Optimal Ordering Policy with Stock-Dependent Demand Rate under Permissible Delay in Payments

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We develop an inventory model to determine optimal ordering policy under permissible delay in payment by considering demand rate to be stock dependent. Mathematical models are derived under two different cases: credit period being greater than or equal to cycle time for settling the account, and credit period being less than or equal to cycle time for settling the account. The results are illustrated with numerical examples. Sensitivity analysis is given for the proposed model.

1. Introduction

In deriving the economic order quantity formula, it is assumed that retailer must pay for items as soon as he receives them from a supplier. But, in practice a supplier will allow a certain fixed period for settling the account corresponding to the supplied items. Usually, interest is not charged for the outstanding amount, if it is paid within a permissible time period. However, if the payment is not paid within the permissible period, interest is charged on the outstanding amount under a previously agreed terms and conditions. Classical inventory models consider the demand rate to be either time dependent or constant, but independent of the stock status. However, for certain types of inventory, particularly consumer goods, the consumption rate may be influenced by the stock levels, i.e., the consumption rate may go up or down with the on hand stock level. One important problem faced in inventory management is the problem of control and maintaining inventories of deteriorating items, such as food items, pharmaceuticals, chemicals and blood. Here, we consider the demand to be stock dependent under a permissible delay in payments. In many real-life situations, for certain types of consumer goods (e.g., fruits, vegetables, coconut, etc.), consumption rate is sometimes influenced by the stock-level. It is usually observed that a large pile of goods on shelves of supermarkets will lead the customer to buy more and then generate a higher demand. The consumption rate may go up or down with the on hand stock level. These phenomena, have attracted many marketing researchers to investigate models related to stock level.

Teng et al. [14] developed an EOQ model for optimal pricing and ordering policy under permissible delay in payments. The authors, by assuming the selling price to be higher than the purchase cost, established an appropriate model for a retailer to determine the optimal price and lot size simultaneously, when the supplier offers a permissible delay in payments. Jaggi et al. [6] developed an optimal order policy for deteriorating items with an inflationary induced demand. The authors considered the demand to increase exponentially due to inflation over a finite horizon using the DCF approach for analysis of optimal inventory policy, taking a constant deterioration rate. Tripathi and Mishra [15] discussed credit financing in economic ordering policies of deteriorating

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items with a quadratic time dependent demand rate. The authors obtained the optimal order quantity with quadratic time dependent demand rate in the presence of trade credit using the DCF approach. Tripathi et al. [16] developed a cash flow oriented EOQ model under permissible delay in payments. The authors developed an inventory for time dependent demand under permissible delay in payment. Discount cash-flow analysis on inventory control under various suppliers' trade credits was discussed by Teng [13]. Related to one level of trade credit. Goyal [3] established a single-item inventory model for deteriorating the economic ordering quantity in the case that the supplier offers the retailer an opportunity to pay his account after a fixed time period. Aggarwal and Jaggi [1] extended Goyal's model [3] to the case of deteriorating items. Jamal et al. [7] and Sarkar et al. [11] discussed the optimal payment time under permissible delay in payment with deterioration.

Liao et al. [8] investigated an inventory model for initial stock dependent demand rate when a delay in payment is permissible. An inventory model for deteriorating items with stock-dependent consumption rate and shortages under inflation and time discounting were discussed by Hou [5]. Hou developed a finite planning horizon inventory model for deteriorating items with stock dependent consumption rate and shortages. In addition, the effects of inflation and time value of money on replenishment policy under instantaneous replenishment with zero lead-time were also considered. Yang et al. [19] developed an inventory model under inflation for deteriorating items with stock dependent consumption rate and partial backlogging shortages. The authors developed an inventory model in which, (a) shortages are partial backlogging to reflect the fact that the more the waiting time, the smaller the backlogging rate, (b) effects of inflation and time value of money are relevant and (c) replenishment cycles and shortage intervals are time varying. Zohou and Yang [20] established a two–warehouse model with a stock–level–dependent demand rate and consideration of transportation cost. Hill and Pakkala [4] establish a discounted cash-flow approach to the base stock inventory model. In this model, authors assumed that a first in first out stock rotation is observed and that customer demands are met on a first come first served basis. Alfares [2] established an inventory model with stock–level dependent demand rate and variable holding cost. The main objective was to determine the minimum inventory policy for an inventory system with inventory level dependent demand rate and a time dependent holding cost. Uthayakumar and Geetha [17] established a replenishment policy for non–instantaneous deteriorating inventory system with partial backlogging. In this model, authors developed a finite planning horizon inventory model for non–instantaneous deteriorating items with a stock–dependent consumption rate. Mandal and Phaujdar [9] developed an inventory model for deteriorating items and stock dependent consumption rate. Mandal and Maiti [10] developed an inventory of damageable items with variable replenishment rate, stock–dependent demand and some units at hand. Sarker et al. [12] established an order–level lot size inventory model with inventory–level dependent demand and deterioration. Vrat et al. [18] developed an inventory model under inflation for stock dependent consumption rate items.

The rest of our work is organized as follows. In Section 2, we give our assumptions and notations. The mathematical formulation id provided in Section 3. In Section 4, numerical examples are given. Sensitivity analysis is given in Section 6 to determine optimal cycle time and optimal annual profit for two cases. Finally, we conclude and provide suggestions for future research in Section 7.

2. Assumptions and Notations

For the mathematical model, the following assumptions are made:
(1) Shortages are not allowed.
(2) Lead time is zero.
(3) Demand for the items is stock–dependent.

(4) Time horizon is infinite.

The following notations are used throughout our work:

D : annual demand, as a decreasing function of inventory; i.e., $D = \alpha\{I(t)\}^{\beta}$, where $\alpha > 1$, $0 < \beta < 1$.

h : unit holding cost per year excluding interest charges.

I_c : interest charged per \$ in stocks per year by the supplier.

I_d : interest earned per \$ per year.

p : selling price per unit.

c : unit purchasing cost, with $c < p$.

s : ordering cost per order.

m : period of permissible delay in settling account; i.e., the trade credit period.

Q : order quantity.

$I(t)$: level of inventory at time t with $0 \le t \le T$.

$Z(T)$: total annual profit.

T : replenishment time interval.

3. Mathematical Formulation

The level of inventory, $I(t)$, decreases only to meet demands. Hence, variation of inventory with respect to time is given by

$$\frac{dI(t)}{dt} = -\alpha\{I(t)\}^{\beta}, \; 0 \le t \le T, \tag{1}$$

with the boundary condition $I(T) = 0$. Consequently, the solution of (1) is given by

$$I(t) = a(T-t)^{1/(1-\beta)} \; 0 \le t \le T, \text{ where } a = \{\alpha(1-\beta)\}^{1/(1-\beta)}, \tag{2}$$

and the order quantity is:

$$Q = I(0) = aT^{1/(1-\beta)} \tag{3}$$

The total annual profit consists of the followings:

(a) Sales revenue $= p\alpha(1-\beta)a^{\beta}T^{\beta/(1-\beta)}$. $\tag{4}$

(b) Cost of placing orders $= \dfrac{s}{T}$. $\tag{5}$

(c) Cost of purchasing $= \dfrac{cQ}{T} = caT^{\beta/(1-\beta)}$. $\tag{6}$

(d) Cost of carrying inventory $= \dfrac{h}{T}\int_0^T I(t)dt = \dfrac{ha(1-\beta)}{(2-\beta)}T^{1/(1-\beta)}$. $\tag{7}$

Regarding interests payable and earned, we have the following two possible cases based on the values of T and m.

Case I : $T \leq m.$

Since, the customer sells $\alpha(1-\beta)a^{\beta}T^{1/(1-\beta)}$ units in total by the end of the replenishment cycle time T, and has $c\alpha(1-\beta)a^{\beta}T^{1/(1-\beta)}$ to pay to the supplier in full by the end of the credit period m, there is no interest payable. However, the interest earned per year is

$$\frac{pI_d}{T}\left[\int_O^T \alpha\{I(t)\}^{\beta} t\, dt + (m-T)\int_O^T \alpha\{I(t)\}^{\beta}\, dt\right] = pI_d\alpha(1-\beta)a^{\beta}\left\{\frac{(1-\beta)}{(2-\beta)}T^{1/(1-\beta)} + (m-T)T^{\beta/(1-\beta)}\right\}.$$

(8)

The total annual profit $Z_1(T)$ is given by

$$Z_1(T) = p\alpha(1-\beta)a^{\beta}T^{\beta/(1-\beta)} - \frac{s}{T} - caT^{\beta/(1-\beta)} - \frac{ha(1-\beta)}{(2-\beta)}T^{1/(1-\beta)}$$

$$+ pI_d\alpha(1-\beta)a^{\beta}\left\{\frac{(1-\beta)}{(2-\beta)}T^{1/(1-\beta)} + (m-T)T^{\beta/(1-\beta)}\right\}.$$

(9)

The optimum value of T is obtained by solving $\dfrac{\partial Z_1(T)}{\partial T} = 0$:

$$\frac{\partial Z_1(T)}{\partial T} = p\alpha\beta a^{\beta}T^{(2\beta-1)/(1-\beta)} + \frac{s}{T^2} - \frac{ca\beta}{1-\beta}T^{(2\beta-1)/(1-\beta)} - \frac{ha}{2-\beta}T^{\beta/(1-\beta)}$$

$$+ \frac{pI_d\alpha a^{\beta}}{(2-\beta)}\left\{m\beta(2-\beta)T^{(2\beta-1)/(1-\beta)} - T^{\beta/(1-\beta)}\right\}$$

(10)

$$\frac{\partial^2 Z_1(T)}{\partial T^2} = p\alpha\beta a^{\beta}\frac{(2\beta-1)}{(1-\beta)}T^{(3\beta-2)/(1-\beta)} - \frac{2s}{T^3} - \frac{ca\beta(2\beta-1)}{(1-\beta)^2}T^{(3\beta-2)/(1-\beta)}$$

$$- \frac{ha\beta}{(1-\beta)(2-\beta)}T^{(2\beta-1)/(1-\beta)} + \frac{pI_d\alpha\beta a^{\beta}}{(1-\beta)(2-\beta)}\left\{m(2-\beta)(2\beta-1)T^{(3\beta-2)/(1-\beta)} - T^{(2\beta-1)/(1-\beta)}\right\} < 0.$$

(11)

The results are is shown in Table 1, by taking numerical values of all parameters used in (11) in Case I for different values of I_d and T.

To solve $\dfrac{\partial Z_1(T)}{\partial T} = 0,$ we get

$$\frac{\beta}{(1-\beta)}\left\{p\alpha(1-\beta)a^{\beta} - ca + pI_d\alpha(1-\beta)a^{\beta}m\right\}T^{(2\beta-1)/(1-\beta)}$$

$$- \frac{1}{(2-\beta)}\left(pI_d\alpha a^{\beta} + ha\right)T^{\beta/(1-\beta)} + \frac{s}{T^2} = 0$$

(12)

If we let $\beta = 0.5$, then (12) is cubic in T, which gives only one positive value of $T = T_1$, by a trail and error method or a numerical technique such Newton–Raphsan's method. The corresponding optimal order quantity is $Q = Q_1 = aT_1^{1/(1-\beta)}$.

Case II: $T \geq m$.

In this case, the buyer sells $\alpha(1-\beta)a^\beta \left\{ T^{1/(1-\beta)} - (T-m)^{1/(1-\beta)} \right\}$ units in total by the end of

the permissible delay m, and has $c\alpha(1-\beta)a^\beta \left\{ T^{1/(1-\beta)} - (T-m)^{1/(1-\beta)} \right\}$ to pay to the supplier. The

items in stock are charged with an interest rate of I_c by the supplier starting at time m. Thereafter, the buyer gradually reduces the amount of the payment from the supplier due to constant sales and the revenue received. The interest payable per year is

$$\frac{cI_c}{T} \int_m^T I(t)dt = \frac{cI_c a(1-\beta)}{T(2-\beta)} (T-m)^{(2-\beta)/(1-\beta)} . \tag{13}$$

Next, during the permissible delay period, the buyer sells products and deposits the revenue into an account that earns I_d per dollar per year. Therefore, the interest earned per year is

$$\frac{pI_d}{T} \int_0^m \alpha\{I(t)\}^\beta t dt = \frac{pI_d\alpha(1-\beta)a^\beta}{T(2-\beta)} \left\{ (1-\beta)T^{(2-\beta)/(1-\beta)} - (1-\beta)(T-m)^{(2-\beta)/(1-\beta)} - m(2-\beta)(T-m)^{1/(1-\beta)} \right\}. \tag{14}$$

The total annual profit is given by

$$Z_2(T) = p\alpha(1-\beta)a^\beta T^{\beta/(1-\beta)} - \frac{s}{T} - caT^{\beta/(1-\beta)} - \frac{ha(1-\beta)}{(2-\beta)} T^{1/(1-\beta)} - \frac{cI_c a(1-\beta)}{T(2-\beta)} (T-m)^{(2-\beta)/(1-\beta)}$$

$$+ \frac{pI_d\alpha(1-\beta)a^\beta}{T(2-\beta)} \left\{ (1-\beta)T^{(2-\beta)/(1-\beta)} - (1-\beta)(T-m)^{(2-\beta)/(1-\beta)} - m(2-\beta)(T-m)^{1/(1-\beta)} \right\}. \tag{15}$$

The optimum value of T is obtained by solving $\dfrac{\partial Z_2(T)}{\partial T} = 0$:

$$\frac{\partial Z_2(T)}{\partial T} = p\alpha\beta a^\beta T^{(2\beta-1)/(1-\beta)} + \frac{s}{T^2} - \frac{ca\beta}{1-\beta} T^{(2\beta-1)/(1-\beta)} - \frac{ha}{2-\beta} T^{\beta/(1-\beta)}$$

$$- \frac{cI_c a}{T} (T-m)^{1/(1-\beta)} + \frac{cI_c a(1-\beta)}{T^2(2-\beta)} (T-m)^{(2-\beta)/(1-\beta)} \tag{16}$$

$$+ \frac{pI_d\alpha a^\beta}{T} \left\{ (1-\beta)T^{1/(1-\beta)} - (1-\beta)(T-m)^{1/(1-\beta)} - m(T-m)^{\beta/(1-\beta)} \right\}$$

$$- \frac{pI_d\alpha(1-\beta)a^\beta}{T^2(2-\beta)} \left\{ (1-\beta)T^{(2-\beta)/(1-\beta)} - (1-\beta)(T-m)^{(2-\beta)/(1-\beta)} - m(2-\beta)(T-m)^{1/(1-\beta)} \right\}$$

Again, we have

$$\frac{\partial^2 Z_2(T)}{\partial T^2} = p\alpha\beta a^\beta \frac{(2\beta-1)}{(1-\beta)} T^{(3\beta-2)/(1-\beta)} - \frac{2s}{T^3} - \frac{ca\beta(2\beta-1)}{(1-\beta)^2} T^{(3\beta-2)/(1-\beta)}$$

$$-\frac{ha\beta}{(1-\beta)(2-\beta)}T^{(2\beta-1)/(1-\beta)} - \frac{cI_c a}{T(1-\beta)}(T-m)^{\beta/(1-\beta)} + \frac{2cI_c a}{T^2}(T-m)^{1/(1-\beta)}$$
$$-\frac{2cI_c a(1-\beta)}{T^3(2-\beta)}(T-m)^{(2-\beta)/(1-\beta)} + \frac{pI_d \alpha a^\beta}{T}\left\{T^{\beta/(1-\beta)} - (T-m)^{\beta/(1-\beta)} - \frac{m\beta}{1-\beta}(T-m)^{(2\beta-1)/(1-\beta)}\right\}$$
$$-\frac{2pI_d \alpha a^\beta}{T^2}\left\{(1-\beta)T^{1/(1-\beta)} - (1-\beta)(T-m)^{1/(1-\beta)} - m(T-m)^{\beta/(1-\beta)}\right\}$$
$$+\frac{2pI_d \alpha(1-\beta)a^\beta}{(2-\beta)T^3}\left\{(1-\beta)T^{(2-\beta)/(1-\beta)} - (1-\beta)(T-m)^{(2-\beta)/(1-\beta)} - m(2-\beta)(T-m)^{1/(1-\beta)}\right\} < 0. \tag{17}$$

The results shown in Table 2, by taking numerical values of all parameters used in (17) in the Case II, for different values of I_d and T.

To solve $\dfrac{\partial Z_2(T)}{\partial T} = 0$, putting $\beta = 0.5$ and $m = 1$ year, (16) turns to

$$(4ha + 4cI_c a)T^3 - (6p\alpha\beta a^\beta - 6ca + 6cI_c a)T^2 - (6s + 2pI_d \alpha a^\beta - 2cI_c a) = 0 \tag{18}$$

Equation (18) is cubic in T, which gives only one positive value of $T = T_2$, by trail and error method or a numerical technique such as Newton–Raphsan's method. The corresponding optimal order quantity is $Q = Q_2 = aT_2^{1/(1-\beta)}$.

4. An Algorithm

Based on the above discussion, we develop the following solution method to determine an optimal solution for the approximating model.

Step 1: Calculate $T = T_1$ using (12). If $T \leq m$, then calculate $Z_1(T_1)$, otherwise go to Step 2.
Step 2: Calculate $T = T_2$ using (18). If $T \geq m$, then calculate $Z_2(T_2)$, otherwise go to step 1.
Step 3: The maximum of $Z_1(T_1)$ and $Z_2(T_2)$ gives the optimal solution.

5. Numerical Example

Case I: $T \leq m$.

Let us take the parameter values of the inventory system as p=$10 per unit, α=50, β=0.5, h=$1.5/unit/year, c=$9.0 per unit, S=50, m=1 year and I_d=0.05/$/year.
Solving (12), we have T_1=0.952871 year and the maximum annual profit is $Z_1(T_1)$=$367.9460. The economic order quantity turns to be Q_1=567.4776 units.

Case II: $T \geq m.$

Let us take the parameter values of the inventory system as $p=$ \$10 per unit, α=50, β=0.5, h=\$2.0/unit/year, $C=$ \$9.0 per unit, S=100, m=1 year, I_d =0.05/\$/year and I_c=0.08/\$/year.

Solving (18), we have T_2 = 1.070451 year and the maximum annual profit is $Z_2(T_2)$ =\$215.998397. The economic order quantity is Q_2=716.1659 units.

6. Sensitivity Analysis

Taking all the parameters as set in the above numerical example, in Case I, the variation of the optimal solution for different values of I_d are given in Table 1.

Table 1. Variation of optimal solution for different values of I_d

I_d	T_1 (years)	Q_1 (units)	$Z_1(T)$ in \$	$\dfrac{\partial^2 Z_1}{\partial T^2}$
0.03	0.98720511	609.1087058	325.080022	-978.9388281
0.04	0.96865232	586.4295732	346.356666	-1068.359563
0.05	0.95287161	567.4776907	367.9460927	-1157.250432
0.06	0.93926674	551.3887555	389.783845	-1245.679407
0.07	0.92742213	537.5698795	411.8222474	-1333.695831

Taking all the parameters as set in the above numerical example, in Case II, the variation of the optimal solution for different values of I_d are given in Table 2.

Table 2. Variation of optimal solution for different values of I_d

I_d	T_2 (years)	Q_2 (units)	$Z_2(T)$ in \$	$\dfrac{\partial^2 Z_2}{\partial T^2}$
0.03	1.01301621	641.3761511	170.9417224	-973.0973133
0.04	1.04313340	680.0795564	192.9204222	-1042.467894
0.05	1.07045111	716.1659868	215.9983975	-1101.641031
0.06	1.09555795	750.1545136	240.0293231	-1153.147212
0.07	1.11886449	782.4110919	264.9004777	-1198.680662

From Table 1, we see that if I_d increases, then optimal order quantity Q_1 and optimal replenishment time T_1 decrease and total annual profit $Z_1(T)$ increases.

From Table 2, we see that if I_d increases, then optimal order quantity Q_2, optimal replenishment time T_2 and total annual profit $Z_2(T)$ increase.

7. Conclusion and Future Research

An inventory system was developed by considering the demand rate being stock–dependent. An algorithm was worked out to maximize retailer's profit per unit time. Numerical example and sensitivity analysis were given to validate the proposed model for case I and case II.

Further research may be carried out by considering variable deterioration in the system. An extension may consider the demand to be a function of time as well as price. Similarly, we can extend the model to allow for shortages.

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