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Sensitivity Analysis of SAW Technique: the Impact of Changing the Decision Making Matrix Elements on the Final Ranking of Alternatives

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Most of data in a multi attribute decision making (MADM) problem are unstable and changeable, and thus sensitivity analysis can effectively contribute to making proper decisions. Here, we offer a new method for sensitivity analysis of multi-attribute decision making problems so that by changing one element of decision making matrix, we can determine changes in the results of a decision making problem. An analysis is made for simple additive weighting method (SAW) technique, a mostly used multi-attribute decision making techniques, and the corresponding formulas are obtained.

Keywords: *Multi-attribute decision making (MADM), SAW Technique, Sensitivity analysis, Ranking methods, Attribute weights.*

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1. Introduction

Multi-attribute decision making models are selector models which are used for evaluating, ranking and selecting the most appropriate alternative from among alternatives. Alternatives of an MADM problem are evaluated by k attributes and the most appropriate alternative is selected or, they are ranked in accordance with attributes' values for the alternatives and the importance of the attributes for the decision maker.

An MADM model is formulated as a decision making matrix as follows:

$$\begin{array}{c c} & \begin{array}{cccc} C_1 & C_2 & \cdots & C_k \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_m \end{array} & \begin{bmatrix} d_{11} & d_{12} & \dots & d_{1k} \\ d_{21} & d_{22} & \dots & d_{2k} \\ \vdots & \vdots & \dots & \vdots \\ d_{m1} & d_{m2} & \dots & d_{mk} \end{bmatrix} \end{array}$$

Where $A_1, A_2, A_3, \cdots, A_m$ are available and predetermined m alternatives and $C_1, C_2, C_3, \cdots, C_k$ are effective k attributes in decision making which are used for measuring the utility of each

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alternative and d_{ij} is special value of the attribute jth for the alternative *i*th, that is, the efficiency of the *i*th alternative versus the *j*th attribute.

The most important issue in MADM models is that the data used are unstable and changeable. Being so, sensitivity analysis after solving the problem can effectively contribute to making accurate decisions.

Early work in this field are the works of Evans [3], Fishburn and Isaacs [4], Schneller and Sphicas [11], which focused on determining decision sensitivity to probabilistic estimation errors. Soofi [13] and Barron and Schmidt [2] proposed sensitivity analysis for additive MADM models. They assumed a set of weights for the attributes and obtained a new set of weights so that the efficiency of alternatives were equal or their order changed. Ma et al. [7] studied the structure of the weight set and conditions that lead to special ranking or priorities of alternatives and discussed additive decision making models. Rios and French [9] by offering a method for sensitivity analysis studied the result of changes in the weights of attributes on the final score of alternatives in MADM models and calculated the required change the weights for changing the optimal solution. These algorithms and methods were revised by Rios et al. [10]. Triantaphyllou and Sanchez [15] studied two types of sensitivity analyse for two MADM methods. First, they determined the most sensitive attribute and calculated the change in the weights that lead to change in the ranking of alternatives and second they measured the sensitivity of the decision making matrix elements .Zavadskas et al. [16] proposed a model to determine sensitivity to changes of separate parameters to increase the reliability of the applied methods. ToloieEshlaghy et al. [14] studied a sensitivity analysis approach to produce complementary information by determination of criterion values in the decision making matrix. Hsingyeh [5] presented a new approach to the selection of compensatory MADM methods for a specific cardinal ranking problem via sensitivity analysis of the attribute weights. Memariani et al. [8] offered a new method for sensitivity analysis of MADM problems so that by using it and changing the weights of the attributes one could determine changes in the final results of a decision making problem. Simanaviciene and Ustinovichius [12] presented sensitivity analysis of TOPSIS and SAW methods. They analyzed the quantitative multiple criteria decision making methods and sensitivity analysis methods used in decision support systems. Both methods are strongly mathematically based. They took notice of these sensitivity methods for the initial data. Monte Carlo method was applied to generate the initial data. Alinezhad and Amini [1] presented a new method for sensitivity analysis in multi-attribute decision making problems in which if the weights of one attribute changed, then changes in the results of the problem was determined. These changes involved changes in the weights of other attributes and changes in the final ranks of alternatives. In line with the context-dependent concept of informational importance, the approach examined the consistency degree among the relative degree of sensitivity of individual attributes using an MADM method and the relative degree of influence of the corresponding attributes indicated by Shannon's entropy concept.

In Section 2, we review the SAW technique and discuss some corresponding formulas and relations. In Section 3, we present a new method for sensitivity analysis of MADM models. We first study the result of change in one entry of the decision making matrix on the final score of alternatives and establish the resulting relations. In Section 4, by working through a numerical example the obtained relations and formulas are verified and their accuracies are confirmed. Finally, we summarize our conclusions and provide suggestions for further researches.

2. The SAW Technique

The SAW technique is one of the most used MADM techniques. It is simple and serves the basis of most MADM techniques such as AHP and PROMETHEE which benefits from additive property to calculate final scores of alternatives. In SAW technique, final score of each alternative is calculated as follows:

$$P_i = \sum_{j=1}^{K} w_j \cdot r_{ij}\,, \quad i = 1, \dots, m, \tag{1}$$

where r_{ij} are normalized values of the decision matrix elements, that is,

$$r_{ij} = \frac{d_{ij}}{d_j^{\max}}, \quad d_j^{\max} = \max_{1 \le i \le m} d_{ij}, \quad j = 1, \dots, k, \tag{2}$$

for profit attributes, or

$$r_{ij} = \frac{d_j^{\min}}{d_{ij}}, \quad d_j^{\min} = \min_{1 \le i \le m} d_{ij}, \quad j = 1, \dots, k, \tag{3}$$

For cost attributes. For any qualitative attributes, one can use appropriate methods to transform qualitative variables to quantitative ones.

3. Identifying the Impact of Change in one Element of Decision Making Matrix on the Final Score of Alternatives

Available sensitivity analysis models for MADM problems mostly focus on determining the most sensitive attribute so that with the least change, the current ranking of alternatives is changed. Here, we consider a new method for sensitivity analysis of MADM problems to calculate the change in the final score of alternatives when a change occurs in one element of the decision making matrix.

In the SAW model, when the kth attribute in lth alternative changes, that is, when the element d_{lk} in the decision matrix changes, the normalized values at kth column in the decision matrix are changed and other values remain unchanged because a linear norm is used and normalization is applied separately for the columns of the decision matrix.

In SAW model, if element d_{lk} in the decision matrix changes to Δ, then eight separate states arise in accordance with the followings:

- whether the attribute is of profit or cost type,
- whether d_{lk} is the most desirable at its column or not,
- after performing the change, whether d'_{lk} or the changed element remains as the most desirable element or not.

The following results distinguish these states.

Theorem 3.1. Assume that the element d_{lk} changes to $d'_{lk} = d_{lk} + \Delta$ and that the kth column is of the profit type and d_{lk} is the most desirable at the column, i.e., $d_{lk} = \max_{1 \le i \le m} d_{ik}$, and after changing d_{lk} to d'_{lk}, d'_{lk} is also the most desirable element at kth column. Then, normalized values of kth column are changed to:

$$r_{ik} = \frac{d_{lk}}{d_{lk}+\Delta}.r_{ik}, \;\; i=1,\dots,m, \;\; i \neq l, \qquad (4)$$
$$r'_{lk} = r_{lk} = 1,$$

and final scores of alternatives are:

$$P'_i = P_i - \frac{\Delta}{d_{lk}+\Delta} \cdot w_k \cdot r_{ik}, \;\; i=1,\dots,m, \;\; i \neq l, \qquad (5)$$
$$P'_l = P_l.$$

Proof. Since d'_{lk} is the most desirable at column k, normalized values of kth column are:

$$r'_{ik} = \frac{d_{ik}}{d'_{lk}} = \frac{d_{ik}}{d_{lk}+\Delta}. \qquad (6)$$

From $r_{ik} = \frac{d_{ik}}{d_{lk}}$, we have $d_{ik} = r_{ik} \cdot d_{lk}$. Replacing this in (6), we have

$$r'_{ik} = \frac{d_{lk}}{d_{lk}+\Delta} \cdot r_{ik}, \;\; i=1,\dots,m, \;\; i \neq l, \qquad (7)$$
$$r'_{lk} = \frac{d'_{lk}}{d'_{lk}} = 1 = r_{lk}, i = 1.$$

Therefore, the final scores of alternatives are:

$$P'_i = \sum_{\substack{j=1 \\ j \neq k}}^{r} r_{ij} \cdot w_j + r'_{ik} \cdot w_k = \sum_{\substack{j=1 \\ j \neq k}}^{r} r_{ij} \cdot w_j + \frac{d_{lk}}{d_{lk}} \cdot r_{ik} \cdot w_k$$
$$= \sum_{\substack{j=1 \\ j \neq k}}^{r} r_{ij} \cdot w_j + \left(1 - \frac{d_{lk}}{d_{lk}+\Delta}\right) \cdot r_{ij} \cdot w_j = \sum_{\substack{j=1 \\ j \neq k}}^{r} r_{ij} \cdot w_j - \frac{\Delta}{d_{lk}+\Delta} \cdot r_{ik} \cdot w_k, \qquad (8)$$
$$P'_l = \sum_{\substack{j=1 \\ j \neq k}}^{r} r_{ij} \cdot w_j + r'_{lk} \cdot w_k = \sum_{j=1}^{r} r_{ij} w_j = P_l.$$

Theorem 3.2. Assume that the element d_{lk} changes to $d'_{lk} = d_{lk} + \Delta$, and that the kth column is of the profit type and element d_{lk} is the most desirable one at the column, i.e., $d_{lk} = \max_{1 \leq i \leq m} d_{ik}$, and after changing d_{lk} to d'_{lk}, d'_{lk} is not the most desirable element at kth column, that is, $d'_{lk} < \max_{\substack{1 \leq i \leq m \\ i \neq l}} d_{ik} = d^*_k$. Then, normalized values of kth column are changed to

$$r'_{ik} = \frac{d_{ik}}{d'^*_k}, \;\; i=1,\dots,m, \;\; i \neq l, \qquad (9)$$
$$r'_{lk} = \frac{d_{lk}+\Delta}{d'^*_k},$$

and the final scores of alternatives are:

$$P_i' = P_i + \left(\frac{d_{ik}}{d_k'^*} - \frac{d_{ik}}{d_{lk}}\right) \cdot w_k, \ \ i = 1, \dots, m, \ \ i \neq l, \\ P_l' = P_l + \left(\frac{d_{lk} + \Delta}{d_k'^*} - 1\right) \cdot w_k. \tag{10}$$

Proof. Since $\max\limits_{1 \leq i \leq m} d_i^k = d_k'^*$, the normalized values of kth column are:

$$r_{ik}' = \frac{d_{ik}}{d_k'^*}, \ \ i = 1, \dots, m, \ \ i \neq l, \\ r_{lk}' = \frac{d_{lk} + \Delta}{d_k'^*}, \tag{11}$$

Therefore, the final scores of alternatives are:

$$\begin{aligned} P_i' &= \sum_{\substack{j=1 \\ j \neq k}}^{r} r_{ij} \cdot w_j + r_{ik}' \cdot w_k = \sum_{j=1}^{r} r_{ij} \cdot w_j + r_{ik}' \cdot w_k - r_{ik} \cdot w_k \\ &= \sum_{j=1}^{r} r_{ij} \cdot w_j + (r_{ik}' - r_{ik}) \cdot w_k = P_i + \left(\frac{d_{ik}}{d_k'^*} - \frac{d_{ik}}{d_{lk}}\right) \cdot w_k, \qquad i = 1, \dots, m, \qquad i \neq l, \\ P_l' &= \sum_{\substack{j=1 \\ j \neq k}}^{r} r_{ij} \cdot w_j + (r_{ik}' - r_{ik}) \cdot w_k = P_l + \left(\frac{d_{lk} + \Delta}{d_k'^*} - 1\right) \cdot w_k. \end{aligned} \tag{12}$$

Theorem 3.3. Assume that the element d_{lk} changes to $d_{lk}' = d_{lk} + \Delta$, and that the kth column is of the profit type and d_{lk} is not the most desirable one at the column, that is, $d_{lk} < \max\limits_{1 \leq i \leq m} d_i^* = d_k'^*$, and after changing d_{lk} to d_{lk}', d_{lk}' is not the most desirable element at kth column, that is, $d_{lk}' < \max\limits_{\substack{1 \leq i \leq m \\ i \neq k}} d_{ik} = d_k'$. Then, normalized values of kth column except for the lth element will not change, that is, we have

$$r_{ik}' = r_{ik}, i = 1, \dots, m, i \neq l, \\ r_{lk}' = r_{lk} + \frac{\Delta}{d_k^*}. \tag{13}$$

and the final scores of alternatives, except for the lth alternative, will not change and we have

$$P_l' = P_l + \frac{\Delta}{d_k^*} \cdot w_k. \tag{14}$$

Proof. Normalized values of kth column are calculated by dividing the values of kth column into d_k^* and then normalized values d_{lk} are computed as:

$$r_{lk}' = \frac{d_{lk}'}{d_k^*} = \frac{d_{lk} + \Delta}{d_k^*} = \frac{d_{lk}}{d_k^*} + \frac{\Delta}{d_k^*} = r_{lk} + \frac{\Delta}{d_k^*}. \tag{15}$$

Therefore, final score of kth alternative would be

$$P_l' = \sum_{\substack{j=1 \\ j \neq k}}^{r} r_{lj} \cdot w_j + r_{lk}' \cdot w_k = \sum_{j=1}^{r} r_{lj} \cdot w_j + \frac{\Delta}{d_k^*} \cdot w_k = P_l + \frac{\Delta}{d_k^*} \cdot w_k. \tag{16}$$

Theorem 3.4. Assume that the element d_{lk} changes to $d'_{lk} = d_{lk} + \Delta$, and that the kth column is of the profit type and element d_{lk} is not the most desirable one at the column, that is, $d_{lk} < \max_{1 \le i \le m} d_{ik} = d'^{*}_{k}$, and after changing d_{lk} to d'_{lk}, d'_{lk} is the most desirable element at kth column that is, $d'_{lk} < \max_{\substack{1 \le i \le m \\ i \ne l}} d_{ik}$. Then, normalized values of kth column are changed to

$$r'_{ik} = \frac{d_k^*}{d_{lk} + \Delta} . d_{lk}, \;\; i = 1, \dots, m, \;\; i \ne l, \qquad (17)$$
$$r_{ik} = 1, i = l,$$

and the final scores of alternatives are:

$$P'_i = P_i + \left(\frac{d_k^*}{d_{lk} + \Delta} - 1 \right) \cdot w_k \cdot r_{ik}, \;\; i = 1, \dots, m, \;\; i \ne l, \qquad (18)$$
$$P'_l = P_l + (1 - r_{lk}) \cdot w_k .$$

Proof. The normalized values of kth column are:

$$r'_{ik} = \frac{d_{ik}}{d'_{lk} + \Delta} = \frac{d_{ik}}{d_{lk} + \Delta}, \qquad (19)$$

and because $r_{ik} = \dfrac{d_{ik}}{d_k^*}$, then $d_{ik} = r_{ik} . d_k^*$ and by replacing it in (19), we have

$$r'_{ik} = \frac{d_{ik}}{d_{lk} + \Delta} = \frac{d_k^*}{d_{lk} + \Delta} . r_{ik}, \;\; i = 1, \dots, m, \;\; i \ne l, \qquad (20)$$
$$r'_{lk} = \frac{d'_{lk}}{d'_{lk}} = 1, i = l.$$

Therefore, the final scores of alternatives are:

$$\begin{aligned} P'_i &= \sum_{\substack{j=1 \\ j \ne k}}^{r} r_{ij} \cdot w_j + r'_{ik} \cdot w_k = \sum_{\substack{j=1 \\ j \ne k}}^{r} r_{ij} \cdot w_j + \frac{d_k^*}{d_{lk} + \Delta} \cdot r_{ik} \cdot w_k \\ &= \sum_{j=1}^{r} r_{ij} \cdot w_j + \left(\frac{d_k^*}{d_{lk} + \Delta} - 1 \right) \cdot r_{ik} \cdot w_k \qquad ; i = 1, 2, \dots, m \;, i \ne l \\ P'_l &= \sum_{\substack{j=1 \\ j \ne k}}^{r} r_{lj} \cdot w_j + w_k = \sum_{j=1}^{r} r_{lj} \cdot w_j + w_k - r_{lk} \cdot w_k = P_l + (1 - r_{lk}) \cdot w_k \qquad ; i = l \end{aligned} \qquad (21)$$

The above four theorems demonstrate the four states corresponding to attribute of the profit type. Next, we consider states corresponding to the cost type.

Theorem 3.5. Assume that the element d_{lk} changes to $d'_{lk} = d_{lk} + \Delta$, and that the kth column is of the cost type and element d_{lk} is the most desirable one at the column, that is, $d_{lk} = \max_{1 \le i \le m} d_{ik}$, and after changing d_{lk} to d'_{lk}, d'_{lk} is also the most desirable element at kth column. Then, normalized values of kth column are changed to:

$$r'_{ik} = r_{ik} + \frac{\Delta}{r_{lk}}, i = 1, \dots, m, i \ne l, \qquad (22)$$
$$r'_{ik} = r_{ik} = 1, \;\; i = l,$$

and the final scores of alternatives are:

$$P_i' = P_i + \frac{\Delta}{d_{ik}} \cdot w_k, \;\; i = 1, \dots, m, \;\; i \neq l, \qquad (23)$$
$$P_l' = P_l.$$

Proof. We have $d_{lk}' = \max\limits_{\substack{1 \le i \le m \\ i \neq l}} d_{ik}$. Then, $d_{lk}' = d_k'^{\min}$ and normalized values of kth column are:

$$r_{ik}' = \frac{d_{lk}'}{d_{ik}} = \frac{d_{lk} + \Delta}{d_{ik}} = \frac{d_{lk}}{d_{ik}} + \frac{\Delta}{d_{ik}} = r_{lk} + \frac{\Delta}{d_{ik}}, \;\; i = 1, \dots, m, \;\; i \neq l, \qquad (24)$$
$$r_{lk} = \frac{d_{lk}'}{d_{lk}'} = 1.$$

Therefore, the final scores of alternatives are:

$$P_i' = \sum_{\substack{j=1 \\ j \neq k}}^{r} r_{ij} \cdot w_j + r_{ij}' \cdot w_k = \sum_{j=1}^{r} r_{lj} \cdot w_j + r_{ik}' \cdot w_k - r_{ik} \cdot w_k$$
$$= \textstyle\sum_{j=1}^{r} r_{lj} \cdot w_j + \frac{\Delta}{d_{ik}} \cdot w_k \quad ; i = 1, \dots, m \;, i \neq l \qquad (25)$$
$$P_l' = P_l \;\; ; i = l$$

Theorem 3.6. Assume that the element d_{lk} changes to $d_{lk}' = d_{lk} + \Delta$, and that the kth column is of the cost type and element d_{lk} is the most desirable one at the column, that is, $d_{lk} = \max\limits_{1 \le i \le m} d_{ik}$, but after changing d_{lk} to d_{lk}', d_{lk}' is not the most desirable element at kth column, that is, $d_{lk}' > \min\limits_{\substack{1 \le i \le m \\ i \neq l}} d_{ik} = r_k'^{\min}$. Then, normalized values of kth column are:

$$r_{ik}' = \frac{d_k'^{\min}}{d_{ik}}, \;\; i = 1, \dots, m, \;\; i \neq l, \qquad (26)$$
$$r_{ik}' = \frac{d_k'^{\min}}{d_{lk} + \Delta}, i = l,$$

and the final scores of alternatives are:

$$P_i' = P_i + \left(\frac{d_k'^{\min}}{d_{ik}} - \frac{d_{lk}}{d_{ik}} \right) \cdot w_k, \;\; i = 1, \dots, m, \;\; i \neq l, \qquad (27)$$
$$P_l' = P_l + \left(\frac{d_k'^{\min}}{d_{lk} + \Delta} - 1 \right) \cdot w_k.$$

Proof. We have $d_k'^{\min} = \max\limits_{1 \le i \le m} d_{ik}$. Then, normalized values of kth column are:

$$r_{ik}' = \frac{d_k'^{\min}}{d_{ik}}, \;\; i = 1, \dots, m, \;\; i \neq l, \qquad (28)$$
$$r_{ik}' = \frac{d_k'^{\min}}{d_{lk} + \Delta}, i = l.$$

Therefore, the final scores of alternatives are:

$$P_i' = \sum_{\substack{j=1 \\ j \neq k}}^{r} r_{ij} \cdot w_j + r_{ij}' \cdot w_k = \sum_{j=1}^{r} r_{lj} \cdot w_j + r_{ik}' \cdot w_k - r_{ik} \cdot w_k$$

$$= P_i + \left(\frac{d_k'^{\min}}{d_{ik}} - \frac{d_{lk}}{d_{ik}}\right) \cdot w_k,\ \ i = 1, \dots, m,\ \ i \neq l,$$

$$P_i' = \sum_{\substack{j=1 \\ j\neq k}}^{r} r_{ij} \cdot w_j + r_{ij}' \cdot w_k = \sum_{j=1}^{r} r_{ij} \cdot w_j + w_k - r_{ik} \cdot w_k = P_i + \left(\frac{d_k'^{\min}}{d_{lk} + \Delta} - 1\right) \cdot w_k\ ; i = l \tag{29}$$

Theorem 3.7. Assume that the element d_{lk} changes to $d_{lk}' = d_{lk} + \Delta$, and that the kth column is of the cost type and element d_{lk} is not the most desirable one at the column, that is, $d_{lk}' > \min_{1\leq i\leq m} d_{ik} = r_k^{\min}$ and after changing d_{lk} to d_{lk}', d_{lk}' is not the most desirable element at kth column, that is, $d_{lk}' \geq \min_{\substack{1\leq i\leq m \\ i\neq l}} d_{ik} = r_k'^{\min}$. Then, normalized values of kth column, except for lth element are changed to:

$$\begin{gathered} r_{ik}' = r_{ik},\ \ i = 1, \dots, m,\ \ i \neq l, \\ r_{lk}' = \left(1 - \frac{\Delta}{d_{lk} + \Delta}\right) \cdot r_{lk}, i = l, \end{gathered} \tag{30}$$

and the final scores of alternatives, except for lth alternative, will not change and we have

$$P_l' = P_l - \left(\frac{\Delta}{d_{lk} + \Delta}\right) \cdot w_k \cdot r_{lk}. \tag{31}$$

Proof. Normalized values of kth column are calculated by dividing the values of kth column into $r_k^{\min}$. Then, the normalized values r_{lk}' are:

$$r_{lk}' = \frac{d_k'^{\min}}{d_{lk} + \Delta}, \tag{32}$$

and since $r_{lk} = \dfrac{d_k^{\min}}{d_{lk}}$ and $d_k^{\min} = r_{lk}.d_{lk}$, we have

$$\begin{gathered} r_{ik}' = r_{ik},\ \ i = 1, \dots, m,\ \ i \neq l, \\ r_{lk}' = \frac{r_{lk} d_{lk}}{d_{lk} + \Delta} = \left(1 - \frac{\Delta}{d_{lk} + \Delta}\right) \cdot r_{lk}, i = l. \end{gathered} \tag{33}$$

Therefore, the final scores of kth alternative are:

$$P_l' = \sum_{\substack{j=1 \\ j\neq k}}^{r} r_{lj} \cdot w_j + r_{lk}' \cdot w_k = \sum_{j=1}^{r} r_{lj} \cdot w_j + r_{ik}' \cdot w_k - \left(\frac{\Delta}{d_{lk} + \Delta}\right) r_{lk} \cdot w_k \tag{34}$$

Theorem 3.8. Assume that the element d_{lk} changes to $d_{lk}' = d_{lk} + \Delta$, and that the kth column is of the cost type and element d_{lk} is not the most desirable one at the column, that is, $d_{lk}' > \min_{1\leq i\leq m} d_{ik} = r_k^{\min}$, but after changing d_{lk} to d_{lk}', d_{lk}' is not the most desirable element at kth column, that is, $d_{lk}' \geq \min_{\substack{1\leq i\leq m \\ i\neq l}} d_{ik}$. Then, normalized values of kth column are changed to:

$$\begin{gathered} r_{ik}' = \frac{d_{lk} + \Delta}{d_k^{\min}} \cdot r_{ik},\ \ i = 1, \dots, m,\ \ i \neq l, \\ r_{lk}' = 1. \end{gathered} \tag{35}$$

Therefore the final scores of alternatives are:

$$P_i' = P_i + \left(\frac{d_{lk} + \Delta}{d_k^{\min}} - 1\right) \cdot w_k \cdot r_{ik}, \;\; i = 1, \dots, m, \;\; i \neq l,$$
$$P_l' = P_l + (1 - r_{lk}) \cdot w_k. \tag{36}$$

Proof. Normalized values of kth column are:

$$r_{ik}' = \frac{d_{lk}'}{d_{ik}} = \frac{d_{lk} + \Delta}{d_{ik}}, i = 1, \dots, m, \;\; i \neq l, \tag{37}$$

and since $r_{ik} = \frac{d_k^{\min}}{d_{ik}}$, $d_{ik} = \frac{d_k^{\min}}{r_{ik}}$ and by replacing it in (37), we have

$$r_{ik}' = \frac{d_{lk} + \Delta}{d_k^{\min}} . r_{ik}, \;\; i = 1, \dots, m, \;\; i \neq l,$$
$$r_{lk}' = 1. \tag{38}$$

Therefore, the final scores of alternatives are:

$$P_i' = \sum_{\substack{j=1 \\ j \neq k}}^{r} r_{ij} \cdot w_j + r_{ik}' \cdot w_k = \sum_{j=1}^{r} r_{ij} \cdot w_j + \frac{d_{lk} + \Delta}{d_k^{\min}} \cdot r_{ik} \cdot w_k - r_{ik} \cdot w_k$$
$$= P_i + \left(\frac{d_{lk} + \Delta}{d_k^{\min}} - 1\right) . w_k \;\; ; i = 1, \dots, m \, , i \neq l \tag{29}$$

$$P_l' = \sum_{\substack{j=1 \\ j \neq k}}^{r} r_{lj} \cdot w_j + r_{lk}' \cdot w_k = \sum_{j=1}^{r} r_{lj} \cdot w_j + w_k - r_{lk} \cdot w_k = P_l + (1 - r_{lk}) \cdot w_k \, ; i = l$$

Now, by considering the above eight states that appear after changing an element of the decision making matrix, we can use one of the above relations and then calculate the resulting change in the final scores of the alternatives. We now present the process in Figure 1.

4. Numerical Example

Consider an MADM problem with three alternatives and four attributes, wherein attributes c_1, c_4 are of cost type and attributes c_2, c_3 are of profit type:

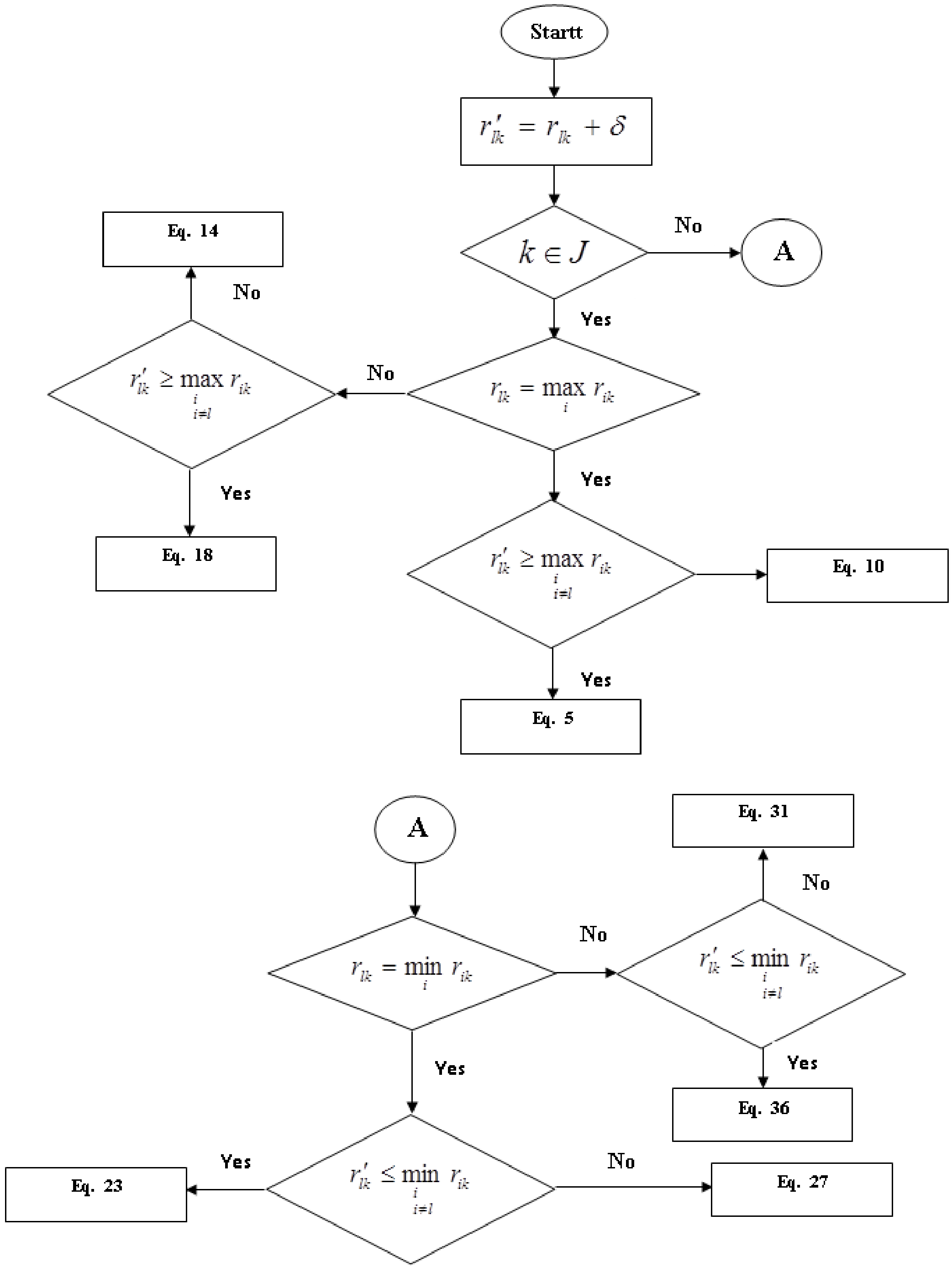

Figure 1. Flowchart to calculate the resulted change in the final score of alternatives

$$R = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{bmatrix} 13 & 9 & 9 & 8 \\ 5 & 3 & 5 & 12 \\ 7 & 5 & 7 & 6 \end{bmatrix} \end{array}$$

$$W^t = (0.4, 0.2, 0.3, 0.1)$$

Using the SAW technique, the normalized matrix, according to the relations given in Section 2, are:

$$R = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{bmatrix} 0.46 & 1 & 1 & 0.75 \\ 1 & 0.33 & 0.56 & 0.50 \\ 0.92 & 0.56 & 0.78 & 1 \end{bmatrix} \end{array}$$

The final scores of alternatives are calculated by $P_\mathrm{i} = \sum_{\mathrm{j}=1}^{\mathrm{k}} w_j \cdot r_{ij}$, $i = 1, \ldots, m$, with $m = 3$ and $k = 4$: $P_1 = 0.758$, $P_2 = 0.683$, $P_3 = 0.811$.
Therefore:

$$A_3 > A_1 > A_2.$$

Now, we assume that the element d_{34} in the decision matrix is increased to $d'_{34} = d_{34} + \Delta = 6 + 3 = 9$. Then 4th column in the new normalized matrix will change and we have:

$$R' = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{bmatrix} 0.46 & 1 & 1 & 0.75 \\ 1 & 0.33 & 0.56 & 0.50 \\ 0.92 & 0.56 & 0.78 & 1 \end{bmatrix} \end{array}$$

By solving the problem again, the new scores of alternatives are:

$$P'_i = \sum_{j=1}^{k} w'_j \cdot r_{ij}\,, \qquad i = 1, \cdots, m,$$

$$P'_1 = 0.783, P'_2 = 0.700, P'_3 = 0.800.$$

Therefore, $A_3 > A_1 > A_2$, and it is clear that the ranking has changed.

Now, instead of solving the problem again, we use the formulas given in Section 3. Since $d'_{34} = d_{34} + \Delta = 6 + 3 = 9$, and the regarded attribute is of the cost type and this element is the most desirable one at its column, but after the change is not the most desirable one, we use the equations corresponding to state 6 for calculating the new scores of alternatives:

$$P'_i = P_i + \left(\frac{d'^{\min}_4}{d_{i4}} - \frac{d_{34}}{d_{i4}}\right) \cdot w_4, \;\; i = 1, 2, 3,$$

$$P'_3 = P_3 + \left(\frac{d'^{\min}_4}{d'_{34} + \Delta} - 1\right) \cdot w_4.$$

Since we have

$$d'^{\min}_4 = 8, d'_{34} = 9, w_4 = 0.1,$$

the final changed scores of alternatives are

$$P'_1 = 0.783, P'_2 = 0.700, P'_3 = 0.800.$$

which are exactly the same as the results obtained before.

5. Conclusion

Decision making is an integral part of human life. Regardless of the variety of decision making problems, we can categorize them into two categories: multi-objective decision making problems in which the decision maker must design an approach that has the most utility by considering limited resources and multi-attribute decision making problems in which the decision maker must select one alternative with most utility from among the available alternatives. Naturally, to select an alternative, one must consider several and often conflicting attributes.

Generally, an MADM problem can be depicted as a matrix. Each row of the matrix correspond to one alternative and each column to one attribute and the elements of the matrix are the efficiency of alternatives against attributes. Generally, the attributes that are chosen for decision making are conflicting. This means that improvement in one attribute may result in the deflation of other attributes. Considering the relative importance of attributes, we can assign weights. Using a vector of weights for the attributes and elements of the decision making matrix, we can solve the MADM problem by available techniques to rank the alternatives or select the best one.

In the classic techniques of MADM, it is often assumed that all the used data (such as weights of attributes, efficiencies of alternatives against attributes, …) are deterministic. Then, final scores or utilities of alternatives are obtained by an MADM solving techniques, whereas in reality, the data of the decision making problem change. After solving the decision making problem, usually a sensitivity analysis is also performed.

Most studies on MADM problems, often determine the most sensitive attribute in the model. This attribute is the one that requires the least change in its weight, as compared to other attributes, to change a ranking of the alternatives. The available studies frequently consider attributes' sensitivity.

Another type of sensitivity analysis, not addressed in the literature, is calculation of the change in the final scores of alternatives corresponding to a change in the weight of a particular attribute. In our proposed sensitivity analysis, for a given change in the weight of one attribute, the changes in the scores of alternatives are calculated.

This type of sensitivity analysis can be implemented in MADM related software to solve decision making problems in a way that by utilizing graphical means, the decision maker may one element of the decision making matrix and observe its effect on the final scores and ranks of the alternatives. The followings are suggested for further research.

- Studying the effect of the change in the weight of one attribute of the decision making matrix on the final scores of alternatives in the SAW technique.

- Studying the effect of simultaneously changing the weight of one attribute and one element of the decision making matrix on the final scores of the alternatives in the SAW technique.

- Applying our proposed sensitivity analysis for other MADM techniques such as AHP and PROMETHEE.

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