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Partial-Fraction Decomposition Approach to the M/H2/2 Queue

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Here, a two server queueing system with Poisson arrivals and two different types of customers (M/H2/2 queue) is analyzed. A novel straightforward method is presented to acquire the exact and explicit forms of the performance measures. First, the steady state equations along with their Z-transforms are derived for the aforementioned queueing system. Using some limiting behaviors of the steady-state probabilities along with partial fraction decomposition as a simple algebraic procedure, the problem reduces to the solution of a system of linear equations.

Keywords*: Queueing systems, Hyper-exponential service times, Z-transform, Partial fraction decomposition.*

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1. Introduction

Consider a queueing system with Poisson arrivals, two different types of customers and two parallel servers. A customer is of type i with probability p_i with $p_i > 0$ and $\sum_i p_i = 1$ for $i = 1, 2$. The inter-arrival and service times of a customer of type i are independent and exponentially distributed random variables with parameters λ_i and μ_i, respectively. Directly from the definition, it is easily deduced that

$$p_i = \frac{\lambda_i}{\lambda} \tag{1}$$

where λ is the overall arrival rate and hence is the sum of the arrival rates of the two types of customers (i.e., $\lambda = \sum_i \lambda_i$). With probability p_i, the service time has an exponential distribution with parameter μ_i and therefore, its probability density function (pdf) is given by

$$f(t) = \sum_i p_i (\mu_i e^{-\mu_i t}). \tag{2}$$

The above pdf is the pdf of hyper-exponential distribution with parameters μ_i and p_i. Therefore, the model is the queue M/H2/2, where H2 denotes a hyper-exponential distribution with two phases.

One of the most challenging problems in queueing theory is to investigate multi-server queues. The queue M/H2/2 is a special case of GI/G/s for which Pollaczeck [7] gives a procedure to calculate

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the mean waiting time when the service time distribution has a rational Laplace transform with distinct poles. However, considerable effort is needed to obtain numerical results. The queue M/H2/2 can also be considered as a special case of M/G/2, which has been studied by some researchers. Hokstad [5] used the supplementary variable technique in order to analyze this queue. Although the approach is noteworthy from the methodological viewpoint, the analytical problem has not been solved. Cohen [2] investigated the queue M/G/2 when the service time distribution is a mixture of negative exponential distributions by analyzing the workloads at various servers. The author obtained an expression for the Laplace-Stieltjes transform of the stationary waiting time distribution. The presented analysis was relatively intricate but useful for developing approximations. Knessl et al. [6] also used the supplementary variable method and an integral equation approach to obtain the stationary distribution of the number of customers in the queue M/G/2. The GI/H2/s queueing system is also a general case for M/H2/2. De Smit [3, 4] presented an approach to identify the distribution of waiting times and queue lengths for the queue GI/H2/s. He reduced the problem to the solution of the Wiener-Hopf-type equations and then used a factorization method to solve the system. Although the approach is methodologically remarkable, but the method is rather complicated and the results are based on some conditions the validity of which validity have not been established. Due to time consuming and intricacy of the proposed approaches for obtaining exact solution, there has been much interest in developing approximation methods for the queues M/G/s and M/Hm/s [1, 8].

Here, a straightforward approach is presented to obtain explicit forms for the performance measures of M/H2/2 queueing system. First, the steady state equations of the system are derived and transformed using Z-transform. Then, based on a probabilistic argument and partial-fraction decomposition method, the problem is reduced to solving a system of linear equations. In addition to the simplicity and directness of the proposed partial-fraction decomposition method, it has the capability to be used in studying more complicated multi-server queues.

The remainder of our work is organized as follows. The steady state equations of the system are derived in Section 2. The details of the Z-transform and partial-fraction decomposition method are presented in Section 3 and the concluding remarks are given in Section 4.

2. Steady State Equations

In order to derive the steady state equations, the state of the system is defined by (nij), where n is the number of customers in the system, i is the number of customers of type 1 being served and j is the number of customers of type 2 being served. For simplicity, let r_n, s_n and q_n be the limiting probabilities of the states $(n20)$, $(n02)$ and $(n11)$, respectively, for $n \geq 2$ and similarly, π_0, r_1 and s_1 be the limiting probabilities of the states (000), (110) and (101), respectively. The steady state equations are

$$\lambda\pi_0 = \mu_1 r_1 + \mu_2 s_1 \tag{3}$$

$$(\lambda + \mu_1) r_1 = \lambda_1 \pi_0 + 2\mu_1 r_2 + \mu_2 q_2 \tag{4}$$

$$(\lambda + \mu_2) s_1 = \lambda_2 \pi_0 + 2\mu_2 s_2 + \mu_1 q_2 \tag{5}$$

$$(\lambda + 2\mu_1) r_2 = \lambda_1 r_1 + 2p_1\mu_1 r_3 + p_1\mu_2 q_3 \tag{6}$$

$$(\lambda + 2\mu_2) s_2 = \lambda_2 s_1 + 2p_2\mu_2 s_3 + p_2\mu_1 q_3 \tag{7}$$

$$(\lambda + \mu_1 + \mu_2) q_2 = \lambda_2 r_1 + \lambda_1 s_1 + 2p_2\mu_1 r_3 + 2p_1\mu_2 s_3 + (p_1\mu_1 + p_2\mu_2) q_3 \tag{8}$$

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$$(\lambda + 2\mu_1)r_n = \lambda r_{n-1} + 2p_1\mu_1 r_{n+1} + p_1\mu_2 q_{n+1}, n \geq 3 \tag{9}$$

$$(\lambda + 2\mu_2)s_n = \lambda s_{n-1} + 2p_2\mu_2 s_{n+1} + p_2\mu_1 q_{n+1}, n \geq 3 \tag{10}$$

$$(\lambda + \mu_1 + \mu_2)q_n = \lambda q_{n-1} + 2p_2\mu_1 r_{n+1} + 2p_1\mu_2 s_{n+1} + (p_1\mu_1 + p_2\mu_2)q_{n+1}, n \geq 3. \tag{11}$$

Since the sum of limiting probabilities is equal to one, we have

$$\pi_0 + r_1 + s_1 + \sum_{n=2}^{\infty}(r_n + s_n + q_n) = 1. \tag{12}$$

3. Z-Transform and Partial-Fraction Decomposition

Four Z-transform functions are defined as follows:

$$R(z) = \sum_{n=1}^{\infty} r_n z^n \tag{13}$$

$$S(z) = \sum_{n=1}^{\infty} s_n z^n \tag{14}$$

$$Q(z) = \sum_{q=1}^{\infty} r_n z^n \tag{15}$$

$$\Pi(z) = \sum_{n=1}^{\infty} \pi_n z^n = R(z) + S(z) + Q(z) + \pi_0. \tag{16}$$

Multiplying Eq. (4) by z, Eq. (6) by z^2 and Eq. (9) by z^n, and then summing the products, we get

$$\begin{aligned}&\left(2\mu_1\left(1 - \frac{p_1}{z}\right) + \lambda(1 - z)\right)R(z)\\ &\qquad = \frac{\mu_2 p_1}{z}Q(z) - 2\mu_1 p_1 r_1 + (\lambda_1 p_1 \pi_0 + (\mu_1(1 + p_2) + \lambda_2)r_1)z - \lambda_2 r_1 z^2.\end{aligned} \tag{17}$$

Similarly, using Eqs. (5), (7) and (10), we obtain:

$$\begin{aligned}&\left(2\mu_2\left(1 - \frac{p_2}{z}\right) + \lambda(1 - z)\right)S(z)\\ &\qquad = \frac{\mu_1 p_2}{z}Q(z) - 2\mu_2 p_2 s_1 + (\lambda_2 p_2 \pi_0 + (\mu_2(1 + p_1) + \lambda_1)s_1)z - \lambda_1 s_1 z^2.\end{aligned} \tag{18}$$

Multiplying Eqs. (4) and (5) by z, Eqs. (6) and (7) by z^2 and Eq. (11) by z^n, and then summing the products, we get

$$\begin{aligned}&\left(\mu_1 + \mu_2 - \frac{\mu_1 p_1 + \mu_2 p_2}{z} + \lambda(1 - z)\right)Q(z)\\ &\qquad = \frac{2\mu_1 p_2}{z}R(z) + \frac{2\mu_2 p_1}{z}S(z) + (\lambda_2 r_1 + \lambda_1 s_1)z^2\\ &\qquad - \left((\lambda + \mu_1)p_2 r_1 + (\lambda + \mu_2)p_1 s_1 - 2\lambda_1 p_2 \pi_0\right)z - 2\mu_1 p_2 r_1 - 2\mu_2 p_1 s_1.\end{aligned} \tag{19}$$

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Considering Eq. (16), the following equation is obtained:

$$\Pi(1) = R(1) + S(1) + Q(1) + \pi_0 = 1. \tag{20}$$

By solving the system of equations (3) and (17)-(20), we can calculate π_0, s_1, $R(z)$, $S(z)$ and $Q(z)$ in terms of r_1. Then, by using Eq. (16), $\Pi(z)$ can be determined as a function of r_1. Hence, we need another equation, in terms of r_1, in order to obtain $\Pi(z)$. The limiting property of the functions $R(z)$, $S(z)$ and $Q(z)$ can be used to obtain the necessary equation. Considering the function $R(z)$, the limiting probability r_n can be obtained using Eq. (13) as follows:

$$r_n = \frac{\partial^n}{\partial z^n} R(z)\Big|_{z=0}. \tag{21}$$

The above equation holds for $n \geq 1$. It is evident that when n tends to infinity, the limiting probability r_n converges to zero. Hence, we can write

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty}\left(\frac{\partial^n}{\partial z^n} R(z)\Big|_{z=0}\right) = 0. \tag{22}$$

As mentioned before, $R(z)$ can be obtained by solving the system of equations (3) and (17)-(20) as a function of r_1. Because of the lengthiness of the $R(z)$ function, it is not presented here and thus, instead of the explicit form, the following simple form is presented:

$$R(z) = \frac{\sum_{i=0}^{5} a_i z^i + r_1 \sum_{i=0}^{4} b_i z^i}{\sum_{i=0}^{4} c_i z^i}, \tag{23}$$

where a_i, b_i and c_i are functions of the parameters λ_1, λ_2, μ_1 and μ_2. In (23), the numerator is a polynomial of degree five and the denominator has four distinct roots as follows:

$$h_1 = \frac{\mu - \sqrt{\lambda^2 + 2(\lambda_2 - \lambda_1)(\mu_1 - \mu_2) + (\mu_1 - \mu_2)^2} + \lambda}{2\lambda}, \tag{24}$$

$$h_2 = \frac{\mu + \sqrt{\lambda^2 + 2(\lambda_2 - \lambda_1)(\mu_1 - \mu_2) + (\mu_1 - \mu_2)^2} + \lambda}{2\lambda}, \tag{25}$$

$$h_3 = \frac{2\mu - \sqrt{\lambda^2 + 4(\lambda_2 - \lambda_1)(\mu_1 - \mu_2) + 4(\mu_1 - \mu_2)^2} + \lambda}{2\lambda}, \tag{26}$$

$$h_4 = \frac{2\mu + \sqrt{\lambda^2 + 4(\lambda_2 - \lambda_1)(\mu_1 - \mu_2) + 4(\mu_1 - \mu_2)^2} + \lambda}{2\lambda}. \tag{27}$$

Lemma 3.1. The roots of the denominator of fraction (23) are real numbers.

The proof of the above lemma is given in the Appendix. Considering this characteristic, it is possible to decompose (23) to the sum of a polynomial of degree one and four simple fractions as follows:

$$R(z) = (d_0 + e_0 r_1 + d_1 z) + \sum_{i=1}^{4} \frac{f_i + g_i r_1}{z - h_i}, \tag{28}$$

where d_0, e_0, d_1, f_i and g_i are functions of the parameters λ_1, λ_2, μ_1 and μ_2. Taking (28) into consideration as the last form of $R(z)$, it is straightforward to obtain the nth derivative:

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$$\frac{d^n}{dz^n}R(z) = n!\,(-1)^n \sum_{i=1}^{4} \frac{f_i + g_i r_1}{(z - h_i)^n} \text{ for } n \geq 2 \tag{29}$$

According to (22), we have

$$\lim_{n\to\infty}\left(n!\,(-1)^n \sum_{i=1}^{4} \frac{f_i + g_i r_1}{h_i^n}\right) = 0. \tag{30}$$

By solving the above equation in terms of r_1, we get

$$r_1 = -\frac{\lim_{n\to\infty}\left(\sum_{i=1}^{4} \frac{f_i}{h_i^n}\right)}{\lim_{n\to\infty}\left(\sum_{i=1}^{4} \frac{g_i}{h_i^n}\right)}. \tag{31}$$

The following lemma is useful for simplifying the above equation.

Lemma 3.2. h_1 is minimum among the roots of the denominator of (23).

See the Appendix for a proof. By dividing the numerator and the denominator of (31) into h_1, we can write

$$r_1 = -\frac{\lim_{n\to\infty}\left(f_1 + \sum_{i=2}^{4} \frac{f_i}{(h_i/h_1)^n}\right)}{\lim_{n\to\infty}\left(g_1 + \sum_{i=2}^{4} \frac{g_i}{(h_i/h_1)^n}\right)}. \tag{32}$$

When n tends to infinity, $(h_i/h_1)^n$, for $i = 2,3,4$, also tend to infinity and hence we have

$$r_1 = \frac{-f_1}{g_1}. \tag{33}$$

As mentioned before, f_1 and g_1 are functions of the parameters λ_1, λ_2, μ_1 and μ_2 and thus r_1 is obtained as a function of these parameters as well. By determining r_1, it is straightforward to obtain $\Pi(z)$, and therefore the performance measures of the queueing system. For example, the mean length of the queue is given by

$$L = \frac{d}{dz}\Pi(z)\Big|_{z=1}. \tag{34}$$

Defining λ, μ, a_i and b_i as

$$\lambda = \lambda_1 + \lambda_2, \tag{35}$$

$$\mu = \mu_1 + \mu_2, \tag{36}$$

$$a_i = \frac{\lambda_1}{\lambda}\ , i = 1,2, \tag{37}$$

$$a_3 = \frac{\mu_1}{\mu}\ , i = 1,2, \tag{38}$$

$$a_3 = a_1 - a_2, \tag{39}$$

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$$b_3 = b_1 - b_2, \tag{40}$$

the explicit form of the average number of customers in the system for the M/H2/2 queueing system is obtained as to be

$$L = \frac{\begin{pmatrix} -\mu_2 b_1^2\big((b_3\lambda+2\mu-4\mu_2 b_1)R_1+4\mu_2 b_1 S_1\big) \\ -a_1^3(2b_1-1)^3\big((3b_1-1)\lambda(R_1+S_1)+2(\mu_1R_1+\mu_2S_1)\big) \\ +(a_1-2a_1b_1)^2\big((3b_1-1)\lambda-2\mu_2 b_1\big)\begin{pmatrix}\lambda(R_1+S_1)\\ -2\mu_2(4b_1-1)(\mu_1R_1+\mu_2S_1)\\ +\lambda\mu\big((2-9b_2b_1)R_1-b_2^2S_1\big)\end{pmatrix}\end{pmatrix}}{\Big((a_1b_3-b_1)\big(\lambda_1 b_3-b_1(\lambda-2\mu_2)\big)^2\Big)}, \tag{41}$$

where

$$R_1 = \frac{-\lambda_1\begin{pmatrix}\lambda_1 b_3\mu\\ +\mu_1(\lambda-2\mu_2)\end{pmatrix}\times \begin{pmatrix} \big(-(\lambda^2-2a_3b_3\lambda\mu+\mu^2)\big) \\ \times\begin{pmatrix} -a_2b_3(a_1b_1+a_2b_2)\lambda^2 \\ +2b_2\begin{pmatrix}a_2^2+4a_2a_1b_1\\+(4a_1^2-3)b_1^2\end{pmatrix}\lambda\mu \\ -4(2-2a_1b_2-b_1)b_2b_1b_3\mu^2\end{pmatrix} \\ +B\begin{pmatrix} a_2b_3(b_2+a_1b_3)\lambda^3 \\ -a_2b_3\begin{pmatrix}1+(3-4b_1)b_1\\+a_1b_3\end{pmatrix}\lambda^2\mu \\ -2b_2\begin{pmatrix}a_2^2+4(a_1-2)a_2b_1\\+(11+4(a_1-4)a_1)b_1^2\end{pmatrix}\lambda\mu^2 \\ -4(2-2a_1b_2-b_1)b_2b_1b_3\mu^3\end{pmatrix}\end{pmatrix}}{2b_2b_1\mu^3\times\begin{pmatrix} -b_2b_1(\lambda^2-2a_3b_3\lambda\mu+\mu^2) \\ \times\begin{pmatrix}(b_2+a_1b_3)\lambda^2\\ +2(b_2-a_2a_1+4a_2a_1b_1-(1+4a_2a_1)b_1^2)\lambda\mu \\ -4b_3\big(-b_2^2+a_1(1-2b_2b_1)\big)\mu^2\end{pmatrix} \\ +B\begin{pmatrix} a_1^3b_3^3\lambda^2(\lambda+2\mu)+b_2^2\mu_1(\lambda+2b_2\mu)(\lambda-2\mu+4\mu_1) \\ +(a_1-2a_1b_1)^2\lambda\begin{pmatrix}(2-3b_1)\lambda^2\\-2(b_1-2+2b_1^2)\lambda\mu\\+6\mu_2\mu_1\end{pmatrix} \\ -a_1b_2b_3\begin{pmatrix}b_3\lambda^3+\big(-2+b_1(8b_1-1)\big)\lambda^2\mu\\+6b_1b_3\lambda\mu^2+4b_1(1-2b_2b_1)\mu^3\end{pmatrix}\end{pmatrix}\end{pmatrix}}, \tag{42}$$

and

$$S_1 = \frac{a_1\lambda^2 b_3 - b_1\lambda(\lambda-2\mu_2) - \mu_2 b_1(\lambda+2\mu_1)R_1}{\mu_2 b_1(\lambda+2\mu_2)}, \tag{43}$$

$$B = \sqrt{\lambda^2 - 2a_3b_3\lambda\mu + \mu^2}. \tag{44}$$

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To clarify the proposed approach, a number of numerical examples are presented. The arrival rates for the two types of customers and the service rates for the two types of services are presented in Table 1. Using (41)-(44), the average number of customers in each queue is calculated and reported in Table 1.

Table 1. Results for the numerical examples

λ_1	λ_2	μ_1	μ_2	L
20	30	30	30	5.4545
20	30	30	40	2.8707
20	30	40	40	2.0513
10	20	15	25	3.2784
10	20	20	30	1.7900
10	20	25	35	1.2783

4. Conclusion

Explicit forms of performance measures of the M/H2/2 queueing system were obtained. First, the steady state equations of the system were derived. Then, based on a probabilistic argument and partial-fraction decomposition method, the problem was reduced to solving a system of linear equations. The advantages of the proposed partial fraction decomposition method are its simplicity as well as its usefulness for solving complicated multi-server queues.

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Appendix

Proof of Lemma 3.1.

Since $\lambda = \lambda_1 + \lambda_2$, we can write

$$\lambda^2 + (\mu_1 - \mu_2)^2 + 2(\lambda_2 - \lambda_1)(\mu_1 - \mu_2) = (\lambda_2 - \lambda_1 + \mu_1 - \mu_2)^2 + 4\lambda_1\lambda_2. \tag{A.1}$$

Since the right side of the above equation is always positive, we have

$$\lambda^2 + (\mu_1 - \mu_2)^2 + 2(\lambda_2 - \lambda_1)(\mu_1 - \mu_2) > 0. \tag{A.2}$$

Hence, h_1 and h_2 are real numbers. According to (A.2), it can be easily seen that

$$\lambda^2 + 4(\lambda_2 - \lambda_1)(\mu_1 - \mu_2) + 4(\mu_1 - \mu_2)^2 > 0. \tag{A.3}$$

Therefore, h_3 and h_4 are also real numbers.

Proof of Lemma 3.2.

It can be easily seen that $h_1 < h_2$ and $h_3 < h_4$. Thus, we just need to prove that $h_1 < h_3$. Two cases may occur. If

$$(\mu_1 - \mu_2)^2 - 2\mu_1\mu_2 + (\lambda_2 - \lambda_1)(\mu_1 - \mu_2) < 0, \tag{A.4}$$

then

$$\begin{aligned}(\mu_1 - \mu_2)^2 - 2\mu_1\mu_2 &+ (\lambda_2 - \lambda_1)(\mu_1 - \mu_2) \\ &< (\mu_1 + \mu_2)\sqrt{\lambda^2 + 2(\lambda_2 - \lambda_1)(\mu_1 - \mu_2) + (\mu_1 - \mu_2)^2}.\end{aligned} \tag{A.5}$$

After some simplifications, we have

$$\begin{aligned}\lambda^2 + 4(\lambda_2 - \lambda_1)(\mu_1 - \mu_2) &+ 4(\mu_1 - \mu_2)^2 \\ &< (\mu_1 + \mu_2)^2 + \lambda^2 + 2(\lambda_2 - \lambda_1)(\mu_1 - \mu_2) + (\mu_1 - \mu_2)^2 \\ &+ 2(\mu_1 + \mu_2)\sqrt{\lambda^2 + 2(\lambda_2 - \lambda_1)(\mu_1 - \mu_2) + (\mu_1 - \mu_2)^2}.\end{aligned} \tag{A.6}$$

Since both sides of the above equation are positive, we can write

$$\begin{aligned}&\sqrt{\lambda^2 + 4(\lambda_2 - \lambda_1)(\mu_1 - \mu_2) + 4(\mu_1 - \mu_2)^2} \\ &\qquad < \mu_1 + \mu_2 + \sqrt{\lambda^2 + 2(\lambda_2 - \lambda_1)(\mu_1 - \mu_2) + (\mu_1 - \mu_2)^2}.\end{aligned} \tag{A.7}$$

The above equation can be easily rewritten as follows

$$\begin{aligned}&\frac{\mu - \sqrt{\lambda^2 + 2(\lambda_2 - \lambda_1)(\mu_1 - \mu_2) + (\mu_1 - \mu_2)^2}}{2\lambda} + \frac{1}{2} \\ &\qquad < \frac{2\mu - \sqrt{\lambda^2 + 4(\lambda_2 - \lambda_1)(\mu_1 - \mu_2) + 4(\mu_1 - \mu_2)^2}}{2\lambda} + \frac{1}{2}.\end{aligned} \tag{A.8}$$

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Hence, $h_1 < h_3$. Considering the case for which Eq. (A.4) does not hold, we have

$$(\mu_1 - \mu_2)^2 - 2\mu_1\mu_2 + (\lambda_2 - \lambda_1)(\mu_1 - \mu_2) \geq 0. \tag{A.9}$$

It is evident that the following equation also holds:

$$(\mu_1 - \mu_2)^2 + (\lambda_2 - \lambda_1)(\mu_1 - \mu_2) - \mu_1\mu_2 \geq 0. \tag{A.10}$$

In addition, according to Eq. (A.2), we have

$$(\lambda_1 + \lambda_2)^2 + (\mu_1 - \mu_2)^2 + 2(\lambda_2 - \lambda_1)(\mu_1 - \mu_2) > 0. \tag{A.11}$$

With respect to Eqs. (A.10) and (A.11), we obtain

$$\begin{aligned}\mu_1\mu_2\big((\lambda_1 + \lambda_2)^2 + (\mu_1 - \mu_2)^2 + 2(\lambda_2 - \lambda_1)(\mu_1 - \mu_2)\big) \\ + \mu_1\mu_2((\mu_1 - \mu_2)^2 + (\lambda_2 - \lambda_1)(\mu_1 - \mu_2) - \mu_1\mu_2) \\ +\lambda_1\lambda_2(\mu_1 - \mu_2)^2 > 0.\end{aligned} \tag{A.12}$$

The above equation can be simplified as follows:

$$\begin{aligned}\big((\mu_1 - \mu_2)^2 - 2\mu_1\mu_2 + (\lambda_2 - \lambda_1)(\mu_1 - \mu_2)\big)^2 \\ < \left((\mu_1 + \mu_2)\sqrt{\lambda^2 + 2(\lambda_2 - \lambda_1)(\mu_1 - \mu_2) + (\mu_1 - \mu_2)^2}\right)^2.\end{aligned} \tag{A.13}$$

According to Eq. (A.9), we have

$$\begin{aligned}(\mu_1 - \mu_2)^2 - 2\mu_1\mu_2 + (\lambda_2 - \lambda_1)(\mu_1 - \mu_2) \\ < (\mu_1 + \mu_2)\sqrt{\lambda^2 + 2(\lambda_2 - \lambda_1)(\mu_1 - \mu_2) + (\mu_1 - \mu_2)^2}.\end{aligned} \tag{A.14}$$

The above equation is the same as Eq. (A.5). Hence, similar to the previous case, we obtain $h_1 < h_3$.

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