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Solving Infinite Horizon Optimal Control Problems of Nonlinear Interconnected Large-Scale Dynamic Systems via a Haar Wavelet Collocation Scheme

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We consider an approximation scheme using Haar wavelets for solving a class of infinite horizon optimal control problems of nonlinear interconnected large-scale dynamical systems. A computational method based on Haar wavelets in the time-domain is proposed for solving the optimal control problem. Haar wavelets integral operational matrix and direct collocation method are utilized to find an approximate optimal trajectory of the original problem. Numerical results are given to demonstrate the applicability and the effectiveness of the proposed method.

***Keywords**: Nonlinear large-scale OCPs, Approximation, Operational matrix, Rationalized Haar functions, Nonlinear programming.*

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1. Introduction

In general, a large-scale system can be considered a dynamical system composed of some lower order interconnected subsystems. These systems are found in many practical applications, such as power systems and physical plants (Sahba [37]; Holland and Diamond [14]). Nevertheless, control of such a system is still challenging, because of the dimensionality problem and high complexity in calculations.

An efficient control strategy for large-scale systems is decentralized control (Huang et al. [16]), which is easier to implement than centralized control. Designing a decentralized controller, however, is more difficult than that of a centralized controller, owing to the interconnections among subsystems. To overcome the difficulties arising from the control of large-scale systems, neural networks have been recognized as a powerful tool, due to their collective computing and parallel processing capabilities (Chen and Li [9]; Padhi and Balakrishnan [29]). Nevertheless, the main drawback of the neural network model is that it can often be trapped at a local minimum.

Recently, some new control strategies have been introduced for large-scale systems. In Chen and Li [8], a decentralized adaptive backstepping neural network control approach was developed for a class of large-scale nonlinear output feedback systems. In this approach, neural networks were employed to approximate the interconnections and a backstepping technique was used to remove the matching condition requirement on interconnections. In Li et al. [23], the decentralized adaptive neural network output feedback stabilization problem was investigated for a class of large-scale stochastic nonlinear strict-feedback systems. In that work, the nonlinear interconnections were assumed to be bounded by some unknown nonlinear functions of the system outputs. Then, in each

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subsystem, only one neural network was employed to compensate for all unknown upper bounding functions, which depended only on the output of the respective subsystem. Using only one neural network, however, may cause loss of precision.

A growing interest has appeared in the field of optimal control. Nevertheless, conventional methods of optimal control are generally impractical for many nonlinear large-scale systems because of the dimensionality problem and high complexity in calculations. One example is the state-dependent Riccati equation (SDRE) method (Chang et al. [4]). Although this scheme has been widely used in many applications, its major limitation is that it needs to solve a sequence of matrix Riccati algebraic equations at each sample state along the trajectory. This may take a long computing time and extensive memory space. Therefore, developing new methods is necessary for solving nonlinear large-scale optimal control problems (OCPs).

A popular trend in handling nonlinear large-scale OCPs is decomposition and coordination, where a large problem is decomposed into small subproblems, based on the problem structure. Then, a proper coordination scheme is carried out to join the subproblems and insure the optimality of the overall solution (Jamshidi [18]). Based on this strategy, hierarchical methods have been proposed (Jamshidi [18]); nevertheless, these methods may take considerable computing time and memory space.

To solve nonlinear large-scale OCPs, in recent years, good results have been gained. For instance, a new successive approximation approach (SAA) was proposed by Tang and Sun [38]. In this approach, instead of directly solving the nonlinear large-scale two-point boundary value problem (TPBVP), derived from the maximum principle, a sequence of nonhomogeneous linear time-varying TPBVPs is solved iteratively. This method has been used in different applications (Tang and Zhang [39]; Zhang et al. [44]). Nevertheless, solving time-varying equations is much more difficult than solving time-invariant ones. Recently, a practical technique, called the extended modal series method, has also been proposed for solving the infinite horizon OCP of nonlinear interconnected large-scale dynamical systems (Jajarmi et al. [17]). This is an indirect method, where the optimal control law and the optimal trajectory are determined in the form of a uniformly convergent series. But, its shortcoming is the high computing complexity due to calculating the coefficients of series in each step where in theory, infinite iterations are required.

Orthogonal functions such as Haar wavelets (Hsiao and Wang [13]; Karimi et al. [19]), Walsh functions (Chen and Hsiao [6]; Razzaghi and Nazarzadeh [32]), block pulse functions (Marzban and Razzaghi [24]; Mashayekhi et al. [27]; Rao [30]), Laguerre polynomials (Wang and Shin [42]), Legendre polynomials (Chang and Wang [5]), Chebyshev functions (Horng and Chou [15]) and Fourier series (Razzaghi [31]), which are often used to represent arbitrary time functions, have frequently been used to deal with various problems of dynamical systems. The main characteristic of this approach is that it reduces the difficulties involved in solving problems described by differential equations, such as in the analysis of linear time-invariant, time-varying systems, model reduction, optimal control, and system identification, to the solution of a system of algebraic equations. Thus, the solution, identification, and optimization procedures are either greatly reduced or much simplified. The available sets of orthogonal functions can be divided into three classes: piecewise constant basis functions such as Haar wavelets, Walsh functions, and block pulse functions; orthogonal polynomials such as Laguerre, Legendre, and Chebyshev polynomials; and sine-cosine functions in Fourier series (Marzban and Razzaghi [25]). Among them, wavelet theory is a relatively new area in mathematical research (Burrus et al. [3]). It has been applied to a wide range of engineering disciplines such as signal processing, pattern recognition, industrial chemical reactors, and computer graphics. Recently, attempts have been made to use wavelet theory to solve surface

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integral equations, improve the finite-difference time-domain method, solve linear differential equations and nonlinear partial differential equations, optimal control problems, and model nonlinear semiconductor devices (Banks [1]; Banks and Burns [2]; Chen and Hsiao [7]; Dai and Cochran [10]; Göllmann et al. [12]; Hsiao and Wang [13]; Karimi et al. [19]; Karimi et al. [20]; Karimi et al. [21]; Karimi [22]; Marzban and Razaghi [26]; Ohkita and Kobayashi [28]; Razzaghi and Ordokhani [33]; Razzaghi and Ordokhani [35]; Teo et al. [40]; Wong et al.[43]).

Motivated by the above discussions, here we consider a particular approximation scheme based on Haar wavelets to be used to solve a class of infinite horizon OCPs of nonlinear interconnected large-scale dynamical systems, where the cost function is assumed to be quadratic and decoupled. First, we transform the infinite horizon problem to a finite horizon one, that is, from the interval [0,∞) to [0, 1). Then, we will assume that the control variables and derivatives of the state variables in the optimal control problems may be expressed in the form of Haar wavelets and unknown coefficients. The state variables can be calculated by using the Haar operational integration matrix. Therefore, all variables in the nonlinear system of equations are expressed as series of the Haar family and its operational matrix. Finally, the task of finding the unknown parameters that optimize the designated performance while satisfying all the constraints is performed the nonlinear programming. The effectiveness of the proposed approach is verified by solving the optimal attitude control problem.

2. Problem Statement and Transformation

Consider a nonlinear interconnected large-scale dynamical system which can be decomposed into N interconnected subsystems. The ith subsystem, for $i = 1,2, \dots, N$, is described by

$$\begin{cases} \dot{x}_i(t) = A_i x_i(t) + B_i u_i(t) + F_i(x(t)), t > 0, \\ x_i(0) = x_{i_0}, \end{cases} \tag{1}$$

where $x_i \in \mathbb{R}^{n_i}$ and $u_i \in \mathbb{R}^{m_i}$ are the state vector and the control vector of the ith subsystem, respectively, $x = (x_1^T, x_2^T, \dots, x_N^T)^T, \sum_{i=1}^{N} n_i = n, F_i: \mathbb{R}^n \to \mathbb{R}^{n_i}, i = 1,2, \dots, N,$ is a nonlinear analytic vector function with $F_i(0) = 0$, and $x_{i_0} \in \mathbb{R}^{n_i}$ is the initial state vector. Also, A_i and B_i are constant matrices of appropriate dimensions such that the pair (A_i, B_i) is completely controllable. Furthermore, the infinite horizon quadratic cost function to be minimized is given by

$$J = \frac{1}{2}\sum_{i=1}^{N} \left\{ \int_0^{\infty} (x_i^T(t) Q_i x_i(t) + u_i^T(t) R_i u_i(t)) dt \right\}, \tag{2}$$

where $Q_i \in \mathbb{R}^{n_i \times n_i}$ and $R_i \in \mathbb{R}^{m_i \times m_i}$ are respectively positive semi-definite and positive definite matrices. Note that the quadratic cost function (2) is assumed to be decoupled as a superposition of the cost functions of the subsystems.

The following time transformation is introduced:

$$t = \frac{\tau}{1-\tau}, t \in [0, \infty). \tag{3}$$

The above problem is transformed into the following finite horizon nonlinear optimal control problem:

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$$minimize \quad J = \frac{1}{2}\sum_{i=1}^{N}\left\{\int_{[0,1)} (x_i^T(\frac{\tau}{1-\tau})Q_i x_i(\frac{\tau}{1-\tau}) + u_i^T(\frac{\tau}{1-\tau})R_i u_i(\frac{\tau}{1-\tau}))\frac{d\tau}{(1-\tau)^2}\right\} \tag{4}$$

$s.t.$

$$\dot{x}_i(\frac{\tau}{1-\tau}) = \frac{1}{(1-\tau)^2}(A_i x_i(\frac{\tau}{1-\tau}) + B_i u_i(\frac{\tau}{1-\tau}) + F_i(x(\frac{\tau}{1-\tau}))), \tau \in [0,1), \tag{5}$$

$$x_i(0) = x_{i_0}. \tag{6}$$

Now, assume

$$\begin{cases} y_i(\tau) = x_i(\frac{\tau}{1-\tau}), & i = 1,2,\dots,N, \\ v_i(\tau) = u_i(\frac{\tau}{1-\tau}), & i = 1,2,\dots,N. \end{cases}$$

Thus, the problem of nonlinear system (1) with the performance index (2) is replaced by

$$minimize \ J = \frac{1}{2}\sum_{i=1}^{N}\left\{\int_{[0,1)} \left(y_i^T(\tau)Q_i y_i(\tau) + v_i^T(\tau)R_i v_i(\tau)\right)\frac{d\tau}{(1-\tau)^2}\right\} \tag{7}$$

$s.t.$

$$\dot{y}_i(\tau) = \frac{1}{(1-\tau)^2}(A_i y_i(\tau) + B_i v_i(\tau) + F_i(y(\tau))), i = 1,2,\dots,N, \tau \in [0,1), \tag{8}$$

$$y_i(0) = y_{i_0}, i = 1,2,\dots,N, \tag{9}$$

where $y(\tau) = (y_1^T(\tau), y_2^T(\tau), \dots, y_N^T(\tau))^T$. In the next section, we will discuss the properties of a direct collocation method based on Haar functions and will use it for solving the finite time horizon problem (7) – (9).

3. Haar Wavelets

3.1. Rationalized Haar Functions

The rationalized Haar (RH) functions $RH(r,\tau), r = 1,2,\dots$, can be defined on the interval [0,1) (e.g., see Marzban and Razzaghi [24]) as

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$$RH(r,\tau)=\begin{cases}1 & J_1 \le \tau < J_{\frac{1}{2}},\\ -1 & J_{\frac{1}{2}} \le \tau < J_0,\\ 0 & otherwise,\end{cases} \tag{10}$$

where

$$J_u=\frac{j-u}{2^i}, u=0,\frac{1}{2},1.$$

The value of r is defined by two parameters i and j via

$$r=2^i+j-1, i=0,1,2,3,\dots, j=1,2,3,\dots,2^i.$$

$RH(0,\tau)$ is defined for $i=j=0$ and is given by

$$RH(0,\tau)=1, \quad 0\le\tau<1.$$

The orthogonality property is given by

$$\int_0^1 RH(r,\tau)RH(\nu,\tau)d\tau=\begin{cases}2^{-i}, & r=\nu,\\ 0, & r\neq\nu,\end{cases} \tag{11}$$

where

$$\nu=2^n+m-1, n=0,1,2,3,\dots, m=1,2,3,\dots,2^n.$$

It should be noted that the set of RH functions is a complete orthogonal set in Hilbert space $L^2[0,1]$. Thus, we can expand any function in this space in terms of RH functions.

3.2. Function Approximation

A function $\mathcal{F}(\tau)\in L^2[0,1]$ may be expanded as an infinite series of RH functions as

$$\mathcal{F}(\tau)=\sum_{r=0}^{\infty} a_r RH(r,\tau), \tag{12}$$

where a_r is given by

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$$a_r = 2^i \int_0^1 \mathcal{F}(\tau) RH(r,\tau) d\tau, \quad r = 0,1,2,\ldots, \tag{13}$$

with $r = 2^i + j - 1, i = 0,1,2,3,\ldots, j = 1,2,3,\ldots,2^i$, and $r = 0$ for $i = j = 0$. If we let $i = 0,1,2,\ldots,\alpha$, then the infinite series in (12) is truncated up to its first K terms as

$$\mathcal{F}(\tau) \simeq \sum_{r=0}^{K-1} a_r RH(r,\tau) = A^T \Phi(\tau), \tag{14}$$

where

$$K = 2^{\alpha+1}, \alpha = 0,1,2,3, \ldots,$$

A and $\Phi(\tau)$ are defined by

$$A = [a_0, a_1, \ldots, a_{K-1}]^T, \tag{15}$$

$$\Phi(\tau) = [\phi_0(\tau), \phi_1(\tau), \ldots, \phi_{K-1}(\tau)]^T, \tag{16}$$

and

$$\phi_r(\tau) = RH(r,\tau), \quad r = 0,1,2,\ldots,K-1.$$

If we set all the collocation points τ_l at the middle of each respective wavelet, then τ_l is defined by

$$\tau_l = \frac{l - 0.5}{K}, \quad l = 1,2,\ldots,K. \tag{17}$$

With these collocation points, the function is discretized over a series of equally spaced nodes. The vector $\Phi(\tau)$ can also be determined at these collocation points. Let the Haar matrix $\widehat{\Phi}_{K\times K}$ be the combination of $\Phi(\tau)$ at all the collocation points. Thus, we get

$$\widehat{\Phi}_{K\times K} = [\Phi(\tau_1), \Phi(\tau_2), \ldots, \Phi(\tau_K)]. \tag{18}$$

For example, if each waveform is divided into eight intervals, the magnitude of the waveform (see Ohkita and Kobayashi [28]) can be represented by

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$$\widehat{\Phi}_{8\times 8} = [\Phi(\tau_1), \Phi(\tau_2), \ldots, \Phi(\tau_8)] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}. \tag{19}$$

Using (14) and (18), we have

$$[\mathcal{F}(\tau_1), \mathcal{F}(\tau_2), \ldots, \mathcal{F}(\tau_K)] = A^T \widehat{\Phi}_{K\times K}. \tag{20}$$

From (20), we get

$$A^T = [\mathcal{F}(\tau_1), \mathcal{F}(\tau_2), \ldots, \mathcal{F}(\tau_K)] \widehat{\Phi}^{-1}_{K\times K}, \tag{21}$$

where

$$\widehat{\Phi}^{-1}_{K\times K} = (\frac{1}{K}) \widehat{\Phi}^{T}_{K\times K} diag(1,1,2,2, \underbrace{2^2, \ldots, 2^2}_{2^2}, \underbrace{2^3, \ldots, 2^3}_{2^3}, \ldots, \underbrace{\frac{K}{2}, \ldots, \frac{K}{2}}_{\frac{K}{2}}). \tag{22}$$

Therefore, the function $\mathcal{F}(\tau)$ is approximated by

$$\mathcal{F}(\tau_l) \approx A^T_{1\times K} \widehat{\Phi}_{K\times K}, \qquad l = 1,2,\ldots,K. \tag{23}$$

It is also expected to approximate the function $\mathcal{F}(\tau)$ with minimum mean integral square error, ε, defined by

$$\varepsilon = \int_0^1 (\mathcal{F}(\tau) - A^T \Phi(\tau))^2 d\tau.$$

Obviously, ε decreases when the level K gets larger and it should converge to zero when K approaches infinity.

3.3. Operational Matrix for Integration

In the solution of optimal control problems, we always need to deal with equations involving differentiation and integration. If the system function is expressed in Haar wavelets, the integration or differentiation operation of Haar series cannot be avoided. The differentiation of step waves will

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generate pulse signals which are difficult to handle, while the integration of step waves will result in constant slope functions which can be calculated by

$$\int_0^\tau \Phi(\tau')d\tau' \simeq P\Phi(\tau), \tag{24}$$

where $P = P_{K\times K}$ is a $K \times K$ operational matrix for integration and is given by in Razzaghi and Ordokhani [34] as

$$P_{K\times K} = \frac{1}{2K}\begin{bmatrix} 2KP_{\frac{K}{2}\times\frac{K}{2}} & -\widehat{\Phi}_{\frac{K}{2}\times\frac{K}{2}} \\ \widehat{\Phi}^{-1}_{\frac{K}{2}\times\frac{K}{2}} & 0 \end{bmatrix}, \tag{25}$$

with $\widehat{\Phi}_{1\times1} = [1], P_{1\times1} = \left[\frac{1}{2}\right]$, $\widehat{\Phi}_{\frac{K}{2}\times\frac{K}{2}}$ and $\widehat{\Phi}^{-1}_{\frac{K}{2}\times\frac{K}{2}}$ are respectively obtained from (18) and (22). The integration of the cross product of the two RH vector is

$$\int_0^1 \Phi(\tau)\Phi^T(\tau)d\tau = D, \tag{26}$$

where

$$D = diag(1,1,\frac{1}{2},\frac{1}{2},\underbrace{\frac{1}{2^2},\dots,\frac{1}{2^2}}_{2^2},\dots,\underbrace{\frac{1}{2^\alpha},\dots,\frac{1}{2^\alpha}}_{2^\alpha}). \tag{27}$$

4. Direct Collocation

4.1. Haar Discretization Method

In the discussion of Haar wavelets, we have already addressed how to approximate a function via Haar wavelets and its corresponding operational integration matrix. We are going to apply this methodology in optimal control problems so that Haar discretization is used in direct collocation (Dai and Cochran [10]). Thus, a continuous solution to a problem will be represented by state and control variables in terms of Haar series and its operational matrix to satisfy the differential equations. The standard interval considered here is denoted by $\tau \in [0,1)$ with the collocation points

$$\tau_l = \frac{l-0.5}{K}, \quad l = 1,2,\dots,K, \tag{28}$$

where K is the number of nodes used in the discretization and also the maximum wavelet index number. Note that the magnitude of K is a power of 2, so that the number of collocation points is also increasing by the same power. All the collocation points are equally distributed over the entire time interval $[0,1)$, with $\frac{1}{K}$ as the time distance between adjacent nodes. We assume that the derivative of the state variables $\dot{y}_i(\tau)$ and control variables $v_i(\tau)$, for $i = 1,2,\dots,N$, can be approximated by Haar wavelets with K collocation points, i.e.,

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$$\dot{y}_i(\tau) \approx C_{y_i}^T \Phi(\tau), \tag{29}$$

$$v_i(\tau) \approx C_{v_i}^T \Phi(\tau), \tag{30}$$

where

$$C_{y_i} = [C_{y_i1}, C_{y_i2}, \dots, C_{y_iK}]^T, \quad C_{v_i} = [C_{v_i1}, C_{v_i2}, \dots, C_{v_iK}]^T, \quad i = 1,2, \dots, N. \tag{31}$$

Using the operational integration matrix P, as defined by (25), the state variables $y_i(\tau)$ can be expressed as

$$y_i(\tau) = \int_0^\tau \dot{y}_i(\tau')d\tau' + y_{i_0} = \int_0^\tau C_{y_i}^T \Phi(\tau')d\tau' + y_{i_0} = C_{y_i}^T P\Phi(\tau) + y_{i_0}, i = 1,2, \dots, N. \tag{32}$$

As stated in (18), the expansion of the matrix $\Phi(\tau)$ at the K collocation points will yield the $K \times K$ Haar matrix $\widehat{\Phi}$. It follows that

$$\begin{gathered} \dot{y}_i(\tau_l) = C_{y_i}^T \Phi(\tau_l), \quad v_i(\tau_l) = C_{v_i}^T \Phi(\tau_l), \quad y_i(\tau_l) = C_{y_i}^T P\Phi(\tau_l) + y_{i_0}, \\ l = 1, \dots, K, \qquad i = 1, \dots, N. \end{gathered} \tag{33}$$

From the above expression, we can evaluate the variables at any collocation point using the product of its coefficient vectors and the corresponding column vector in the Haar matrix.

4.2. Nonlinear Programming

When the Haar collocation method is applied to optimal control problems, the nonlinear programming variables can be set as the unknown coefficient vectors of the derivatives of the state variables and control variables

$$y_i = [C_{y_i1}, C_{y_i2}, \dots, C_{y_iK}, C_{v_i1}, C_{v_i2}, \dots, C_{v_iK}]^T, i = 1,2, \dots, N. \tag{34}$$

The performance index (7) is then restated by

$$\begin{aligned} J = \frac{1}{2}\sum_{i=1}^{N} \Bigg\{ \int_{[0,1)} \Big(& ((C_{y_i}^T P\Phi(\tau) + y_{i_0})^T Q_i \big(C_{y_i}^T P\Phi(\tau) + y_{i_0}\big) \\ & + (C_{v_i}^T \Phi(\tau))^T R_i (C_{v_i}^T \Phi(\tau))\Big) \frac{d\tau}{(1-\tau)^2} \Bigg\}. \end{aligned} \tag{35}$$

Since the Haar wavelets are expected to be constant steps at each time interval, the above equation can be simplified as

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$$J = \frac{1}{2K}\sum_{i=1}^{N}\sum_{l=1}^{K}\Big((C_{y_i}^T P\Phi(\tau_l) + y_{i_0})^T Q_i\big(C_{y_i}^T P\Phi(\tau_l) + y_{i_0}\big) + (C_{v_i}^T\Phi(\tau_l))^T R_i(C_{v_i}^T\Phi(\tau_l))\Big)\frac{1}{(1-\tau_l)^2}. \tag{36}$$

Substituting $\dot{y}_i, v_i$ and y_i, for $i = 1,2,\dots,N$, in (8) with the Haar wavelet expressions in (33), we get

$$C_{y_i}^T\Phi(\tau_l) = \frac{1}{(1-\tau_l)^2}\Big(A_i\big(C_{y_i}^T P\Phi(\tau_l) + y_{i_0}\big) + B_i\left(C_{v_i}^T\Phi(\tau_l)\right) + F_i\big(C_y^T P\Phi(\tau_l) + y_0\big)\Big), \quad l = 1,\dots,K. \tag{37}$$

The system equation constraints are all treated as nonlinear constraints in a nonlinear programming solver. The boundary constraints need more attention. Since the first and last collocation points are not set as the initial and final time, respectively, the initial and final state variables are calculated according to

$$y_{i_0} = y_i(\tau_1) - \frac{\dot{y}_i(\tau_1)}{2K}, \qquad y_{i_1} = y_i(\tau_K) + \frac{\dot{y}_i(\tau_K)}{2K}, \qquad i = 1,2,\dots,N. \tag{38}$$

This way, the optimal control problems are transformed into nonlinear programming problems in a structured form which is solved by GAMS software (Rosenthal and Brooke [36]).

5. An Illustrative Example

The development of control laws to regulate the attitude of spacecraft and aircraft has been the focus of many research projects (Chang et al. [4] and Tsiotras [41]). From among this class of problems, the optimal attitude control problem has proven to be challenging due to its cascade nature. In this section, the effectiveness and high accuracy of our proposed approach are verified by solving the optimal attitude control problem. To this end, consider the Euler dynamics and kinematics of a rigid body as follows (Tsiotras [41]):

$$\begin{cases}\dot{\rho}(t) = H(\rho(t))\omega(t),\\ \dot{\omega}(t) = J^{-1}S(\omega(t))J\omega(t) + J^{-1}u(t),\end{cases} \tag{39}$$

where $J = diag(10,6.3,8.5), \rho = (\rho_1,\rho_2,\rho_3)^T \in \mathbb{R}^3$ is the vector of Rodrigues parameters, $\omega = (\omega_1,\omega_2,\omega_3)^T \in \mathbb{R}^3$ is the angular velocity, and $u = (u_1,u_2,u_3)^T \in \mathbb{R}^3$ is the control torque. The symbol $S(\cdot)$ is a skew symmetric matrix of the form $S(\omega) = \begin{bmatrix}0 & \omega_3 & -\omega_2\\ -\omega_3 & 0 & \omega_1\\ \omega_2 & -\omega_1 & 0\end{bmatrix}$, and the matrix valued function $H(\rho)$ is given by $H(\rho) = \frac{1}{2}(I - S(\rho) + \rho\rho^T)$. In addition, the initial conditions are $\rho(0) = (0.3735,0.4115,0.2521)^T$ and $\omega(0) = (0,0,0)^T$.

Let us define the state vector $x_i(t) \triangleq (\rho_i(t),\omega_i(t))^T$, for $i = 1,2,3$. Therefore, the 6th-order nonlinear interconnected dynamical system (39) is decomposed into three interconnected subsystems as (1), where

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$$A_1 = A_2 = A_3 = \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ \frac{1}{10} \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ \frac{10}{63} \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 \\ \frac{2}{17} \end{bmatrix},$$

$$F_1(x(t)) = \begin{bmatrix} \frac{1}{2}(x_{1,2}x_{1,1}^2 - x_{2,2}x_{3,1} + x_{2,2}x_{1,1}x_{2,1} + x_{3,2}x_{2,1} + x_{3,2}x_{1,1}x_{3,1}) \\ -\frac{11}{50}x_{3,2}x_{2,2} \end{bmatrix},$$

$$F_2(x(t)) = \begin{bmatrix} \frac{1}{2}(x_{1,2}x_{3,1} + x_{1,2}x_{1,1}x_{2,1} + x_{2,2}x_{2,1}^2 - x_{3,2}x_{1,1} + x_{3,2}x_{2,1}x_{3,1}) \\ -\frac{5}{21}x_{3,2}x_{1,2} \end{bmatrix},$$

$$F_3(x(t)) = \begin{bmatrix} \frac{1}{2}(-x_{1,2}x_{2,1} + x_{1,2}x_{1,1}x_{3,1} + x_{2,2}x_{1,1} + x_{2,2}x_{2,1}x_{3,1} + x_{3,2}x_{3,1}^2) \\ \frac{37}{85}x_{2,2}x_{1,2} \end{bmatrix},$$

$$x_1(0) = (0.3735,0)^T, x_2(0) = (0.4115,0)^T, x_3(0) = (0.2521,0)^T,$$

with $x_{i,j}$ being the jth element of vector x_i. The infinite horizon quadratic cost function to be minimized is given by (2), where $N = 3, Q_i = I_{2\times 2}$, and $R_i = 1$ for $i = 1, 2, 3$. Now, by change of variable (3), the above problem is transformed into the following problem:

$$minimize \quad J = \frac{1}{2}\sum_{i=1}^{3}\left\{\int_{[0,1)}\left(y_i^T(\tau)Q_i y_i(\tau) + v_i^T(\tau)R_i v_i(\tau)\right)\frac{d\tau}{(1-\tau)^2}\right\}$$

s.t.

$$\dot{y}_i(\tau) = \frac{1}{(1-\tau)^2}(A_i y_i(\tau) + B_i v_i(\tau) + F_i(y(\tau))), \tau \in [0,1), i = 1,2,3,$$

$$y_1(0) = (0.3735,0)^T, y_2(0) = (0.4115,0)^T, y_3(0) = (0.2521,0)^T,$$

where

$$F_1(y(\tau)) = \begin{bmatrix} \frac{1}{2}(y_{1,2}y_{1,1}^2 - y_{2,2}y_{3,1} + y_{2,2}y_{1,1}y_{2,1} + y_{3,2}y_{2,1} + y_{3,2}y_{1,1}y_{3,1}) \\ -\frac{11}{50}y_{3,2}y_{2,2} \end{bmatrix},$$

$$F_2(y(\tau)) = \begin{bmatrix} \frac{1}{2}(y_{1,2}y_{3,1} + y_{1,2}y_{1,1}y_{2,1} + y_{2,2}y_{2,1}^2 - y_{3,2}y_{1,1} + y_{3,2}y_{2,1}y_{3,1}) \\ -\frac{5}{21}y_{3,2}y_{1,2} \end{bmatrix},$$

$$F_3(y(\tau)) = \begin{bmatrix} \frac{1}{2}(-y_{1,2}x_{2,1} + y_{1,2}x_{1,1}y_{3,1} + y_{2,2}x_{1,1} + y_{2,2}y_{2,1}y_{3,1} + y_{3,2}y_{3,1}^2) \\ \frac{37}{85}y_{2,2}y_{1,2} \end{bmatrix},$$

with $y_{i,j}$ being the jth element of vector y_i. In order to obtain an accurate enough suboptimal trajectory-control pair, we applied the proposed method for $K = 1024$. The results are depicted in Figures 1-6.

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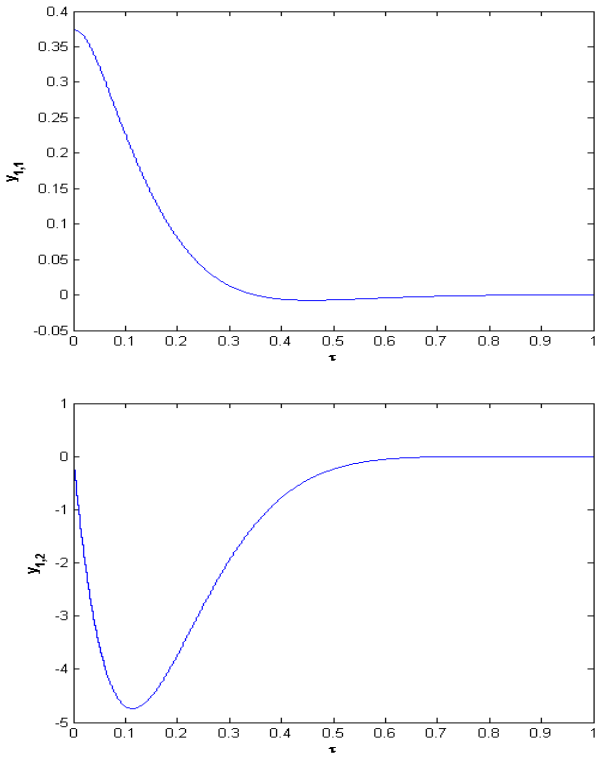


Figure 1. Approximate optimal trajectory obtained of $y_1(\tau)$.

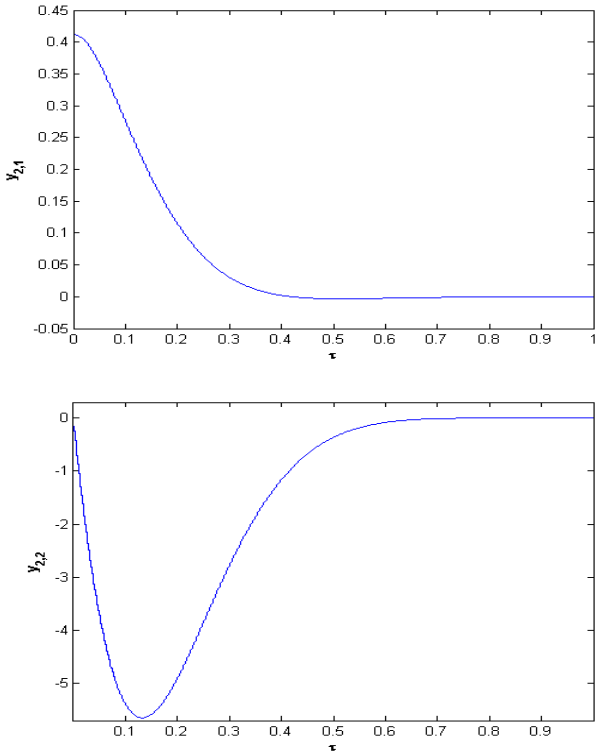


Figure 2. Approximate optimal trajectory obtained of $y_2(\tau)$.

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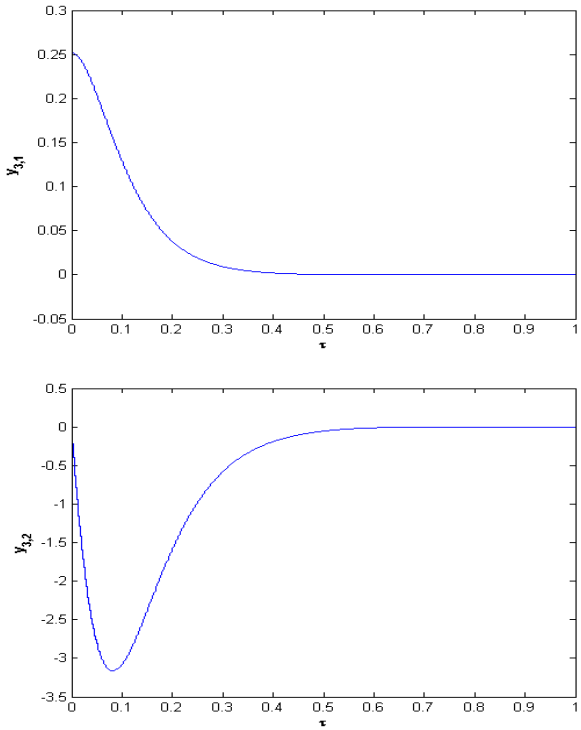


Figure 3. Approximate optimal trajectory obtained of $y_3(\tau)$.

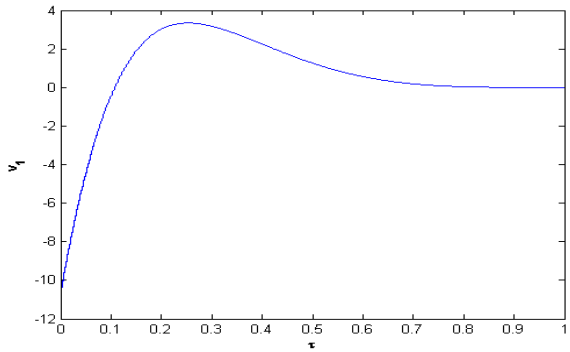


Figure 4. Approximate optimal control obtained of $v_1(\tau)$.

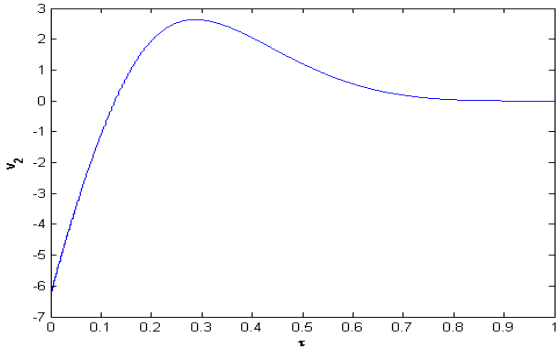


Figure 5. Approximate optimal control obtained of $v_2(\tau)$.

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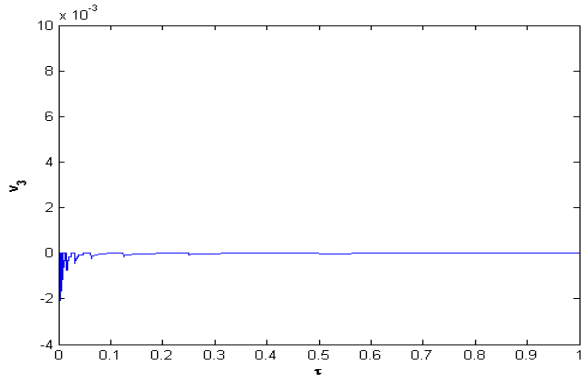


Figure 6. Approximate optimal control obtained of $v_3(\tau)$.

Finally, a natural question arises: are there advantages of the proposed collocation method as compared to the existing ones? To answer this, we summarize what we have observed from numerical experiments and theoretical results as follows.

• A main advantage of using Haar wavelets is that the matrices $\widehat{\Phi}_{K\times K}, \widehat{\Phi}_{K\times K}^{-1}$ and D introduced in (18), (22) and (27), have large numbers of zero elements, i.e., they are sparse; hence, the proposed method is very attractive to reduce the CPU time and computer memory while preserving the accuracy of the solution.

• The simple implementation of Haar wavelet-based optimal control in real applications is interesting.

• Haar functions are also noted for their rapid convergence of the expansion of functions.

• The proposed method also produces results similar to other collocation methods for continuous optimal control problems and shows advantages in discrete optimal control problems when the switching time is unknown.

• The proposed orthogonal collocation method leads to rapid convergence as the number of collocation points increases.

• With $\tau_l = \frac{l-0.5}{K} \neq 1, l = 1,2,\ldots,K$, there is no numerical difficulties. In fact, we do not apply numerical integration methods such as Simpson's rule for calculation of the integral (7), since it leads to some problems at the right end-point. We use the formula (36) to calculate the integral in (7) which does not require $\tau = 1$. Thus, the integration on the finite-time interval will be convergent.

• As real-time applications of the developed control, three example problems can be solved to illustrate various elements of the main points of the proposed ideas: first is a standard linear–quadratic regulator problem. The second example is a nonlinear control problem of stabilizing NPSAT1, an experimental spacecraft designed. The inverted pendulum problem, with all its nonlinearities and saturation constraints, can be considered as the third real-time application of the infinite-horizon control (see Fahroo and Ross [11]).

6. Conclusion

Some nonlinear large-scale optimal control problems (OCPs), were approximately solved by a combined parameter and function optimization algorithm. To this end and on the basis of the

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approximation of dynamical systems and performance index into Haar series, an efficient and accurate method was then applied to solve a class of infinite horizon OCPs of nonlinear interconnected large-scale dynamical systems. An illustrative example was worked through to demonstrate the validity and applicability of the proposed method.

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