patterns are usually designed grid by perpendicular vertical and horizontal lines in order to simplify the motion and reducing traffic. Moreover, l_1 is an e_2-block norm to be defined later. These norms have special propertythat the distance along the vertical and horizontal lines are the same. Also, l_1 is contained in both the l_p and block norm sets. These properties justifythe use of the l_1 norm favorably.

Assume the distance measures in the regions S_1 and S_2 are l_1 and l_p norms, respectively. In Lemma 2.1, we state a property as an the extension of the special case provided by Brimberg et al. [1].

Lemma 2.1. *Let* $A = (x_1,\dots,x_n) \in S_1$ *and* $D = (y_1,\dots,y_n) \in S_2$ *be two arbitrary points. The shortest path from* A *to* D *passes through the point* $B = (x_1,\dots,x_{i-1},\alpha,x_{i+1},\dots,x_n)$, *which is the projection of* A *onto the hyperplane* $H = \{x : e_i^T x = \alpha\}$.

Proof. Let $C=(x'_1,\ldots,x'_{i-1},\alpha,x'_{i+1},\ldots,x'_n)$, $C\neq B$, be an arbitrary point on the hyperplane H. We would show

$$\|A-B\|_1+\|B-D\|_p\leq\|A-C\|_1+\|C-D\|_p. \tag{3}$$

By definition of the l_p norm,we have,

$$\|A-B\|_1+\|B-D\|_p=|x_i-\alpha|+(|x_1-y_1|^p+\ldots+|x_{i-1}-y_{i-1}|^p$$

$$+|\alpha-y_i|^p+|x_{i+1}-y_{i+1}|^p+\ldots+|x_n-y_n|^p)^{\frac{1}{p}}$$

$$=|x_i-\alpha|+(|x_1-x'_1+x'_1-y_1|^p+\ldots+|x_{i-1}-x'_{i-1}+x'_{i-1}-y_{i-1}|^p$$

$$+|\alpha-y_i|^p+|x_{i+1}-x'_{i+1}+x'_{i+1}-y_{i+1}|^p+\ldots+|x_n-x'_n+x'_n-y_n|^p)^{\frac{1}{p}}.$$

Employing the triangle inequality, we get

$$\|A-B\|_1+\|B-D\|_p\leq|x_i-\alpha|+(|x_1-x'_1|^p+\ldots+|x_{i-1}-x'_{i-1}|^p$$

$$+|x_{i+1}-x'_{i+1}|^p+\ldots+|x_n-x'_n|^p)^{\frac{1}{p}}$$

$$+(|x'_1-y_1|^p+\ldots+|x'_{i-1}-y_{i-1}|^p+|\alpha-y_i|^p+|x'_{i+1}-y_{i+1}|^p+\ldots+|x'_n-y_n|^p)^{\frac{1}{p}}$$

$$\leq|x_i-\alpha|+(|x_1-x'_1|+\ldots+|x_{i-1}-x'_{i-1}|+|x_{i+1}-x'_{i+1}|+\ldots+|x_n-x'_n|)$$

$$+(|x'_1-y_1|^p+\ldots+|x'_{i-1}-y_{i-1}|^p+|\alpha-y_i|^p+|x'_{i+1}-y_{i+1}|^p+\ldots+|x'_n-y_n|^p)^{\frac{1}{p}}$$

$$=\|A-C\|_1+\|C-D\|_p,$$

completing the proof. □

Now, consider a different case in which the points are given at the plane R^2 and the distance measures in the regions S_1 and S_2 are l_1 and a block norm, l_B, respectively. The block norms are those whose contours are polytopes. Ward and Wendell [8, 9] demonstrated that a block norm could be characterized as follows:

$$\|x\|_B=\min\{\sum_{g=1}^{r}|\lambda_g|: x=\sum_{g=1}^{r}\lambda_g b_g\}, \tag{4}$$

where the vector points b_g and $-b_g$, $g=1,\ldots,r$, constitute the extreme points of the polytope corresponding to the unit contour. They also presented another characterization of the block norms based on the concept of polar sets as follows:

$$\| x \|_B = \max\{| x b_g^0 |: g = 1,...,r^0 \} \tag{5}$$

where b_g^0 and $-b_g^0$, $g = 1,...,r^0$, are the extreme vector points of the polar set $B^0 = \{v : b_g v \le 1, g = \pm 1, \pm 2, ..., \pm r\}$.

Definition 2.2. *Let v be a vector in the plane R^2 and l_B be a block norm satisfiying in the following conditions:*

(1) $\sum_{g=1}^{r} \lambda_g b_g = v$,

(2) $\sum_{g=1}^{r} \lambda_g = 1$.

Then, l_B is called a v-*block norm*.

Specially, if v is $e_2 = (0,1)$, then the point $(0,1)$ is on the contour of any considered e_2-block norm (see Figure 1). Furthermore, the distance between each pair of points on the vertical axis using any e_2-block norm is the same. This is a key point that we use for the remainder of our Work.

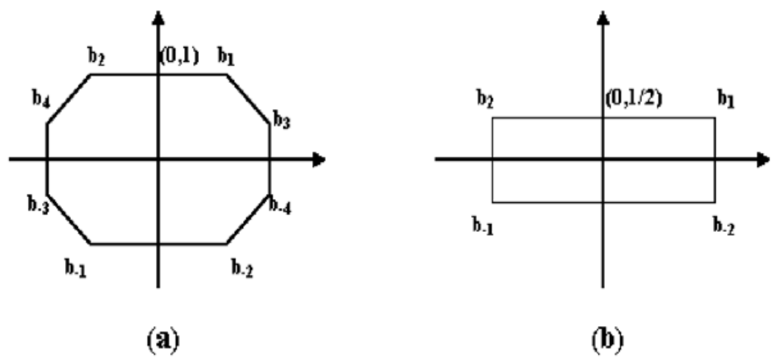

Figure 1. *(a) Contour of an e_2-block norm with 8 extreme points,and(b) contour of a non e_2-block norm*

Lemma 2.3. *Let R^2 be divided into two regions S_1 and S_2 by a straight line $L = \{x : x = \alpha\}$,and the distance measure in S_1 and S_2 are l_1 and e_2-block norm l_B ,respectively. Also, let $A \in S_1$ and $D \in S_2$ be two arbitrary points. Then, the shortest path from A to D passes through the point E which is the projection of A onto the line L.*

Proof. Similar to the proof of Lemma 2.1, let C be an arbitrary distinct point from E onto the line L. We want to show that

$$\| A - E \|_1 + \| E - D \|_B \le \| A - C \|_1 + \| C - D \|_B . \tag{6}$$

Using the triangle inequality, the following relationship is obtained from (4):

$$\| A-E \|_1 + \| E-D \|_B \le \| A-E \|_1 + \| E-C \|_B + \| C-D \|_B .$$

Since l_1 is an e_2-block norm, and for all e_2-block norms, the distance between each pair of points on the vertical line L is the same then the following equality is attained:

$$\| A-E \|_1 + \| E-C \|_1 + \| C-D \|_B = \| A-C \|_1 + \| C-D \|_B ,$$

completing the proof. □

3. Minisum and Minimax Problems

Here,we consider the single facility minisum and minimax location problems on the plane R^2 being divided into two regions S_1 and S_2 by a straight line $L = \{x : x = \alpha\}$. In all of the following subsections,consider m points $a_1,...,a_m$ with weights given two separate sets $J_1 = \{i \mid a_i \in S_1\}$ and $J_2 = \{i \mid a_i \in S_2\}$ respectively.

At first, the minisum and minimax problems in three special different cases are studied in detail and some relaxation methods are given to effectively solve the problems. Then, in the next section some explanatory examples are given to clarify the discussions.

3.1 l_1 and l_p Norms

Let S_1 and S_2 be measured by l_1 and l_p norms, respectively. We consider two different cases in which the optimal solution x^* belongs to the regions S_1 and S_2 respectively (see Figure 2).

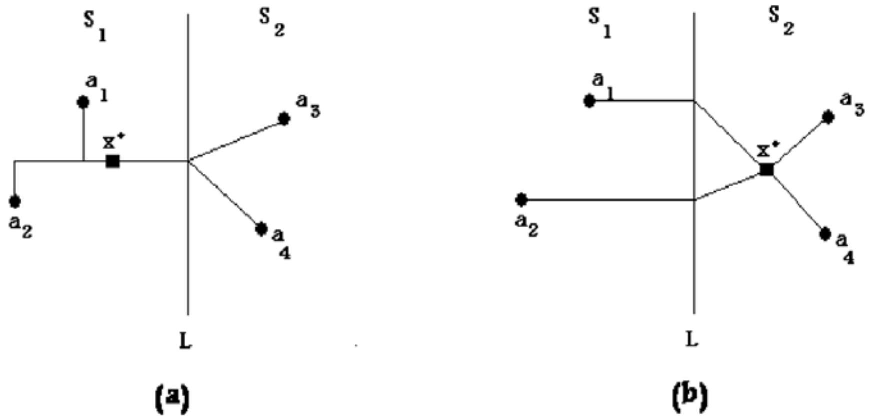

Figure 2. (a) x^* is in S_1, and (b) x^* is in S_2

First, consider the optimal solution x^* being in the region S_2. By Lemma 2.1, the minisum problem can be written as:

$$\min\ z=\sum_{i\in S_2} w_i(|x_1-a_{i1}|^p+|x_2-a_{i2}|^p)^{\frac{1}{p}}+\sum_{i\in S_1} w_i\,|\alpha-a_{i1}|+\sum_{i\in S_1} w_i(|x_1-\alpha|^p+|x_2-a_{i2}|^p)^{\frac{1}{p}}$$

or

$$\min\ z=\sum_{i\in S_1\cup S_2} w_i(|x_1-c_{i1}|^p+|x_2-a_{i2}|^p)^{\frac{1}{p}}+K\,, \tag{7}$$

where c_{i1} and K are constants.The solution of this problem is obtained by using a modified Weiszfeld procedure given in [3]. Similar to the minisum problem the minimax problem, can be formulated as:

$$\min\ z=\max\{\max_{i\in S_2} w_i[|x_1-a_{i1}|^p+|x_2-a_{i2}|^p]^{\frac{1}{p}},$$

$$\max_{i\in S_1}\ w_i(|\alpha-a_{i1}|+[|x_1-\alpha|^p+|x_2-a_{i2}|^p]^{\frac{1}{p}})\}.$$

The solution of this problem is obtainedwith regard to the following minimax problem which is concerned with only one type of l_p norm and basically is a nonlinear problem.

$$\min\ z=\max_{i\in S_1\cup S_2}\ w_i[|x_1-c_{i1}|^p+|x_2-a_{i2}|^p]^{\frac{1}{p}}+K_i\,, \tag{8}$$

where c_{i1} and K_i are appropriately defined based on the previous relation. Now, consider the optimal solution, x^*, lying in the region S_1. Then, the minisum problem is given by:

$$\min\ z=\sum_{i\in S_1} w_i(|x_1-a_{i1}|+|x_2-a_{i2}|)+\sum_{i\in S_2} w_i\,|x_1-\alpha|+\sum_{i\in S_2} w_i(|\alpha-a_{i1}|^p+|x_2-a_{i2}|^p)^{\frac{1}{p}}. \tag{9}$$

The problem (9) is separable with respect to the variables x_1 and x_2. Therefore, the optimal solutions of the two following one variable problems give the optimal solution of the problem (9):

$$\min\ z_1=\sum_{i\in S_1} w_i\,|x_1-a_{i1}|+\sum_{i\in S_2} w_i\,|x_1-\alpha| \tag{10}$$

and

$$\min\ z_2=\sum_{i\in S_1} w_i\,|x_2-a_{i2}|+\sum_{i\in S_2} w_i[|\alpha-a_{i1}|^p+|x_2-a_{i2}|^p]^{\frac{1}{p}}. \tag{11}$$

After considering the minisum problem, we turn our attention back to the minimax problem:

$$\min z\ =\max\{\max_{i\in S_1} w_i\,[|x_1-a_{i1}|+|x_2-a_{i2}|],\max_{i\in S_2} w_i\,(|x_1-\alpha|+[|\alpha-a_{i1}|^p+|x_2-a_{i2}|^p]^{\frac{1}{p}})\}.$$

This problem can be clearly reformulated as an almost linear programming problem with only one type of nonlinear constraints.

$$\begin{aligned}
&\min\ z \\
&s.t. \\
&w_i(y_1+y_2)\le z, \quad i\in S_1 \\
&x_1-a_{i1}\le y_1, \quad i\in S_1 \\
&-x_1+a_{i1}\le y_1, \quad i\in S_1 \\
&x_2-a_{i2}\le y_2, \quad i\in S_1 \\
&-x_2+a_{i2}\le y_2, \quad i\in S_1 \\
&x_1-\alpha\le y_{3,} \quad i\in S_2 \\
&-x_1+\alpha\le y_3, \quad i\in S_2 \\
&w_i([|\alpha-a_{i1}|^p+|x_2-a_{i2}|^p]^{\frac{1}{p}}+y_3)\le z, \quad i\in S_2.
\end{aligned} \tag{12}$$

Since this problem is a convex optimization problem, then any local solution is also the global solution and convex optimization methods can be used to solve this problem.

3.2 l_1 and e_2-Block Norms

Let the regions S_1 and S_2 be measured with l_1 and e_2-block norm l_B, and a_i' be the projection of a_i onto the line L. If the optimal solution lies in the region S_2 then the minisum and minimax problems can be respectively stated as follows:

$$\min\ z=\sum_{i\in J_1} w_i|\alpha-a_{i1}|+\sum_{i\in S_1} w_i\|x-a_i'\|_B+\sum_{i\in S_2} w_i\|x-a_i\|_B \tag{13}$$

and

$$\min\ z=\max\{\max_{i\in S_1} w_i(|\alpha-a_{i1}|+\|x-a_i'\|_B),\max_{i\in S_2} w_i\|x-a_i\|_B\}. \tag{14}$$

Note that $|\alpha-a_{i1}|$ is a constant, and value therefore both problems (13) and (14) can be rewritten as linear programming problems by using the relations (4) and (5) (see [9]).

Now, consider the case that the optimal solution lies in region S_1. Let x' be the projection of x onto the line $L: x=\alpha$. Then, the minisum and minimax formulas are devised as follows respectively:

$$\min\ z=\sum_{i\in S_1} w_i[|x_1-a_{i1}|+|x_2-a_{i2}|]+\sum_{i\in S_2} w_i(\|x'-a_i\|_B+|x_1-\alpha|) \tag{15}$$

and

$$\min\ z=\max\{\max_{i\in S_1} w_i(|x_1-a_{i1}|+|x_2-a_{i2}|),\max_{i\in S_2} w_i(\|x'-a_i\|_B+|x_1-\alpha|)\}. \tag{16}$$

Since $x_1'=\alpha$, then the solution of the problem (15)canbe obtained by the investigation of problems (10) and (17):

$$\min\ z=\sum_{i\in S_1} w_i\,|\,x_2-a_{i2}\,|+\sum_{i\in S_2} w_i\,\|\,x'-a_i\,\|_B\ . \tag{17}$$

The minimax problem (16) can be rewritten in the following convenient form:

$$\begin{aligned}
&\min\ z && (18)\\
&s.t.\\
&\quad w_i(y_1+y_2)\le z, \quad i\in S_1\\
&\quad x_1-a_{i1}\le y_1, \quad i\in S_1\\
&\quad -x_1+a_{i1}\le y_1, \quad i\in S_1\\
&\quad x_2-a_{i2}\le y_2, \quad i\in S_1\\
&\quad -x_2+a_{i2}\le y_2, \quad i\in S_1\\
&\quad x_1-\alpha\le y_3, \quad i\in S_2\\
&\quad -x_1+\alpha\le y_3, \quad i\in S_2\\
&\quad w_i(\|\,x'-a_i\,\|_B+y_3)\le z, \quad i\in S_2.
\end{aligned}$$

Problems (17) and (18)can be easily changed to linear and can readily be solved (see Section 3.3).

3.3. Two Block Norms

Let S_1 and S_2, the regions beside line L, be measured by two block norms, l_{B_1} and l_{B_2}, respectively. With out loss of generality assume that the optimal solution x^* be included in the region S_1. Let $u_i=(\alpha,u_{i2})$ be a point on the line L such that the shortest path from x^* to a_i passes through z_i. The minisum and minimax problems are given as follows:

$$\min\ z=\sum_{i\in S_1} w_i\,\|\,x-a_i\,\|_{B_1}+\sum_{i\in S_2} w_i(\|\,u_i-a_i\,\|_{B_2}+\|\,x-u_i\,\|_{B_1}) \tag{19}$$

and

$$\min\ z=\max\{\max_{i\in S_1} w_i\,\|\,x-a_i\,\|_{B_1},\max_{i\in S_2} w_i(\|\,u_i-a_i\,\|_{B_2}+\|\,x-u_i\,\|_{B_1})\}\ . \tag{20}$$

The block norms within the problems (19) and (20) can be replaced by the following equalities with respect to the relation (5):

$$y_i=\|\,x-a_i\,\|_{B_1}=\max\{|\,(x-a_i)b^0_{g_1}\,|: g_1=1,\ldots,r_1^0\}$$

$$y'_i=\|\,u_i-a_i\,\|_{B_2}=\max\{|\,(u_i-a_i)b^0_{g_2}\,|: g_2=1,\ldots,r_2^0\}$$

$$y_i'' = \| x - u_i \|_{B_1} = \max\{|(x - u_i)b_{g_1}^0| : g_1 = 1,\ldots,r_1^0\}.$$

So, the two following minisum and minimax linear programming problems, (21) and (22), are respectively derived:

$$\min\ z = \sum_{i \in S_1} w_i y_i + \sum_{i \in S_2} w_i (y_i' + y_i'') \tag{21}$$

$$\begin{aligned}
s.t.\quad & \\
& (x - a_i)b_{g_1}^0 \le y_i, \quad i \in J_1, g_1 = 1,\ldots,r_1^0 \\
& -(x - a_i)b_{g_1}^0 \le y_i, \quad i \in J_1, g_1 = 1,\ldots,r_1^0 \\
& (u_i - a_i)b_{g_2}^0 \le y_i', \quad i \in J_2, g_2 = 1,\ldots,r_2^0 \\
& -(u_i - a_i)b_{g_2}^0 \le y_i', \quad i \in J_2, g_2 = 1,\ldots,r_2^0 \\
& (u_i - x)b_{g_1}^0 \le y_i'', \quad i \in J_2, g_1 = 1,\ldots,r_1^0 \\
& -(u_i - x)b_{g_1}^0 \le y_i'', \quad i \in J_2, g_1 = 1,\ldots,r_1^0
\end{aligned}$$

and

$$\min\ z \tag{22}$$

$$\begin{aligned}
s.t.\quad & \\
& w_i(x - a_i)b_{g_1}^0 \le z, \quad i \in J_1, g_1 = 1,\ldots,r_1^0 \\
& -w_i(x - a_i)b_{g_1}^0 \le z, \quad i \in J_1, g_1 = 1,\ldots,r_1^0 \\
& w_i(y_i' - y_i'') \le z, \quad i \in J_2 \\
& (u_i - a_i)b_{g_2}^0 \le y_i', \quad i \in J_2, g_2 = 1,\ldots,r_2^0 \\
& -(u_i - a_i)b_{g_2}^0 \le y_i', \quad i \in J_2, g_2 = 1,\ldots,r_2^0 \\
& (u_i - x)b_{g_1}^0 \le y_i'', \quad i \in J_2, g_1 = 1,\ldots,r_1^0 \\
& -(u_i - x)b_{g_1}^0 \le y_i'', \quad i \in J_2, g_1 = 1,\ldots,r_1^0.
\end{aligned}$$

Problems (21) and (22) can be readily solved using linear programming.

3.4. Sufficient Conditions

Note that for all of the considered problems, in order to check whether the optimal solution Lies in the region S_1 or S_2, we need to solve two problems, one for each region, and compare the obtained results to get the optimal solution. Fortunately, in a special case discussed in Lemma 3.1 below, sufficient conditionsexist,under under which solution of only one problem givethe optimal solution.

Lemma 3.1. *Let the occupying norm in the region* S_1 *be* l_1 *and in* S_2 *be* l_p *or* e_2*-block norm* $, W_1 = \sum_{i \in S_1} w_i$ *and* $W_2 = \sum_{i \in S_2} w_i$. *If* $W_2 \geq W_1$, *then* x^*, *the optimal solution of the single facility minisum problem, is included in* $S_2 \cup L$.

Proof. If $x^* \in S_1$, then the solution of (10) implies $x_1^* = \alpha$. Thus, $x^* \in S_2 \cup L$. □

3.5. Examples

Now, we give some illustrating examples to clarify the above discussion. First,a asmall example with 3 points is given to specify the problem formulation. Then, an example with 18 points is considered. For these two examples, we assume that the given points are distributed in the plane R^2 and the dividing line, $L : x = 0$, belongs to S_1 which is measured with the l_1 norm. For both examples, two cases of the region S_2 being measured with l_p and l_B norms are considered. The considered l_B norm is an e_2-block norm with the extreme points of its contour and its polar set as shown in Table 1.

Table 1. The extreme points of l_B and corresponding polar set

extreme points of l_B	extreme points of polar set
$b_1 = (0,1)$	$b_1^0 = (\frac{\sqrt{3}}{3},1)$
$b_2 = (\frac{\sqrt{3}}{2},\frac{1}{2})$	$b_2^0 = (1,2-\sqrt{3})$
$b_3 = (1,0)$	$b_3^0 = (1,-2+\sqrt{3})$
$b_4 = (\frac{\sqrt{3}}{2},-\frac{1}{2})$	$b_4^0 = (\frac{\sqrt{3}}{3},-1)$
$b_{-1} = (0,-1)$	$b_{-1}^0 = (-\frac{\sqrt{3}}{3},-1)$
$b_{-2} = (-\frac{\sqrt{3}}{2},-\frac{1}{2})$	$b_{-2}^0 = (-1,-2+\sqrt{3})$
$b_{-3} = (-1,0)$	$b_{-3}^0 = (-1,2-\sqrt{3})$
$b_{-4} = (-\frac{\sqrt{3}}{2},\frac{1}{2})$	$b_{-4}^0 = (-\frac{\sqrt{3}}{3},1)$

Example 1. Let $a_1 = (-1,1) \in S_1$, $a_2 = (-2,0) \in S_1$ and $a_3 = (1,1) \in S_2$ with relevant weights $w_1 = w_2 = 1$ be given. We solve the minisum problem in three different cases with $w_3 = 1, 1.5, 2$.

In the two cases of $w_3 = 1$ and $w_3 = 1.5$, the sum of the weights, $W = \sum_{i=1}^{3} w_i$, does not satisfy the condition of Lemma 3.1, and so obtaining the optimal solution reguiressolving the two problems and getting the best solution of the tow regions S_1 and S_2. But for the case $w_3 = 2$, the condition of Lemma 3.1 is satisfied,and therefore an optimal solution lies in the region S_2. Consider the following two cases.

(1) Let the region S_2 be measured by an l_p norm. It is necessary to solve the problem in the two foregoing cases to get the optimal solution.

(a) To obtain the best solution $x \in S_1$, it is needed to solve the followin problems:

$$\min\ z_{11} = |x_1 + 1| + |x_1 + 2| + w_3 |x_1|$$

and

$$\min\ z_{12} = |x_2 - 1| + |x_2| + w_3 (1 + |x_2 - 1|^p)^{\frac{1}{p}}.$$

We assume the cases, corresponding to $p = 2, 3, 10, 100, \infty$ and $w_3 = 1, 1.5, 2$. In all cases the best solution in the region S_1 is $x = (-1,1)$ with the objective function values as presented in Table 2.

Table 2. The results of Example 1 in the case $x \in S_1$

	p = 2,3,10,100 , ∞		
w_3	z_{11}	z_{12}	$z_1 = z_{11} + z_{12}$
1	2	2	4
1.5	2.5	2.5	5
2	3	3	6

(b) To obtain the best solution $x \in S_2$, the following problem needs to be solved:

$$\min\ z_2 = (x_1^p + x_2^p)^{\frac{1}{p}} + (x_1^p + |x_2 - 1|^p)^{\frac{1}{p}} + w_3 (|x_1 - 1|^p + |x_2 - 1|^p)^{\frac{1}{p}} + 3.$$

Table 3 contains the results for this problem using different values of p and w_3. In the three cases, where $w_3 = 1, 1.5, 2$ and $p = \infty$,the two points $(0.5, 0.5)$ and $(1,1)$ are optimal solutions lying in the region S_2.

Table 3. Results of the l_1 and l_p for the case $x \in S_2$ in Example 1

	$w_3 = 1$			$w_3 = 1.5$			$w_3 = 2$		
p	x_1	x_2	z_2	x_1	x_2	z_2	x_1	x_2	z_2
2	0.21	0.79	4.93	0.46	0.81	5.29	1	1	5.41
3	0.35	0.65	4.81	0.51	0.70	5.11	1	1	5.26
10	0.47	0.53	4.59	0.51	0.55	4.86	1	1	5.07
100	0.49	0.50	4.51	0.50	0.50	4.76	1	1	5
∞	0.50	0.50	4.50	0.50	0.50	4.75	1	1	5

Table 4 contains the optimal solutions of the main problem, i.e., $z^* = \min\{z_1, z_2\}$. In the case $w_3 = 2$, the optimal solution lies in the region S_2, as expected. For the values $p = 10, 100, \infty$ and $w_3 = 1.5$, in spite of the fact that the sum of the weights in the region S_1 is more than the weights in the region S_2, the optimal solutions are included in the region S_2. This result illustrates that the condition of Lemma 3.1 is sufficient but not necessary.

Table 4. Optimal solutions of Example 1 with l_1 and l_p norms

	$w_3 = 1$			$w_3 = 1.5$			$w_3 = 2$		
p	x_1^*	x_2^*	z^*	x_1^*	x_2^*	z^*	x_1^*	x_2^*	z^*
2	-1	1	4	-1	1	5	1	1	5.41
3	-1	1	4	-1	1	5	1	1	5.26
10	-1	1	4	0.51	0.55	4.86	1	1	5.07
100	-1	1	4	0.50	0.50	4.76	1	1	5.00
∞	-1	1	4	0.50	0.50	4.75	1	1	5

(2) Let the measuring norm in the region S_2 be the foregoing mentioned l_B norm. The two following cases must be traced.

(a) Find the best point $x \in S_1$, obtained by solving the following two problems:

$$\min \ z_{11} = |x_1 + 1| + |x_1 + 2| + w_3 |x_1|$$

and

$$\min \ z_{12} = |x_2 - 1| + |x_2| + w_3 y$$

s.t.:

$$-\frac{\sqrt{3}}{3} - 1 + x_2 \le y$$

$$\frac{\sqrt{3}}{3} + 1 - x_2 \le y$$

$$\frac{\sqrt{3}}{3} - 1 + x_2 \le y$$

$$
\begin{gathered}
-\frac{\sqrt{3}}{3}+1-x_2 \le y \\
-1+(x_2-1)(2-\sqrt{3}) \le y \\
1-(x_2-1)(2-\sqrt{3}) \le y \\
-1+(x_2-1)(-2+\sqrt{3}) \le y \\
1+(x_2-1)(-2+\sqrt{3}) \le y \\
y \ge 0\,.
\end{gathered}
$$

(b) Finding the best point $x \in S_2$, obtained by solving the following problem:

$$
\begin{aligned}
&\min \ z_2 = y_1 + y_2 + w_3 y_3 + 3 \\
&\text{s.t.:} \\
&\pm(x_1, x_2-1)\begin{pmatrix}\frac{\sqrt{3}}{3}\\ 1\end{pmatrix} \le y_1 \\
&\pm(x_1, x_2-1)\begin{pmatrix}\frac{\sqrt{3}}{3}\\ -1\end{pmatrix} \le y_1 \\
&\pm(x_1, x_2-1)\begin{pmatrix}1\\ 2-\sqrt{3}\end{pmatrix} \le y_1 \\
&\pm(x_1, x_2-1)\begin{pmatrix}-1\\ 2-\sqrt{3}\end{pmatrix} \le y_1 \\
&\pm(x_1, x_2)\begin{pmatrix}\frac{\sqrt{3}}{3}\\ 1\end{pmatrix} \le y_2 \\
&\pm(x_1, x_2)\begin{pmatrix}\frac{\sqrt{3}}{3}\\ -1\end{pmatrix} \le y_2 \\
&\pm(x_1, x_2)\begin{pmatrix}1\\ 2-\sqrt{3}\end{pmatrix} \le y_2 \\
&\pm(x_1, x_2)\begin{pmatrix}-1\\ 2-\sqrt{3}\end{pmatrix} \le y_2 \\
&\pm(x_1-1, x_2-1)\begin{pmatrix}\frac{\sqrt{3}}{3}\\ 1\end{pmatrix} \le y_3 \\
&\pm(x_1-1, x_2-1)\begin{pmatrix}\frac{\sqrt{3}}{3}\\ -1\end{pmatrix} \le y_3 \\
&\pm(x_1-1, x_2-1)\begin{pmatrix}1\\ 2-\sqrt{3}\end{pmatrix} \le y_3 \\
&\pm(x_1-1, x_2-1)\begin{pmatrix}-1\\ 2-\sqrt{3}\end{pmatrix} \le y_3 \\
&y_1, y_2, y_3 \ge 0\,.
\end{aligned}
$$

Table 5 shows the obtained results for the different regions S_1 and S_2. In the case $w_3 = 2$, the weights of the points satisfy in the condition in Lemma 3.1, and therefore the optimal solution lies in the region S_2, as shown in the table.

Table 5. The results for Example 1

	$x \in S_1$				$x \in S_2$					
w_3	x_1	z_{11}	x_2	z_{12}	x_1	x_2	z_2	x_1^*	x_2^*	z^*
1	-1	2	1	2	0	1	5	-1	1	4
1.50	-1	2.50	1	2.50	0.50	0.71	5.43	-1	1	5
2	-1	3	1	3	1	1	5.57	1	1	5.57

Example 2. consider 18 points as shown in Table 6.

Table 6. The points for Example 2

points in S_1	(-3,3)	(-3,0)	(-3,-2)	(-2,2)	(-2,-1)	(-1,4)	(-1,1)	(-1,0)	(-1,-2)
points in S_2	(1,3)	(1,1)	(1,-1)	(2,2)	(2,0)	(2,-2)	(3,4)	(3,-1)	(4,0)

Two cases are considered: (1) the weights of all points equal to one, (2) the wights of all points equal to except for the point $a_1 = (-3,3)$ with $w_1 = 5$. In the first case, the sum of the weights satisfies the condition of Lemma 3.1, and so the optimal solution x^* lies in the region S_2. But in the second case, the condition of Lemma 3.1 is not satisfied, and therefore it is not known where the optimal solution may lie.

(1) Let the region S_2 be measured by the norm l_p with $p = 2,3,10,100$ .The problem needs to be solved in two cases to get the optimal solution.
To find the best solution $x \in S_1$, we have to consider the models (10) and (11). The model (10) is independent of p and for all cases of p the solution is the same. To find the best point $x \in S_2$, it is required to consider model (7). The results for $w_1 = 1$ and $w_1 = 5$ and different values of p are presented in Tables 7 and 8.

Table 7. The results corresponding to the l_1 and l_p normsfor Example 2 with $w_1 = 1$

	$x \in S_1$					$x \in S_2$		
p	x_1	z_{11}	x_2	z_{12}	$z_1 = z_{11} + z_{12}$	x_1	x_2	z_2
2	-0.9967	17	0.1961	40.7674	57.7674	0.8444	0.5192	55.2776
3	-0.9967	17	0.2790	39.0373	56.0373	0.9135	0.66398	53.1641
10	-0.9967	17	0.1427	37.3880	54.3880	0.9930	0.9149	51.4627
100	-0.9967	17	0.01517	37.0346	54.0346	0.9979	0.9817	51.0323

Table 8. The results corresponding tothe l_1 and l_p normsfor Example 2 with $w_1 = 5$.

	$x \in S_1$					$x \in S_2$		
p	x_1	z_{11}	x_2	z_{12}	$z_1 = z_{11} + z_{12}$	x_1	x_2	z_2
2	-1	25	1.03	49.52	74.03	0.71	1.13	76.43
3	-1	25	1.18	47.63	72.63	0.89	1.12	73.68
10	-1	25	1.44	46.10	71.10	1.01	1.07	71.48
100	-1	25	1.46	46.00	71.00	1.00	1.02	71.03

(2) Let the norm in the region S_2 be the previously mentioned l_B norm. To obtain the optimal point, the two following cases are considered. First, the best solution belongs to region S_1 is found by employing the two problems (10) and (17), Then, the best solution of the region S_2 is obtained from the problem (13). Table 9 give the results.

Table 9. The results corresponding to l_1 and l_B for the Example 2

	$x \in S_1$					$x \in S_2$		
w_1	x_1	z_{11}	x_2	z_{12}	$z_1 = z_{11} + z_{12}$	x_1	x_2	z_2
1	-0.10	17	0.42	42.96	59.96	0.86	0.50	59.04
5	-1.00	25	1.00	51.58	76.58	0.00	1.00	80.57

The results show that for the case $w_1 = 1$, the optimal solutions lie in the region S_2, satisfying the condition of Lemma 3.1, and for the case $w_1 = 5$, the optimal solutions lies in the region S_1 .

4. Summary and Conclusion

We considered the single facility minisum and minimax location problems on the plane R^2 being divided into two regions S_1 and S_2 by a straight line $L : x = \alpha$. The regions S_1 and S_2 were considered to be equipped with different norms. First, a special case with the distance measure in one region being the l_1 norm and in the other region being the l_p norm was discussed. Then other cases considering the l_1 norm and a block norm or two block norms measuring the, two regions were studied. Our formulation policy was to replace the l_1 norm and the block norms by linear formulations. I llastrating examples were given to clarify the solution methods.

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