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On SOCP/SDP Formulation of the Extended Trust Region Subproblem

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We consider the extended trust region subproblem (eTRS) as the minimization of an indefinite quadratic function subject to the intersection of unit ball with a single linear inequality constraint. Using a variation of the S-Lemma, we derive the necessary and sufficient optimality conditions for eTRS. Then, an SOCP/SDP formulation is introduced for the problem. Finally, several illustrative examples are provided.

Keywords: *Extended trust region subproblem, S-Lemma, Semidefinite program, Second order cone program.*

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1. Introduction

Consider the following extended trust region subproblem (eTRS)

$$\begin{aligned} \min\ & x^T A x + 2a^T x \\ \text{s.t.}\ & \|x\|^2 \le 1 \\ & b^T x \le \beta, \end{aligned} \tag{1}$$

where $A^T = A \in \mathbb{R}^{n\times n}$ is indefinite, $a, b \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$. Since A is indefinite, this is a nonconvex optimization problem and semidefinite programming (SDP) relaxation is not tight in general. If $b = 0$ and $\beta = 0$, then eTRS reduces to the well-known trust region subproblem (TRS) which is the key subproblem in solving nonlinear optimization problems [4]. Although TRS is a nonconvex problem, it enjoys strong duality and exact SDP relaxation exists for it [5]. However, the following classical SDP relaxation is not exact for eTRS as it will also be shown in the numerical results section:

$$\begin{aligned} \min\ & A\cdot X + 2a^T x \\ \text{s.t.}\ & \operatorname{trace}(X) \le 1, \\ & b^T x \le \beta, \\ & X \succcurlyeq xx^T. \end{aligned} \tag{2}$$

First, the authors of [11] studied eTRS and proposed an exact SOCP/SDP (Second order cone program/Semidefinite program) formulation for it. Due to the importance of eTRS also in solving general nonlinear optimization problems, several variants of it have been the focus of current research

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[1, 2, 3, 6, 7, 9]. Beck and Eldar [1] studied eTRS under the condition that $\dim\big(\mathrm{Ker}(A-\lambda_1 I)\big) \geq 2$, which is equivalent to

$$\lambda_1 = \lambda_2, \tag{3}$$

where λ_1 and λ_2 are the two smallest eigenvalues of A. Under this condition, they showed the necessary and sufficient optimality conditions for eTRS:

(i) $2(A+\lambda I)x = -(2a+\mu b)$,
(ii) $(A+\lambda I) \succcurlyeq 0$,
(iii) $\lambda(\|x\|^2-1)=0, \quad \mu(b^T x-\beta)=0$,
(iv) $\lambda, \mu \geq 0$.

Jeyakumar and Li [7] showed that $\dim\big(\mathrm{Ker}(A-\lambda_1 I_n)\big) \geq 2$, together with the Slater's condition ensures that a set of combined first and second-order Lagrange multiplier conditions are necessary and sufficient for the global optimality of eTRS and consequently for strong duality. In [6], the authors improved upon the dimension condition by Jeyakumar and Li under which eTRS admits an exact semidefinite relaxation. They provided the following condition:

$$\mathrm{rank}([A-\lambda_1 I_n \quad b]) \leq n-1. \tag{4}$$

It should be noted that TRS has at most one local-nonglobal minimum (LNGM) [8], which is a candidate for the optimal solution of eTRS if it is feasible. An efficient algorithm for computing LNGM is given in [10]. All the above rank conditions guarantee that the global solution of eTRS does not happen at the LNGM of TRS. Most recently, in [2] the authors derived the SOCP/SDP reformulation of [11] by a different approach and extended it to the cases where more than one linear constraint exist. Here, using a variant of the S-Lemma, we first derive the necessary and sufficient optimality conditions for eTRS leading to an SOCP/SDP formulation. Then, we establish that our derived formulation is the dual of the formulation given in [2, 11]. Finally, we present several numerical examples illustrating various cases of the optimal solution of eTRS.

2. Global Optimality Conditions for eTRS

We define the dual cone of S as $S^* = \{y : \langle y, x\rangle \geq 0, \ \forall x \in S\}$. The following proposition, which is a variant of the S-Lemma, plays a key role in the proof of the optimality conditions.

Proposition 2.1. Let $f, g\colon \mathbb{R}^n \to \mathbb{R}$ be quadratic functions with $g(x) = x^T A_g x + a_g^T x + c_g$, $b \in \mathbb{R}^n$, and $\beta \in \mathbb{R}$. Moreover, assume that $g(x)$ is convex and there exists an $\bar{x} \in \mathbb{R}^n$ such that $b^T\bar{x} < \beta$ and $g(\bar{x}) < 0$. Then, the following two statements are equivalent:

1. The system

$$\begin{gathered} f(x) < 0, \\ g(x) \leq 0, \\ b^T x \leq \beta, \\ x \in \mathbb{R}^n, \end{gathered}$$

is not solvable.

2. There is a nonnegative multiplier $y \geq 0$, a scalar $u_0 \in \mathbb{R}$ and a vector $u \in \mathbb{R}^n$ such that

$$f(x) + yg(x) + (u^T x - u_0)(b^T x - \beta) \geq 0, \qquad \forall x \in \mathbb{R}^n,$$
$$u \in \left\{x \in \mathbb{R}^n : x^T A_g x \leq 0, \qquad a_g^T x \leq 0\right\}^*,$$
$$\begin{pmatrix} u_0 \\ u \end{pmatrix} \in \left\{\begin{pmatrix} x_0 \\ x \end{pmatrix} : \ x_0 = -1, \qquad g(x) \leq 0, \qquad c_g + a_g^T x \leq 0\right\}^*.$$

Proof. See [11, Corollary 7].

Corollary 2.1. If $g(x) = \|x\|^2 - 1$, then item 2 in Proposition 1 is equivalent to

$$f(x) + yg(x) + (u^T x - u_0)(b^T x - \beta) \geq 0, \qquad \forall x \in \mathbb{R}^n,$$
$$\begin{pmatrix} -u_0 \\ u \end{pmatrix} \in L_{n+1}$$

where L_{n+1} is the Lorentz cone defined as follows:

$$L_{n+1} = \{x = (x_0; \bar{x}) \in \mathbb{R}^{n+1} \mid \|\bar{x}\| \leq x_0\}.$$

In the following theorem, we give the optimality conditions for eTRS. Our proof follows the idea of [7].

Theorem 2.1. Suppose that the strict feasibility constraint holds for eTRS, i.e.,

$$\exists \hat{x} \in \mathbb{R}^n \ \text{ with } \ \|\hat{x}\|^2 - 1 < 0, \qquad b^T \hat{x} - \beta < 0.$$

Moreover, let x^* be a feasible point for eTRS. Then, x^* is a global minimizer of eTRS if, and only if, there exist $\lambda_0 \in \mathbb{R}_+$ and $(-u_0, u) \in L^{n+1}$ such that the following conditions hold:

- (i) $(2A + 2\lambda_0 I + bu^T + ub^T)x^* = -(2a - \beta u - bu_0)$,
- (ii) $\lambda_0(\|x^*\|^2 - 1) = 0, (u^T x^* - u_0)(b^T x^* - \beta) = 0$,
- (iii) $(2A + 2\lambda_0 I + bu^T + ub^T) \succcurlyeq 0$.

Proof. [**Necessity**] Let x^* be a global minimizer of eTRS. Then, the following system of inequalities has no solution:

$$x^T A\, x + 2a^T x + \gamma < 0,$$
$$\|x\|^2 - 1 \leq 0,$$
$$b^T x \leq \beta,$$

where $\gamma = -\left(x^{*T} A\, x^* + 2a^T x^*\right)$. Thus, by Proposition 2.1 there exist $\lambda_0 \geq 0$ and a vector $(u_0; u) \in \mathbb{R}^{n+1}$ such that

$$x^T A x + 2a^T x + \gamma + \lambda_0(\|x\|^2 - 1) + (u^T x - u_0)(b^T x - \beta) \geq 0, \qquad \forall x \in \mathbb{R}^n$$

and $u^T x - u_0 \geq 0,\ \forall x \in \mathbb{R}^n : \|x\|^2 \leq \delta^2$. Let $x = x^*$. Then, we have

$$\lambda_0(\|x^*\|^2 - 1) + (u^T x^* - u_0)(b^T x^* - \beta) \geq 0.$$

Now, as x^* is feasible for eTRS, $\lambda_0 \geq 0$, and $(u^T x^* - u_0) \geq 0$ it follows that

$$\lambda_0(\|x^*\|^2 - 1) = 0, \;\; (u^T x^* - u_0)(b^T x^* - \beta) = 0.$$

Let

$$h(x) = x^T A\, x + 2a^T x + \lambda_0(\|x\|^2 - 1) + (u^T x - u_0)(b^T x - \beta).$$

Then, obviously x^* is a global minimizer of h, and so $\nabla h(x^*) = 0$ and $\nabla^2 h(x^*) \succcurlyeq 0$, i.e.,

$$\begin{gathered}(2A + 2\lambda_0\, I + bu^T + ub^T)x^* = -(2a - \beta u - bu_0),\\ (2A + 2\lambda_0 I + bu^T + ub^T) \succcurlyeq 0.\end{gathered}$$

Thus, all conditions (i), (ii) and (iii) hold.

[**Sufficiency**] If the optimality conditions hold, then from (ii) we see that

$$h(x) = x^T A x + 2a^T x + \lambda_0(\|x\|^2 - 1) + (u^T x - u_0)(b^T x - \beta)$$

is convex. Moreover, from condition (i), we have $\nabla h(x^*) = 0$ and therefore, x^* is a global minimizer of h. Thus, for the given λ_0 and $(u_0; u)$ in the optimality conditions and for any feasible solution of eTRS, we have

$$\begin{aligned}x^T A\, x + 2a^T x &\geq x^T A\, x + 2a^T x + \lambda_0(\|x\|^2 - 1) + (u^T x - u_0)(b^T x - \beta)\\ &\geq {x^*}^T A\, x^* + 2a^T x^* + \lambda_0(\|x^*\|^2 - 1) + (u^T x^* - u_0)(b^T x^* - \beta)\\ &= {x^*}^T A x^* + 2a^T x^*.\end{aligned}$$

This implies that x^* is a global minimizer of eTRS. ■

Theorem 2.2. Suppose that there exists $\bar{x} \in \mathbb{R}^n$ with $\|\bar{x}\|^2 - 1 < 0$ and $b^T \bar{x} - \beta < 0$. Then, we have

$$\begin{aligned}&\min\{x^T\, A\, x + 2a^T x : \|x\|^2 \leq 1, b^T x \leq \beta\}\\ &= \max_{\lambda_0 \geq 0, (u_0, u) \in S} \min_x \{x^T\, A\, x + 2a^T x + \lambda_0(\|x\|^2 - 1) + (u^T x - u_0)(b^T x - \beta)\},\end{aligned}$$

where

$$S = \{(u_0; u) \mid (u^T x - u_0) \geq 0, \qquad \forall x \colon \|x\|^2 \leq 1\},$$

and the maximum is attained.

Proof. It is easy to see that for every feasible point of eTRS, and every $\lambda_0 \geq 0$ and $(u_0, u) \in S$,

$$x^T A\, x + 2a^T x \geq x^T A x + 2a^T x + \lambda_0(\|x\|^2 - 1) + (u^T x - u_0)(b^T x - \beta).$$

Therefore,

$$\begin{aligned}&\min\{x^T A x + 2a^T x : \|x\|^2 \leq 1, \& b^T x \leq \beta\}\\ &\geq \max_{\lambda_0 \geq 0, (u_0, u) \in S} \min_{\mathrm{x}} \{x^T A\, x + 2a^T x + \lambda_0(\|x\|^2 - 1) + (u^T x - u_0)(b^T x - \beta)\}.\end{aligned}$$

To show the reverse inequality, let x^* be a global minimizer of eTRS. Then, there exist $\lambda_0 \in \mathbb{R}_+$ and $(u_0, u) \in S$ such that the following conditions hold:

- $(2A + 2\lambda_0 I + bu^T + ub^T)x^* = -(2a - \beta u - bu_0)$,
- $\lambda_0(\|x^*\|^2 - 1) = 0$ and $(u^T x^* - u_0)(b^T x^* - \beta) = 0$,
- $(2A + 2\lambda_0 I + bu^T + ub^T) \succcurlyeq 0$.

We see that

$$h(x) = x^T A\, x + 2a^T x + \lambda_0(\|x\|^2 - 1) + (u^T x - u_0)(b^T x - \beta)$$

is convex, $\nabla h(x^*) = 0$ and $\nabla^2 h(x^*) \succcurlyeq 0$. Therefore, x^* is a global minimizer of h, i.e.,

$$\begin{aligned}
& x^T A\, x + 2a^T x + \lambda_0(\|x\|^2 - \delta^2) + (u^T x - u_0)(b^T x - \beta) \\
& \geq x^{*T} A x^* + 2a^T x^* + \lambda_0(\|x^*\|^2 - 1) + (u^T x^* - u_0)(b^T x^* - \beta) \\
& = x^{*T} A x^* + 2a^T\, x^*.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \min\{x^T A x + 2a^T x : \|x\|^2 \leq 1, b^T x \leq \beta\} \\
& \leq \max_{\lambda_0 \geq 0, (u_0;\, u) \in S} \min_x \{x^T A x + 2a^T x + \lambda_0(\|x\|^2 - 1) + (u^T x - u_0)(b^T x - \beta)\}.
\end{aligned}$$

■

As we see, in general strong duality does not hold for eTRS which is the reason to consider conditions given in [1, 6, 7] to guarantee it.

Corollary 2.2. If $u = 0$ and $u_0 \neq 0$, then strong duality holds for eTRS.

Proof. This follows from Theorem 2.2. ■

Form Theorem 2.2, we further have

$$\begin{aligned}
& \max_{\lambda_0 \geq 0, (u_0; u) \in S} \min_x \{x^T A\, x + 2a^T x + \lambda_0(\|x\|^2 - 1) + (u^T x - u_0)(b^T x - \beta)\} \\
& \quad = \max \;\; z \\
& \qquad \text{s.t.} \begin{pmatrix} -\lambda_0 + \beta u_0 - z & \frac{1}{2}(2a - \beta u - bu_0)^T \\ \frac{1}{2}(2a - \beta u - bu_0) & A + \lambda_0\, I + \frac{1}{2}(\, bu^T + ub^T) \end{pmatrix} \succcurlyeq 0, \\
& \qquad \|u\| \leq -u_0, \\
& \qquad \lambda_0 \geq 0,
\end{aligned} \tag{5}$$

which is an SOCP/SDP formulation of eTRS. In what follows, we show that this formulation is the dual of the SOCP/SDP formulation given in [2, 11]. Consider the Lagrange function of (5):

$$\mathcal{L}(Y, v, u, u_0, \lambda_0, z) = z + \begin{pmatrix} -\lambda_0 + \beta u_0 - z & \frac{1}{2}(2a - \beta u - b u_0)^T \\ \frac{1}{2}(2a - \beta u - b u_0) & A + \lambda_0 I + \frac{1}{2}(bu^T + ub^T) \end{pmatrix} \cdot Y + v^T \begin{pmatrix} -u_0 \\ u \end{pmatrix},$$

where $Y \succcurlyeq 0$ and $\|\bar{v}\| \leq v_0$. Let

$$Y = \begin{pmatrix} \alpha & x^T \\ x & X \end{pmatrix}.$$

Thus, the Lagrangian can be written as

$$\begin{aligned} \mathcal{L}(Y, v, u, u_0, \lambda_0, z) &= z + \left(A + \lambda_0 I + \frac{1}{2}(bu^T + ub^T)\right) \cdot X + (2a - \beta u - b u_0)^T x \\ &\quad + \alpha(-\lambda_0 + \beta u_0 - z) + \bar{v}^T u - v_0 u_0 \\ &= A \cdot X + 2a^T x + (1 - \alpha) z + \lambda_0(\mathrm{trace}(X) - \alpha) + (Xb - \beta x + \bar{v})^T u \\ &\quad + (-b^T x - v_0 + \beta) u_0. \end{aligned}$$

Therefore, the Lagrangian dual becomes

$$\begin{aligned} & \min_{\begin{pmatrix} \alpha & x^T \\ x & X \end{pmatrix} \succcurlyeq 0, \|\bar{v}\| \leq v_0} \ \max_{\lambda_0 \geq 0, \|u\| \leq -u_0} \mathcal{L}(Y, v, u, u_0, \lambda_0, z) \\ = & \min_{\begin{pmatrix} \alpha & x^T \\ x & X \end{pmatrix} \succcurlyeq 0, \|\bar{v}\| \leq v_0} \ \max_{\lambda_0 \geq 0, \|u\| \leq -u_0} A \cdot X + 2a^T x + (1 - \alpha) z + \lambda_0(\mathrm{trace}(X) - \alpha) \\ & \qquad + (Xb - \beta x + \bar{v})^T u + (-b^T x - v_0 + \beta) u_0 \\ = & \min_{\begin{pmatrix} \alpha & x^T \\ x & X \end{pmatrix} \succcurlyeq 0, \|\bar{v}\| \leq v_0} \mathcal{G}(X, x, \alpha), \end{aligned}$$

where

$$\begin{aligned} \mathcal{G}(X, x, \alpha) = \max_{\lambda_0 \geq 0, \|u\| \leq -u_0} & A \cdot X + 2a^T x + (1 - \alpha) z + \lambda_0(\mathrm{trace}(X) - \alpha) \\ & + (Xb - \beta x + \bar{v})^T u + (-b^T x - v_0 + \beta) u_0 \end{aligned}$$

We further have

$$\mathcal{G}(X, x, \alpha) = \begin{cases} A \cdot X + 2a^T x, & \text{if } 1 - \alpha = 0, \ \mathrm{trace}(X) - \alpha \leq 0 \\ & \quad Xb - \beta x + \bar{v} = 0, \ \ -b^T x - v_0 + \beta \geq 0 \\ \infty, & \text{o.w.} \end{cases}$$

Thus, the Lagrangian dual becomes

$$\begin{aligned} \min \quad & A \cdot X + 2a^T x \\ \text{s.t.} \quad & \mathrm{trace}(X) - 1 \leq 0, \\ & Xb - \beta x + \bar{v} = 0, \\ & -b^T x - v_0 + \beta \geq 0, \\ & \|\bar{v}\| \leq v_0, \\ & X \succcurlyeq x x^T. \end{aligned} \tag{6}$$

From (6), we have

$$\begin{aligned}&\bar{v} = \beta x - Xb,\\ &v_0 \leq -b^T x + \beta,\\ &\|\bar{v}\| \leq v_0 \Rightarrow \|\beta x - Xb\| \leq v_0 < -b^T x + \beta.\end{aligned}$$

Therefore, (6) can be written as

$$\begin{array}{ll}\min & A \cdot X + 2a^T x\\ \text{s.t.} & \operatorname{trace}(X) \leq 1,\\ & \|\beta x - Xb\| \leq -b^T x + \beta,\\ & X \succcurlyeq xx^T.\end{array} \tag{7}$$

This SOCP/SDP formulation is exactly the one given in [2, 11], but our derivation is completely different.

Corollary 2.3. If at the optimal solution of (7), $X^* = x^*_{socp/sdp}\left(x^*_{socp/sdp}\right)^T$, then $x^*_{socp/sdp}$ is optimal for (1).

2.1. Rank One Decomposition Procedure

In order to derive an optimal solution for eTRS from a solution of (7) that is not rank one, here we give a rank one decomposition approach similar to the one in [12]. The following lemma is used for this derivation.

Lemma 2.1. [12] Let G be an arbitrary symmetric matrix and X be a positive semidefinite matrix with rank r. Further, suppose that $G \cdot X \geq 0$. Then, there exists a rank-one decomposition of X such that

$$X = \sum_{i=1}^{r} x_i x_i^T$$

and $x_i^T G x_i \geq 0$, for $i = 1, \cdots, r$. If, in particular, $G \cdot X = 0$, then $x_i^T G x_i = 0$, for $i = 1, \cdots, r$.

Let X^* be an optimal solution for (7) which is not rank one and consider the following notations:

$$Y^* = \begin{pmatrix} 1 & \left(x^*_{socp/sdp}\right)^T \\ x^*_{socp/sdp} & X^* \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -I_n \end{pmatrix}, \quad g = \begin{pmatrix} \beta \\ -b \end{pmatrix}.$$

Obviously, we have

$$\begin{aligned}&\left\|\beta x^*_{socp/sdp} - X^* b\right\| \leq -b^T x^*_{socp/sdp} + \beta \Leftrightarrow \begin{pmatrix} \beta - b^T x^*_{socp/sdp} \\ \beta x^*_{socp/sdp} - X^* b \end{pmatrix} = Y^* g \in L_{n+1},\\ &\operatorname{trace}(X^*) \leq 1 \Leftrightarrow J \cdot Y^* \geq 0.\end{aligned}$$

The following cases may occur.

Case 1. $Y^* g = 0$. From Lemma 2.1, there exists a rank one decomposition of Y^* as follows:

$$Y^* = \sum_{i=1}^{r} (y_i^*)(y_i^*)^T,$$

where r is the rank of Y^* such that $J \cdot [(y_i^*)(y_i^*)^T] \geq 0$, for $i = 1, \dots, r$. Moreover, $J \cdot [(y_i^*)(y_i^*)^T] = 0$, for $i = 1, \dots, r$, if $J \cdot Y^* = 0$. We may choose the sign of the y_i^* to ensure that $y_i^* \in L_{n+1}$, $i = 1, \dots, r$.

By linear independence of the y_i^*, we get $g^T y_i^* = 0, i = 1, \dots, r$. Let $y_i^* = \begin{pmatrix} t_i^* \\ \bar{y}_i^* \end{pmatrix}$, $i = 1, \dots, r$. Since $y_i^* \in L_{n+1}$ and $y_i^* \neq 0$, we have $t_i^* > 0$, $i = 1, \dots, r$. Take any $1 \leq j \leq r$; it follows that $\begin{pmatrix} 1 \\ \bar{y}_i^*/t_i^* \end{pmatrix} (1 \quad [\bar{y}_i^*/t_i^*]^T)$ is optimal for (7).

Case 2. $J \cdot Y^* > 0$ and $Y^* g \neq 0$. Due to the complementarity condition, we must have $\lambda_0 = 0$. Let $y_g^* := Y^* g = \begin{pmatrix} t_g^* \\ \bar{y}_g^* \end{pmatrix}$. Since $y_g^* \in L_{n+1}$ by feasibility, we know that $t_g^* > 0$. Moreover, $J \cdot \left[y_g^* (y_g^*)^T\right] = (t_g^*)^2 - \|\bar{y}_g^*\|^2 \geq 0$, and $y_g^* (y_g^*)^T g = (g^T Y^* a) Y^* a \in L_{n+1}$. Therefore, $y_g^* (y_g^*)^T / (t_g^*)^2$ is optimal for (7) as it is feasible and satisfies the complementarity conditions.

Case 3. $J \cdot Y^* = 0$ and $Y^* g \neq 0$. Denote $y_g^* := Y^* g \neq 0$. Let $\tilde{Y} = Y^* - \frac{Y^* g g^T Y^*}{g^T Y^* g} \succcurlyeq 0$. It is easy to see that $\tilde{Y} g = 0$.

Case 3.1. $J \cdot [y_g^* (y_g^*)^T] = 0$. In this subcase, we have that $\frac{y_g^* (y_g^*)^T}{(t_g^*)^2}$ is optimal for (7).

Case 3.2. $J \cdot [y_g^* \, (y_g^*)^T] > 0$. In this subcase, we have

$$J \cdot \tilde{Y} = J \cdot Y^* - J \cdot \left[y_g^* (y_g^*)^T\right] / (g^T Y^* g) < 0. \tag{8}$$

Now, let us decompose $\tilde{Y}$ as

$$\tilde{Y} = \sum_{i=1}^{s} \tilde{y}_i y_i^T,$$

where $s = \operatorname{rank}(\tilde{Y}) > 0$. Since $\tilde{Y} g = 0$, we have $\tilde{y}_i^T g = 0$, for $i = 1, \dots, s$. Choose j such that

$$J \cdot \tilde{y}_j (\tilde{y}_j)^T < 0.$$

Such a j must exist due to (8). Consider the following quadratic equation:

$$J \cdot \left[(y_g^* + \alpha \, \tilde{y}_j)(\, y_g^* + \alpha \tilde{y}_j)^T\right] = 0.$$

This equation has two distinct real roots having opposite signs. Choose the one such that the first component of $y_g^* + \alpha\tilde{y}_j$ is positive. Denote

$$y_g^* + \alpha\tilde{y}_j := \begin{pmatrix} t^* \\ \bar{y}^* \end{pmatrix}.$$

In this case, since $J \cdot \left[y_g^*\left(y_g^*\right)^T\right] > 0$, it follows that y_g^* is in the strict interior of the cone L_{n+1}. Due to the complementarity, we must have $(u_0^*; u^*) = 0$. Let us consider the solution $\begin{pmatrix} 1 \\ \bar{y}^*/t^* \end{pmatrix}(1 \quad (\bar{y}^*/t^*)^T)$. It is easy to check that this solution is both feasible and complementary to the dual optimal solution $(\lambda_0^*; u_0^*; u^*)$, and thus optimal for (7).

3. Numerical Examples

Here, we are to provide various examples explaining different cases that might occur for the optimal solution of eTRS.

Example 3.1. Consider the following example:

$$A = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 11 \end{pmatrix}, a = \begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix}, b = \begin{pmatrix} 20 \\ 8 \\ -14 \end{pmatrix}, \delta = 1, \beta = 5.$$

We have $\lambda_1 = -4$ and $\dim\left(\text{Ker }(A - \lambda_{\min}(A)I_n)\right) = 1 \ngeq 2$. Thus, the dimension condition (4) fails to hold. Moreover, the new dimension condition given in [6] also fails to hold, since

$$\text{rank}([A - \lambda_1 I_n \quad b]) = \text{rank}\begin{pmatrix} 0 & 0 & 0 & 20 \\ 0 & 8 & 0 & 8 \\ 0 & 0 & 7 & -14 \end{pmatrix} = 3 \nleq 2.$$

The optimal objective value of SDP relaxation (2) is -7.6827. The global solution of TRS is $x_g^* = [1, 0, 0]^T$, which is infeasible for eTRS, and LNGM of TRS is $x_l^* = [-1, 0, 0]^T$, which is feasible for eTRS with the objective value of 4.0000. Moreover, for (7), the optimal solution is $x_{socp/sdp}^* = [0.6266, -0.2169,\ 0.4140]^T$ and $X^* = x_{socp/sdp}^*\left(x_{socp/sdp}^*\right)^T$. Thus, $x_{socp/sdp}^*$ is optimal for (1) with the objective value of -4.1329. As we see, the classical SDP relaxation (2) is not exact for this example and subsequently strong duality fails to hold. Also, it is worth to note that at the optimal solution, the linear constraint is active while the trust region constraint is not active.

Example 3.2. Consider the following example:

$$A = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}, a = \begin{pmatrix} 0.5714 \\ 0 \\ 0 \end{pmatrix}, b = \begin{pmatrix} -17 \\ 14 \\ -2 \end{pmatrix}, \delta = 1, \beta = 4.4.$$

We have $\lambda_1 = -4$ and $\dim\left(\text{Ker }(A - \lambda_{\min}(A)I_n)\right) = 1 \ngeq 2$. Thus, the dimension condition (4) fails to hold for this example as well. Also, the new dimension condition of [6] fails to hold here, since

$$\mathrm{rank}([A-\lambda_1 I_n \quad b]) = \mathrm{rank}\begin{pmatrix} 0 & 0 & 0 & -17 \\ 0 & 5 & 0 & 14 \\ 0 & 0 & 3 & -2 \end{pmatrix} = 3 \nleq 2.$$

The global solution of TRS is $x_g^* = [-1, 0, 0]^T$, which is infeasible for eTRS, and LNGM of TRS is $x_l^* = [1, 0, 0]^T$, which is feasible for eTRS with the objective value of -2.4972. The optimal objective value of the SDP relaxation (2) is -5.4326 and the optimal objective value of the SOCP/SDP formulation (7) is -2.4972, which is also the optimal objective value of (1). Moreover, for (7), the optimal solution is $x_{socp/sdp}^* = [1, 0, 0]^T$ and $X^* = x_{socp/sdp}^*\left(x_{socp/sdp}^*\right)^T$, and thus $x_{socp/sdp}^*$ is optimal for (1). Here also strong duality fails to hold like the previous example. Finally, at the optimal solution, the linear constraint is not active while the trust region constraint is active.

Example 3.3. Consider the following example where at optimality both constraints are active:

$$A = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 2 \end{pmatrix}, a = \begin{pmatrix} 0 \\ 2.2857 \\ 0 \end{pmatrix}, b = \begin{pmatrix} 4 \\ -15 \\ 18 \end{pmatrix}, \delta = 1, \beta = 4.$$

Here, we have $\lambda_1 = -8$ and $\dim\left(\mathrm{Ker}\,(A - \lambda_{\min}(A) I_n)\right) = 1 \ngeq 2$. Thus, the dimension condition (4) fails to hold. Moreover, the new dimension condition of [6] also fails to hold, since

$$\mathrm{rank}([A-\lambda_1 I_n \quad b]) = \mathrm{rank}\begin{pmatrix} 4 & 0 & 0 & 4 \\ 0 & 0 & 0 & -15 \\ 0 & 0 & 10 & 18 \end{pmatrix} = 3 \nleq 2.$$

The global solution of TRS is $x_g^* = [0, -1, 0]^T$, which is infeasible for eTRS, and LNGM of TRS is $x_l^* = [0, 1, 0]^T$, which is feasible for eTRS with the objective value of -3.4286. The optimal objective value of the SDP relaxation (2) is -11.0642 and the optimal objective value of the SOCP/SDP formulation (7) is -9.7551, which is also the optimal objective value of (1). The optimal solution of (7) is $x_{socp/sdp}^* = [-0.2885, -0.8567, -0.4276]^T$ and $X^* = x_{socp/sdp}^*\left(x_{socp/sdp}^*\right)^T$, and thus $x_{socp/sdp}^*$ is optimal for (1).

In all three examples above, the optimal solution of (7) is rank one, and thus we easily have the solution of (1). However, this is not the case, in general, as illustrated by the following example.

Example 3.4. Let

$$A = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{pmatrix}, a = \begin{pmatrix} 0.5714 \\ 0 \\ 0 \end{pmatrix}, b = \begin{pmatrix} -6 \\ -3 \\ 0 \end{pmatrix}, \delta = 1, \beta = 2.2.$$

We have $\lambda_1 = -4$ and $\dim\left(\mathrm{Ker}\,(A - \lambda_{\min}(A) I_n)\right) = 1 \ngeq 2$. Thus, the dimension condition (4) does not hold. Moreover, the new dimension condition of [6] also fails to hold, since

$$\mathrm{rank}([A-\lambda_1 I_n \quad b]) = \mathrm{rank}\begin{pmatrix} 0 & 0 & 0 & -6 \\ 0 & 5 & 0 & -3 \\ 0 & 0 & 1 & 0 \end{pmatrix} = 3 \nleq 2.$$

The global solution of TRS is $\bar{x}^* = [-1, 0, 0]^T$, which is again infeasible for eTRS, and LNGM of TRS is $\bar{x} = [1, 0, 0]^T$, which is feasible for eTRS with the objective value of -2.8572. The optimal objective value of the SDP relaxation (2) is -5.4354 and the optimal objective value of the SOCP/SDP formulation (7) is -3.6121, which is also the optimal objective value of (1). The optimal solution of (7) is

$$X^* = \begin{pmatrix} 0.1842 & -0.0537 & 0 \\ -0.053 & 0.0156 & 0 \\ 0 & 0 & 0.8001 \end{pmatrix}, x^*_{socp/sdp} = \begin{pmatrix} -0.4292 \\ 0.1251 \\ 0 \end{pmatrix},$$

obviously showing $X^* \neq x^*_{socp/sdp}\left(x^*_{socp/sdp}\right)^T$. By the rank-one decomposition procedure discussed in the previous section, one gets the optimal solution $x^* = [-0.4292, 0.1251,\ -0.8945]^T$ for (1).

4. Conclusions

Using a variant of the S-Lemma, we presented the necessary and sufficient optimality conditions for the extended trust region subproblem leading to an SOCP/SDP formulation of it. Our derived formulation turned out to be the dual of the SOCP/SDP formulation given in [2, 11] using a completely different approach. Extending this idea for several linear inequality constraints could be an interesting future research direction.

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