Oligopoly with Capacity Constraints and Thresholds

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Abstract

Extended Oligopoly models will be introduced and examined in which the firms might face capacity limits, thresholds for minimal and maximal moves, and antitrust thresholds in the case of partial cooperation. Similar situation occurs when there is an additional cost of output adjustment, which is discontinuous at zero due to set-up costs. In these cases the payoff functions of the firms are nondifferentiable and in some cases even discontinuous. Under the usual concavity assumptions Cournot oligopolies have monotonic response functions and unique Cournot-Nash equilibrium. However the introduction of these more realistic additions into the oligopoly models creates a fundamentally new situation: the existence of no equilibrium or the presence of multiple, in some cases even infinitely many, equilibria. It also results in a very different asymptotic behavior of the dynamic extensions. The paper gives a brief survey of the relevant models, derives the response functions of the firms, and examines the existence and the number of equilibria. In the case of infinitely many equilibria the equilibrium-set will be also determined and characterized.

1. Introduction

Since the pioneering work of Cournot [3] oligopoly theory became one of the most frequently discussed topics in the literature of mathematical economics. Many different variants of oligopolies are known including single-product models without and with product differentiation, multi-product oligopolies, labor managed and rent-seeking games to mention only a few. A comprehensive summary of the different model types and some applications are given in Okuguchi and Szidarovszky [8]. In examining the static and dynamic versions of these models the authors make several simplifying assumptions and ignore certain important facts in order to make the mathematical development relatively simple.

In this paper we will introduce thresholds for output changes, capacity limits, antitrust thresholds and production adjustment costs into oligopolies. For our discussion we select the most simple oligopoly model in which the inverse demand function and all cost functions are linear. Even in this special case the existence and uniqueness of the equilibrium can be lost, there are cases without equilibrium and also cases with multiple, sometimes infinitely many equilibria.

The paper is organized as follows. Section 2 introduces the classical Cournot model with nonnegativity constraints and capacity limits. Output change thresholds are added to the model in Sections 3 and 4. Antitrust thresholds are introduced in Section 5, and the effect of output adjustment cost is examined in Section 6. The last section concludes the paper.

2. The Classical Cournot Model

Consider an industry with n firms that produce a single product or offers the same service to a homogeneous market. Let x_k be the output of firm k, $s = \sum_{k=1}^{n} x_k$ the total output of the industry, $p(s)$ the unit price and $c_k(x_k)$ the cost of firm k. Then the profit of this firm can be given as the difference of its revenue and cost:

$$\pi_k = x_k p(s) - c_k(x_k) = x_k p(x_k + s_k) - c_k(x_k), \qquad (1)$$

where $s_k = \sum_{l \neq k} x_l$ is the output of the rest of the industry.

For the sake of simplicity assume that

$$p(s) = A - Bs \quad \text{and} \quad C_k(x_k) = c_k x_k + d_k \qquad (2)$$

with positive values of A, B, c_k and d_k. In this special case the profit of firm k has the special form

$$\pi_k = x_k(A - Bx_k - Bs_k) - (c_k x_k + d_k). \qquad (3)$$

With given values of s_k, the best response of firm k is the output level that maximizes its profit. The first order conditions imply that

$$\frac{\partial \pi_k}{\partial x_k} = A - 2Bx_k - Bs_k - c_k = 0$$

so

$$R_k(s_k) = -\frac{s_k}{2} + \frac{A - c_k}{2B}. \qquad (4)$$

The usual dynamic process is adaptive adjustment:

$$x_k(t+1) = x_k(t) + K_k\Big(R_k\big(\sum_{l \neq k} x_l(t)\big) - x_k(t)\Big), \quad (1 \le k \le n) \qquad (5)$$

where $K_k > 0$ is the speed of adjustment of firm k.

Negative x_k value has no economic meaning. Some authors solve this problem by determining the confinement set C, which is a subset of the feasible output space with the following property. If the initial output is selected from C, then in the corresponding dynamic model the output will remain nonnegative for all future times. Then they simply ignore all trajectories with initial outputs selected outside C. A better approach is to require the nonnegativity of the output as a constraint in maximizing the profit. In this case (see Figure 1),

$$R_k(s_k) = \begin{cases} 0 & \text{if } A - Bs_k - c_k \le 0 \\ -\dfrac{s_k}{2} + \dfrac{A - c_k}{2B} & \text{otherwise.} \end{cases} \tag{6}$$

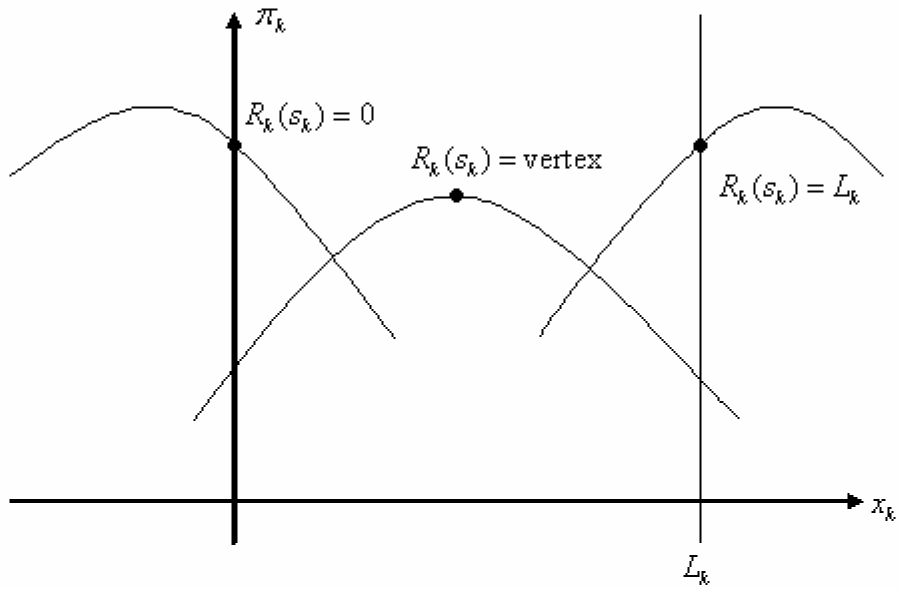

Figure 1. Shape of the profit function of firm k

If firm k has a capacity limit L_k, then the best response of firm k can be obtained by maximizing its profit subject to the constraint $0 \le x_k \le L_k$:

$$R_k(s_k) = \begin{cases} 0 & \text{if } A - Bs_k - c_k \le 0 \\ L_k & \text{if } A - 2BL_k - Bs_k - c_k \ge 0 \\ -\dfrac{s_k}{2} + \dfrac{A - c_k}{2B} & \text{otherwise.} \end{cases} \tag{7}$$

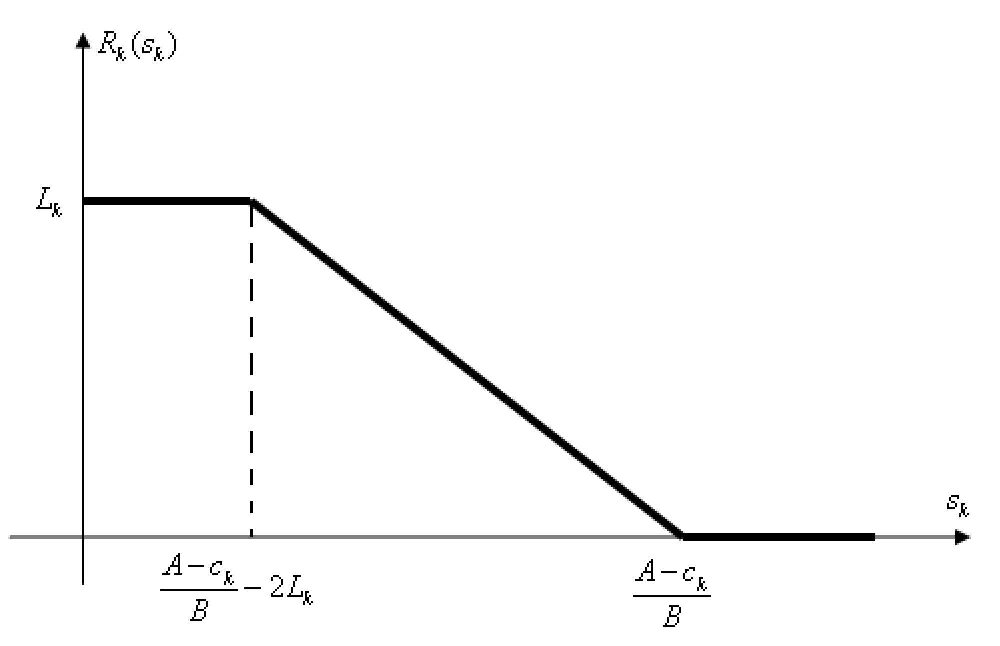

Figure 2. Best response function of firm k

Notice that $R_k(s_k)$ is continuous but not differentiable. If equilibrium is a break-point, Jacobian does not exist, so stability analysis becomes very difficult. It can be shown that there is always a unique equilibrium in the linear-linear case, so adding the nonnegativity constrait and capacity limits does not change the number of equilibria. Bischi *et al.* [1] presents a large collection of linear and nonlinear examples and the main theoretical results.

3. Oligopolies with Thresholds for Minimal Move

Assume next that firm k has a positive threshold ε_k such that if at any time period it has to make an output change below ε_k, then the firm does not make that change, keeps the current output. This constraint can be mathematically described as

$$x_k(t+1) = \begin{cases} x_k(t) & \text{if } K_k \left| R_k(\sum_{l \neq k} x_l(t)) - x_k(t) \right| < \varepsilon_k \\ x_k(t) + K_k(R_k(\sum_{l \neq k} x_l(t)) - x_k(t)) & \text{otherwise.} \end{cases} \quad (8)$$

With this threshold we lose the continuity of the system. Figure 3 shows the right hand side with $K_k = 1$.

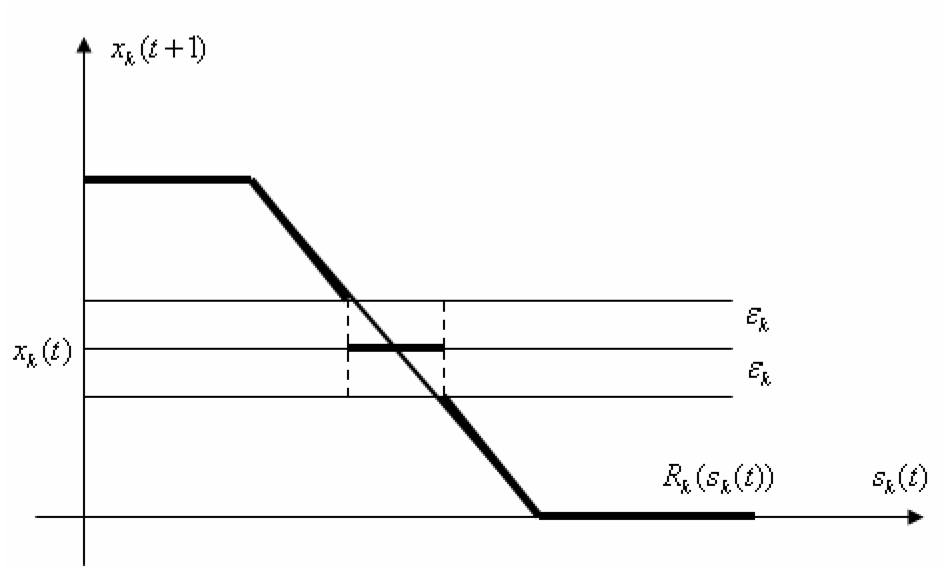

Figure 3. Dynamic rule (8)

Clearly a vector $(x_1^*, \ldots, x_n^*)$ is a steady state (or equilibrium) of the system if and only if for all k,

$$-\frac{\varepsilon_k}{K_k} < R_k\Big(\sum_{l \neq k} x_l^*\Big) - x_k^* < \frac{\varepsilon_k}{K_k}, \tag{9}$$

which system of inequalities usually has infinitely many solutions. Figure 4 shows the equilibrium-set in the duopoly case, which is characterized by the following system of linear inequalities:

$$\begin{aligned} &0 \le x \le L_1,\ 0 \le y \le L_2 \\ &R_1(y) - \frac{\varepsilon_1}{K_1} < x < R_1(y) + \frac{\varepsilon_1}{K_1} \\ &R_2(x) - \frac{\varepsilon_2}{K_2} < y < R_2(x) + \frac{\varepsilon_2}{K_2} \end{aligned} \tag{10}$$

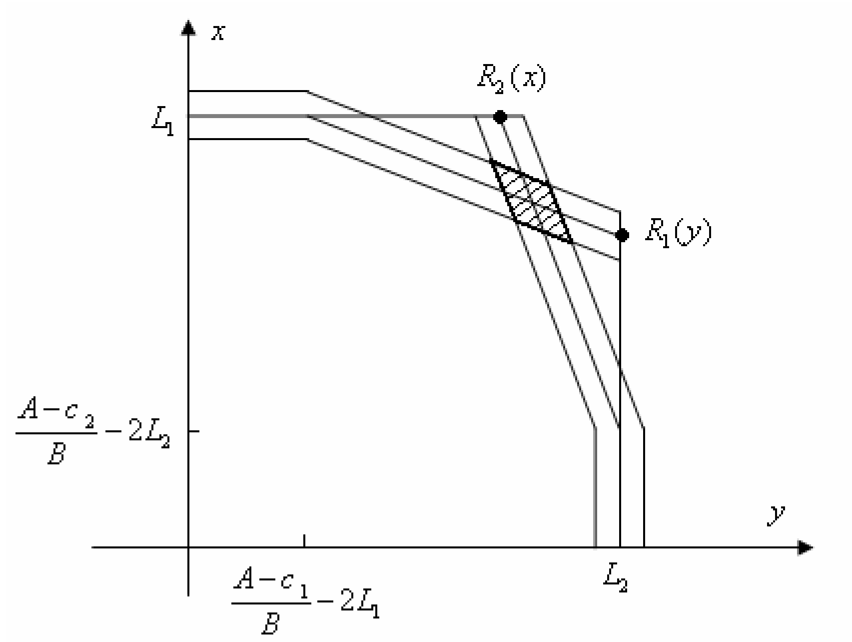

Figure 4. Illustration of the equilibrium set

Notice that the equilibrium of the game without the threshold remains equilibrium, so the existence of the equilibrium is guaranteed.

4. Oligopolies with Thresholds for Maximal Move

Assume next that the firms cannot make large output change, and each firm has a positive threshold Δ_k such that output change above Δ_k cannot be made. In this case system (8) is modified as follows:

$$x_k(t+1) = \begin{cases} x_k(t) - \Delta_k & \text{if } R_k(\sum_{l \neq k} x_l(t)) - x_k(t) < -\dfrac{\Delta_k}{K_k} \\ x_k(t) + \Delta_k & \text{if } R_k(\sum_{l \neq k} x_l(t)) - x_k(t) > \dfrac{\Delta_k}{K_k} \\ x_k(t) + K_k(R_k(\sum_{l \neq k} x_l(t)) - x_k(t)) & \text{otherwise.} \end{cases} \tag{11}$$

With this threshold we lose the differentiability of the system. Figure 5 shows the right hand side with $K_k = 1$. This kind of threshold does not change the steady state, so it exists and is unique.

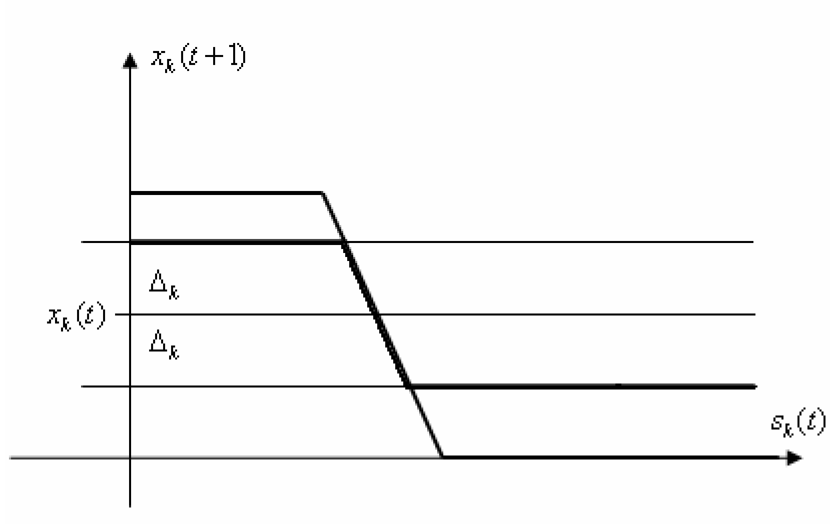

Figure 5. Dynamic Rule (11)

5. Antitrust Threshold

In this section we assume that the firms partially cooperate in order to increase profit. If π_l is the profit of firm l $(1 \le l \le n)$, then we assume that the payoff function of firm k is the following:

$$\varphi_k = \pi_k + \sum_{l \ne k} \gamma_{kl} \pi_l \,, \tag{12}$$

where γ_{kl} is the cooperation level of firm k toward its competitor l. In the linear-linear case

$$\varphi_k = x_k(A - Bs_k - Bx_k) - (c_k x_k + d_k) + \sum_{l \ne k} \gamma_{kl}(x_l(A - Bs_k - Bx_k) - (c_l x_l + d_l)).$$

If we make the simplifying assumption that $\gamma_{kl} \equiv \gamma_k$ for all l, that is , each firm has the same cooperation level toward its competitors, then

$$\varphi_k = x_k(A - Bs_k - Bx_k) - (c_k x_k + d_k) + \gamma_k s_k (A - Bs_k - Bx_k) - \gamma_k \sum_{l \ne k} (c_l x_l + d_l) \,, \tag{13}$$

By simple differentiation

$$\frac{\partial \varphi_k}{\partial x_k} = A - 2Bx_k - B(1+\gamma_k)s_k - c_k \,,$$

so the best response of firm k is the following:

$$\overline{R}_k(s_k) = \begin{cases} 0 \quad \text{if } A - B(1+\gamma_k)s_k - c_k \le 0 \\ L_k \;\; \text{if } A - 2BL_k - B(1+\gamma_k)s_k - c_k \ge 0 \\ -\dfrac{(1+\gamma_k)s_k}{2} + \dfrac{A - c_k}{2B} \quad \text{otherwise,} \end{cases} \tag{14}$$

which is shown in Figure 6.

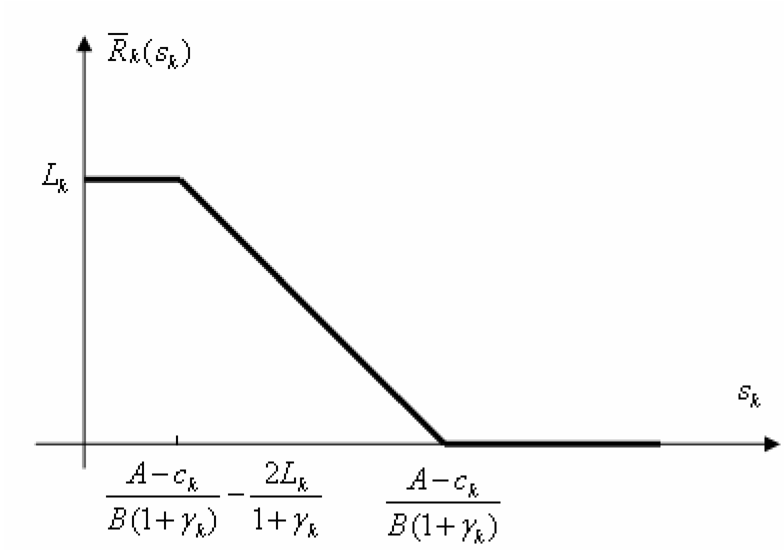

Figure 6. Best response of firm k with partial cooperation

Clearly, $\overline{R_k}(s_k) \leq R_k(s_k)$ for all s_k and k, that is, the best response of the firms decrease by introducing partial cooperation. It is easy to see that in case of $\gamma_k < 1$ for all k, there is a unique equilibrium, and the equilibrium industry output decreases with increasing cooperation levels. Consider next the special case of $\gamma_k \equiv 1$ and $c_k \equiv c$. In this case each firm maximizes the common payoff function

$$\begin{aligned}\varphi &= \sum_{k=1}^{n}(x_k(A-Bs)-(cx_k+d_k)) \\ &= s(A\text{-}Bs)-cs-\sum_{k=1}^{n}d_k\end{aligned} \tag{15}$$

where $s=\sum_{k=1}^{n}x_k$ is the industry output. This is a concave parabola in s which is maximal if

$$s^* = \begin{cases} 0 \ \text{ if } A-c \leq 0 \\ \sum_{k=1}^{n} L_k \ \text{ if } A-2B\sum_{k=1}^{n} L_k - c \geq 0 \\ \dfrac{A-c}{2B} \ \text{ otherwise.} \end{cases}$$

The equilibrium is unique, $x_1^* = \ldots = x_n^* = 0$, in the first case. Otherwise there are infinitely many equilibria forming the set

$$X^* = \left\{ (x_1^*, \ldots, x_n^*) \mid 0 \le x_k^* \le L_k \text{ for all } k, \sum_{k=1}^{n} x_k^* = s^* \right\}.$$

Consider next the following dynamic process. If at any time period t, the industry output is at least $\overline{S}$, where $\overline{S}$ is a given threshold, then the firms continue their partial cooperation. Otherwise they stop cooperating, and for the next time period each firm adjusts its output toward its noncooperative best response. If $R_k(s_k)$ and $\overline{R}_k(s_k)$ denote the noncooperative and partially cooperative best responses, then this dynamic rule can be formulated as

$$x_k(t+1) = \begin{cases} x_k(t) + K_k\left(R_k\left(\sum_{l \ne k} x_l(t)\right) - x_k(t)\right) & \text{if } \sum_{l=1}^{n} x_l(t) < \overline{S} \\ x_k(t) + K_k\left(\overline{R_k}\left(\sum_{l \ne k} x_l(t)\right) - x_k(t)\right) & \text{otherwise.} \end{cases} \tag{16}$$

Figure 7 shows the right hand side with $K_k = 1$.

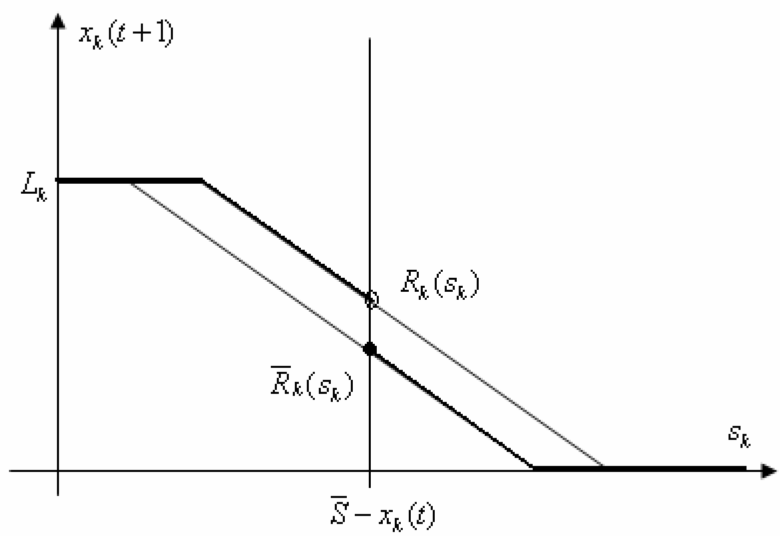

Figure 7. Dynamic rule (16)

Let $(x_1^*, \ldots, x_n^*)$ denote the noncooperative equilibrium and $(x_1^{**}, \ldots, x_n^{**})$ the partially cooperative equilibrium, which is unique if $\gamma_k < 1$ for all k. From the dynamic equation (16) we conclude that $(x_1^*, \ldots, x_n^*)$ is a steady state (or equilibrium) if and only if

$\sum_{k=1}^{n} x_k^* < \overline{S}$, and $(x_1^{**}, \ldots, x_n^{**})$ is a steady state (or equilibrium) if and only if $\sum_{k=1}^{n} x_k^{**} \geq \overline{S}$. All steady states can be obtained in this way. Therefore we have the following cases:

(i) If $\sum_{k=1}^{n} x_k^* < \overline{S}$, then $(x_1^*, \ldots, x_n^*)$ is the unique equilibrium;

(ii) If $\sum_{k=1}^{n} x_k^{**} \geq \overline{S}$, then $(x_1^{**}, \ldots, x_n^{**})$ is the unique equilibrium;

(iii) If $\sum_{k=1}^{n} x_k^{**} < \overline{S} \leq \sum_{k=1}^{n} x_k^*$, then there is no equilibrium.

Since $\sum_{k=1}^{n} x_k^{**} \leq \sum_{k=1}^{n} x_k^*$, exactly one of cases (i), (ii), (iii) occurs with any particular selection of the threshold $\overline{S}$.

For the fundamentals of partial cooperation and antitrust thresholds see for example, Cyert and DeGroot [4], Chiarella and Szidarovszky[2] and Matsumoto *et al.* [6].

6. Oligopolies with Output Adjustment Costs

In addition to the classical model discussed in Section 2 we will now assume that the firms have additional cost due to output increase: $K_k(x - x(t))$, where x is the planned output at time period $t+1$. Function K_k is assumed to be linear:

$$K_k(x - x(t)) = \begin{cases} 0 & \text{if } x \leq x(t) \\ \alpha_k (x - x(t)) & \text{otherwise.} \end{cases} \tag{17}$$

In this case K_k is continuous, no set-up cost is assumed that would result in a jump of the function at zero.

Consider now the payoff function of firm k at time period $t+1$:

$$\pi_k = x_k(A - Bs_k - Bx_k) - (c_k x_k + d_k) - \begin{cases} 0 & \text{if } x_k \leq x_k(t) \\ \alpha_k (x_k - x_k(t)) & \text{otherwise.} \end{cases} \tag{18}$$

The best response depends on both s_k and $x_k(t)$, and we have several cases. Notice that

$$\frac{\partial \pi_k}{\partial x_k} = A - Bs_k - 2Bx_k - c_k - \begin{cases} 0 & \text{if } x_k < x_k(t) \\ \alpha_k & \text{if } x_k > x_k(t), \end{cases} \tag{19}$$

and if $x_k = x_k(t)$, then the left and right hand side derivatives are different.

(i) $R_k(s_k, x_k(t)) = 0$, if

$$\left.\frac{\partial \pi_k}{\partial x_k}\right|_{x_k = 0+} \leq 0$$

This is the case when

$$A - Bs_k - c_k \le 0.$$

Here $x_k = 0+$ means the right hand side derivative at zero. That is,

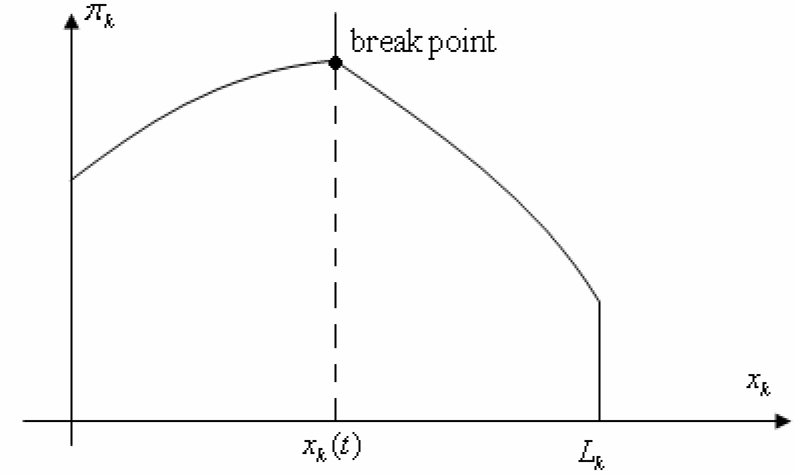

Figure 8. Graph of π_k with output adjustment cost

$$s_k \ge \frac{A - c_k}{B}. \tag{20}$$

(ii) $R_k(s_k, x_k(t)) \in (0, x_k(\mathrm{t}))$, if

$$\left.\frac{\partial \pi_k}{\partial x_k}\right|_{x_k = x_k(t)-} < 0 < \left.\frac{\partial \pi_k}{\partial x_k}\right|_{x_k = 0+}$$

which can be rewritten as

$$\frac{A - c_k}{B} - 2x_k(t) < s_k < \frac{A - c_k}{B}. \tag{21}$$

In this case the vertex in interval $(0, x_k(t))$ is the profit maximizing output:

$$R_k(s_k, x_k(t)) = \frac{A - c_k}{2B} - \frac{s_k}{2}. \tag{22}$$

(iii) $R_k(s_k, x_k(t)) = x_k(t)$, if

$$\left.\frac{\partial \pi_k}{\partial x_k}\right|_{x_k = x_k(t)-} \ge 0 \ge \left.\frac{\partial \pi_k}{\partial x_k}\right|_{x_k = x_k(t)+}.$$

This condition can be rewritten as

$$\frac{A - c_k - \alpha_k}{B} - 2x_k(t) \le s_k \le \frac{A - c_k}{B} - 2x_k(t). \tag{23}$$

(iv) $R_k(s_k, x_k(t)) \in (x_k(t), L_k)$, if

$$\left.\frac{\partial \pi_k}{\partial x_k}\right|_{x_k = x_k(t)+0} > 0 > \left.\frac{\partial \pi_k}{\partial x_k}\right|_{x_k = L_k -}$$

which can be written as

$$\frac{A - c_k - \alpha_k}{B} - 2L_k < s_k < \frac{A - c_k - \alpha_k}{B} - 2x_k(t). \tag{24}$$

In this case the vertex in interval $(x_k(t), L_k)$ is the best response:

$$R_k(s_k, x_k(t)) = \frac{A - c_k - \alpha_k}{2B} - \frac{s_k}{2}. \tag{25}$$

(v) $R_k(s_k, x_k(t)) = L_k$, if

$$\left.\frac{\partial \pi_k}{\partial x_k}\right|_{x_k = L_k - 0} \geq 0,$$

which can be rewritten as

$$s_k \leq \frac{A - c_k - \alpha_k}{B} - 2L_k. \tag{26}$$

Figure 9 shows the graph of the best response function.

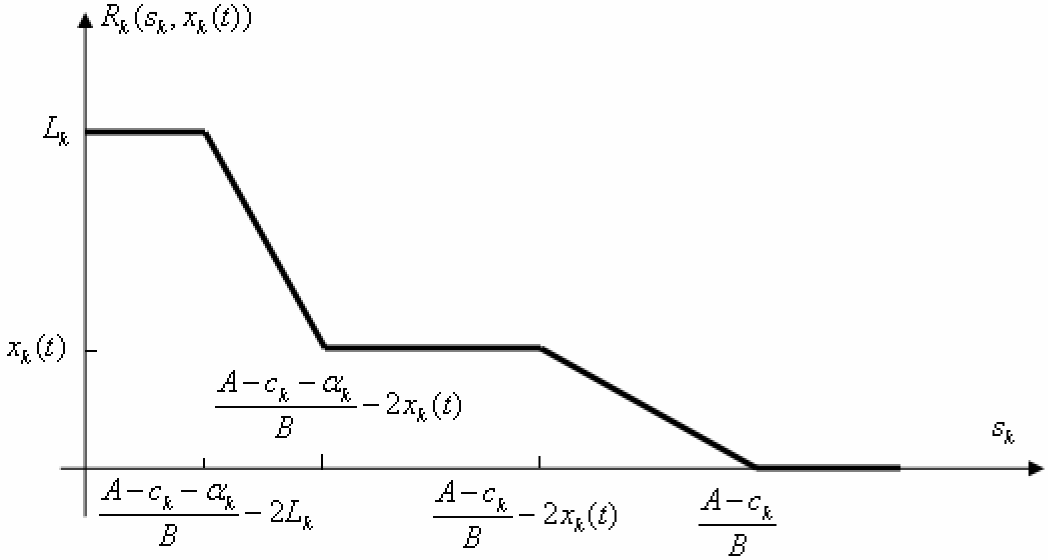

Figure 9. Best response with output adjustment costs

An output vector $(x_1^*, \ldots, x_n^*)$ is a steady state (or equilibrium) if and only if for all k,

$$\left.\frac{\partial \pi_k}{\partial x_k}\right|_{x_k = x_k^* + 0} \leq 0 \leq \left.\frac{\partial \pi_k}{\partial x_k}\right|_{x_k = x_k^* - 0} \tag{27}$$

where all other coordinates are at equilibrium level. If $x_k^* = 0$, then only the left hand side of the relation is required, and in the case of $x_k^* = L_k$, only the right hand side has to be satisfied. Simple algebra shows that these conditions can be rewritten as follows:

$$\frac{A - c_k - \alpha_k}{2B} - \frac{1}{2}\sum_{l \neq k} x_l^* \leq x_k^* \leq \frac{A - c_k}{2B} - \frac{1}{2}\sum_{l \neq k} x_l^* \tag{28}$$

with the additional constraint

$$0 \leq x_k^* \leq L_k \ .$$

There are usually infinitely many equilibria and they form a polyhedron. Figure 10 illustrates the equilibrium set in a duopoly, where $A = 20, B = 1, L_1 = L_2 = 10$ and $c_1 = c_2 = \alpha_1 = \alpha_2 = 5$.

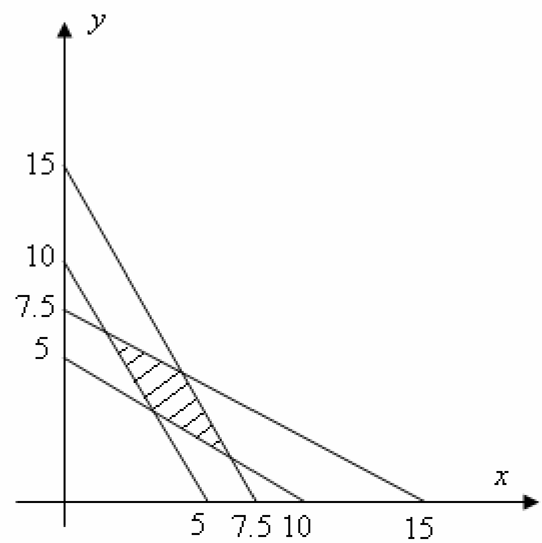

Figure 10. Equilibrium set in a duopoly with output adjustment costs

In the case of discontinuous additional costs, the situation becomes much more complicated. By assuming positive set-up costs,

$$K_k(x - x(t)) = \begin{cases} 0 & \text{if} \quad x \leq x(t) \\ \alpha_k(x - x(t)) + \beta_k & \text{otherwise.} \end{cases} \tag{29}$$

Figure 11 shows the payoff function of firm k. There are many possibilities for the location of the profit maximizing output. Figure 12 shows the graph of the best response functions.

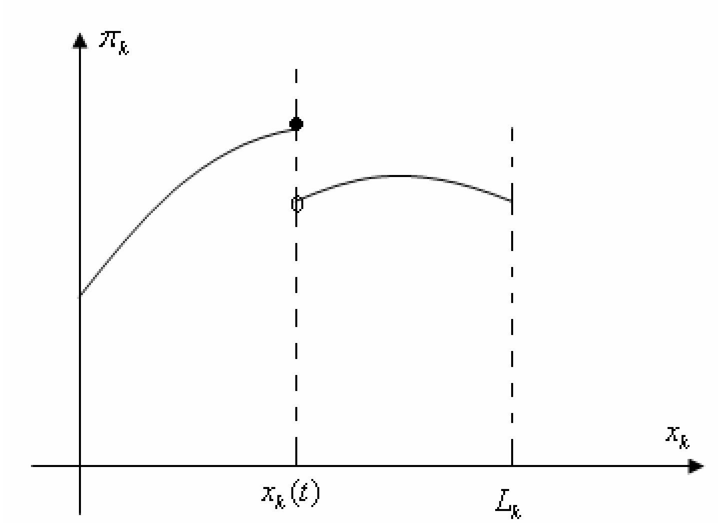

Figure 11. Shape of the profit function of firm k

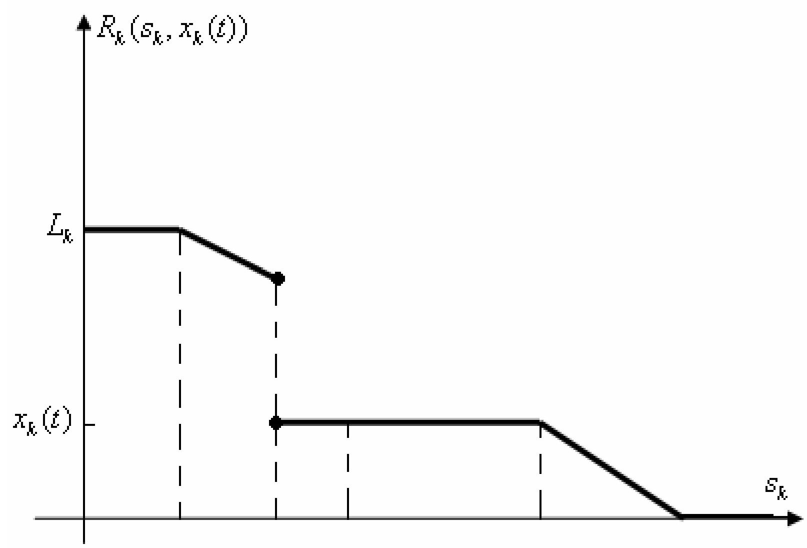

Figure 12. Best response with discontinuous output adjustment cost

Notice that there is the possibility of two best responses, so response selection becomes here a new issue. The equilibrium set also becomes more complicated and defined by nonlinear inequalities. A comprehensive literature review and a summary of the different possible cases in nonlinear oligopolies with output adjustment costs are presented in Szidarovszky[9].

7. Conclusions

Extended oligopoly models were introduced and examined in which the firms' outputs were assumed to be nonnegative and bounded in addition to thresholds for minimum and maximum output change, antitrust thresholds and output adjustment costs. We illustrated how each of these additional factors alter the best responses of the firms and the number of equilibria. We can lose the existence, the uniqueness of the equilibrium, the best response might become multiple even in the most simple cases of linear price and linear cost functions. In each case we were able to derive the best response functions of the firms and to characterize the set of equilibria. Some initial simulation studies (Matsumoto *et al.* [6], Dabkowski, [5]) show that the asymptotical behavior of the dynamic extensions of these models may become very complex.

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