

Some Results on Necessary Conditions for Two Quasidifferentiable Optimization Problems*

Zun-Quan Xia¹
 Chun-Ling Song²
 Li-Wei Zhang³

Two main results of necessary conditions of optimality for two kinds of problems, bilevel optimization and quasidifferentiable MPEC, are presented via Demyanov sum of quasidifferentials. The result that Lagrange multipliers are independent of the choices of quasidifferentials and supergradients is given.

Keywords: Nonsmooth optimization; quasidifferentiable function; bilevel optimization; MPEC; optimality conditions; Demyanov difference (sum).

1 – Introduction

We consider the following two classes of problems,

$$(P_1) \quad \begin{aligned} \min \quad & \theta(x) = f(x, v_1(x), \dots, v_m(x)) \\ \text{s. t.} \quad & v_i(x) = \max\{\varphi_i(x, y_i) \mid G_i(x, y_i) \leq 0, H_i(x, y_i) = 0\}, \quad i = 1, \dots, m, \end{aligned}$$

where $f : R^{n+m} \rightarrow R^1$ is quasidifferentiable, $x \in R^n$, $y_i \in R^{s_i}$, and

$$\begin{aligned} \varphi_i, g_{ij}, h_{ik} & \in C^2, \quad i = 1, \dots, m, j = 1, \dots, p_i, k = p_i + 1, \dots, q_i, \\ G_i(x, y_i) & = (g_{i1}(x, y_i), \dots, g_{ip_i}(x, y_i))^T, \\ H_i(x, y_i) & = (h_{i(p_i+1)}(x, y_i), \dots, h_{iq_i}(x, y_i))^T, \end{aligned}$$

*Invited Paper

¹CORA, Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, China,
 zxiazhh@dlut.edu.cn.

²Department of Mathematics, School of Sciences, Foshang University, Foshang 528313, China, China,
 sohu@163.com.

³CORA, Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, China,
 lwzhang@dlut.edu.cn.

and

$$(P_2) \quad \begin{aligned} & \min f(x, y) \\ & \text{s. t. } z_j(x, y) \leq 0, \quad j = 1, \dots, r, \\ & \quad y = (y_1, \dots, y_m)^T \in S(x) = \prod_{i=1}^m S_i(x), \end{aligned}$$

where $f, z_j : R^n \times R^{s_1} \times \dots \times R^{s_m} \rightarrow R^1$, $j = 1, \dots, r$, are quasidifferentiable, $x \in R^n$, $y_i \in S_i(x) \subseteq R^{s_i}$, and

$$\begin{aligned} & \varphi_i, g_{ij}, h_{ik} \in C^2, \quad i = 1, \dots, m, j = 1, \dots, p_i, k = p_i + 1, \dots, q_i, \\ & G_i(x, y_i) = (g_{i1}(x, y_i), \dots, g_{ip_i}(x, y_i))^T, \\ & H_i(x, y_i) = (h_{i(p_i+1)}(x, y_i), \dots, h_{iq_i}(x, y_i))^T, \\ & S_i(x) = \arg \max \{ \varphi_i(x, y_i) \mid G_i(x, y_i) \leq 0, H_i(x, y_i) = 0 \}, \quad i = 1, \dots, m. \end{aligned}$$

A function $f : R^n \rightarrow R^1$ is said to be quasidifferentiable at x in the sense of Demyanov and Rubinov (1980), if f is directionally differentiable at $x \in R^n$ and there exists a pair of compact convex sets, $\underline{\partial}f(x), \bar{\partial}f(x) \subseteq R^n$, such that

$$f'(x; d) = \max_{v \in \underline{\partial}f(x)} \langle v, d \rangle + \min_{w \in \bar{\partial}f(x)} \langle w, d \rangle, \quad \forall d \in R^n;$$

see [2]. $Df(x) = [\underline{\partial}f(x), \bar{\partial}f(x)]$ is called a quasidifferential of f at x , $\underline{\partial}f(x)$ and $\bar{\partial}f(x)$ are called subdifferential and superdifferential of f at x , respectively. Elements of a subdifferential and a superdifferential are called subgradients and supergradients, respectively.

The following assumptions will be used in this paper for ensuring the quasidifferentiability of $v_i(\cdot)$, $i = 1, \dots, m$, at x^0 and the validity of optimality conditions. Define:

$$Y_i(x) = \arg \max \{ \varphi_i(x, y_i) \mid G_i(x, y_i) \leq 0, H_i(x, y_i) = 0 \}, \quad i = 1, \dots, m.$$

Assumption 1.1. (Uniform Boundedness) $Y_i(x)$ is uniformly bounded in some neighborhood of x^0 ; i.e., there exists a neighborhood N_i of x^0 and a bounded set $T_i \subseteq R^{s_i}$ such that $Y_i(x) \subseteq T_i$, for any $x \in N_i$.

Assumption 1.2. (M-F Constraint Qualification) For every $y_i^0 \in Y_i(x^0)$, the lower problem,

$$\begin{aligned} & \max \varphi_i(x, y_i) \\ & \text{s.t. } G_i(x, y_i) \leq 0, H_i(x, y_i) = 0, \quad i = 1, \dots, m, \end{aligned} \tag{1}$$

satisfies M-F constraint qualification:

1. The vectors $\nabla_y h_{ij}(x^0, y_i^0)$, $j = p_i + 1, \dots, q_i$, are linearly independent.
2. There exists a $w_i \in R^{s_i}$ satisfying,

$$w_i^T \nabla_y g_{ij}(x^0, y_i^0) < 0, \quad \forall j \in \{\nu \mid g_{i\nu}(x^0, y_i^0) = 0, \nu = 1, \dots, p\}, \tag{2}$$

$$w_i^T \nabla_y h_{ij}(x^0, y_i^0) = 0, \quad j = p + 1, \dots, q. \tag{3}$$

Assumption 1.3. (Second Order Sufficient Conditions) For any $\alpha_i \in A_i(x^0, y_i^0)$ and $d \neq 0$ satisfying the following conditions,

$$\nabla_y g_{ij}(x^0, y_i^0)^T d = 0, \quad \forall j \in J(\alpha_i) = \{j \mid \alpha_{ij} > 0, j = 1, \dots, p\}, \quad (4)$$

$$\nabla_y h_{ij}(x^0, y_i^0)^T d = 0, \quad j = p+1, \dots, q, \quad (5)$$

one has,

$$d^T \nabla_{yy}^2 L_i(x^0, y_i^0, \alpha_i) d > 0, \quad (6)$$

where,

$$A_i(x^0, y_i^0) = \left\{ \alpha \in R^q \left| \begin{array}{ll} \nabla_y L_i(x^0, y_i^0, \alpha) = 0 \\ \alpha_j \geq 0, & j = 1, \dots, p_i \\ \alpha_j g_{ij}(x^0, y_i^0) = 0, & j = 1, \dots, p_i \end{array} \right. \right\}, \quad (7)$$

$$L_i(x, y_i, \alpha_i) = \varphi_i(x, y_i) + \sum_{k=1}^{p_i} \alpha_k g_{ik} + \sum_{k=p_i+1}^{q_i} \alpha_k h_{ik}, \quad (8)$$

$$\alpha_i = (\alpha_1, \dots, \alpha_{p_i}, \alpha_{p_i+1}, \dots, \alpha_{q_i}). \quad (9)$$

Here, $\text{co}C$ denotes the convex hull of C . In the next section, necessary conditions for problem (P_1) , i. e., for a class of quasidifferentiable bilevel optimization, are given, and necessary conditions for problem (P_2) , i. e., for a class of quasidifferentiable MPEC problems, are presented in Section 3.

2 – The Case of (P_1)

For every $y_i^0 \in Y_i(x^0)$, the set of Lagrange multiplier vectors of lower level problem is nonempty if and only if M-F constraint qualification holds at y_i^0 ; see [6, 10]. Moreover, the following theorem holds.

Theorem 2.1. [1, 7, 8, 11, 12, 13] Suppose that the M-F constraint qualification and second order sufficient conditions hold for every $y_i^0 \in Y_i(x^0)$. Then, $v_i(\cdot)$ is (local) Lipschitzian, directionally differentiable and

$$v'_i(x^0; d) = \sup_{y_i \in Y_i(x^0)} \inf_{\alpha \in A_i(x^0, y_i)} d^T \nabla_x L_i(x^0, y_i, \alpha). \quad (1)$$

If for every i , $i = 1, \dots, m$, $Y_i(x^0)$ is finite, i. e., $Y_i(x^0) = \{y_i^1, \dots, y_i^{\beta_i}\}$, then (1) can be written as:

$$v'_i(x^0; d) = \max\{\langle d, e \rangle \mid e \in C_i^1\} - \max\{\langle d, e \rangle \mid e \in C_i^2\}, \quad (2)$$

i. e., $v_i(\cdot)$ is quasidifferentiable at x^0 , and $[C_i^1, -C_i^2]$ is a quasidifferential of v_i at x^0 , where,

$$\begin{aligned}
C_i^1 &= \text{co}(\bigcup_{l=1}^{\beta_i} (\sum_{\mu \neq l} B_i^\mu)), & C_i^2 &= \sum_{l=1}^{\beta_i} B_i^l, \\
B_i^l &= -\text{co}(\bigcup_{\alpha \in A(x^0, y_i^l)} \{\nabla_x L_i(x^0, y_i^l, \alpha)\}), & \forall l = 1, \dots, \beta_i, \\
y_i^j &\in R^{S_i}, \quad \forall j = 1, \dots, \beta_i. & \square
\end{aligned}$$

In the remainder of the paper, we assume that for every i , $i = 1, \dots, m$, $Y_i(x^0)$ is finite. Consider problem (P_1) . Since $v(\cdot)$ is Lipschitzian, it is uniformly directionally differentiable. By the quasidifferential calculus of composition functions [3], one has that θ is quasidifferentiable at x^0 , and $[\underline{\partial}\theta(x^0), \bar{\partial}\theta(x^0)]$ is formulated by:

$$\begin{aligned}
\underline{\partial}\theta(x^0) &= \{w \mid w = (\underline{v}^{(1)}, \dots, \underline{v}^{(n)}) + \sum_{i=n+1}^{n+m} [v^{(i)}(\lambda_i + \mu_i) - v'^{(i)}\lambda_i - v''^{(i)}\mu_i], \\
&\quad v = (v^{(1)}, \dots, v^{(n+m)}) \in \underline{\partial}f(y^0), \lambda_i \in C_{i-n}^1, \mu_i \in -C_{i-n}^2\}, \\
\bar{\partial}\theta(x^0) &= \{w \mid w = (\bar{v}^{(1)}, \dots, \bar{v}^{(n)}) + \sum_{i=n+1}^{n+m} [u^{(i)}(\lambda_i + \mu_i) + v'^{(i)}\lambda_i + v''^{(i)}\mu_i], \\
&\quad u = (u^{(1)}, \dots, u^{(n+m)}) \in \bar{\partial}f(y^0), \lambda_i \in C_{i-n}^1, \mu_i \in -C_{i-n}^2\}, \\
v' &\leq v \leq v'', \quad v' \leq 0, \quad v'' \geq 0, \quad y^0 = (x^0, v_1(x^0), \dots, v_m(x^0))^T, \\
\underline{v}^{(i)} &= v^{(i)} - v'^{(i)}, \quad \bar{v}^{(i)} = u^{(i)} + v'^{(i)}, \quad i = 1, \dots, n.
\end{aligned}$$

Let $A, B \subseteq R^n$ be convex compact. The Demyanov difference of A and B , our basic operation here, is defined by:

$$A \dot{-} B = \text{clco}\{\nabla\delta^*(h \mid A) - \nabla\delta^*(h \mid B) \mid h \in T\},$$

where, $T = \{h \in R^n \mid \nabla\delta^*(\cdot \mid A)(h) \text{ and } \nabla\delta^*(\cdot \mid B)(h) \text{ exist}\}$. The form of Demyanov difference, $\underline{\partial}f(x) \dot{-} (-\bar{\partial}f(x))$, will play the main role and is also denoted by $\partial^+f(x)$.

Lemma 2.1. [4] *Let $f : R^n \rightarrow R^1$ be quasidifferentiable. If $x^0 \in \arg \min_{x \in R^n} f(x)$, then $0 \in \partial^+f(x^0)$.* \square

The following theorem can be obtained in terms of Lemma 2.1.

Theorem 2.2. *Suppose Assumptions 1, 2 and 3 hold, and for i , $i = 1, \dots, m$, $Y_i(x^0)$ is finite. If x^0 is a minimizer of (P_1) , then $0 \in \partial^+\theta(x^0)$.* \square

In what follows, assume that θ is a maximal function, i. e.,

$$\begin{aligned}
\min \quad \theta(x) &= \max\{v_1(x), \dots, v_m(x)\} \\
\text{s. t.} \quad v_i(x) &= \max\{\varphi_i(x, y_i) \mid G_i(x, y_i) \leq 0, H_i(x, y_i) = 0\}, \quad i = 1, \dots, m.
\end{aligned} \tag{3}$$

One has from the quasidifferential calculus of maximal functions that the quasidifferential of $\theta(\cdot)$ at x^0 is given by:

$$\begin{aligned}
\underline{\partial}\theta(x^0) &= \text{co} \bigcup_{k \in R(x^0)} (C_k^1 + \sum_{i \in R(x^0) \setminus \{k\}} C_i^2), \\
\bar{\partial}\theta(x^0) &= - \sum_{i \in R(x^0)} C_i^2,
\end{aligned}$$

where, $R(x^0) = \{i \in 1 : m \mid \theta(x^0) = v_i(x^0)\}$.

Lemma 2.2. [3] Let $A, B \subseteq R^n$ be convex compact. Then, $(A + B) \dot{-} B = A$. \square

Lemma 2.3. Let $A_i, B \subseteq R^n, i = 1, \dots, m$, be convex compact. Then, $\text{co}(\bigcup_{i=1}^m A_i) \dot{-} B \subseteq \text{co} \bigcup_{i=1}^m (A_i \dot{-} B)$.

Proof: straightforward. \square

Theorem 2.3. Suppose Assumptions 1, 2 and 3 hold, and for $i, i = 1, \dots, m$, $Y_i(x^0)$ is finite. If x^0 is a minimizer of (3), then there exists a finite number of $\alpha \in A_0(x^0, y_k^l) \subseteq A(x^0, y_k^l), l = 1, \dots, \beta_k, k \in R(x^0)$ and $\lambda(\alpha, l, k) \geq 0$, such that

$$\sum_{\alpha \in A_0(x^0, y_k^l), l=1, \dots, \beta_k, k \in R(x^0)} \lambda(\alpha, l, k) \nabla_x L_k(x^0, y_k^l, \alpha) = 0.$$

Proof: One has from Theorem 2.2 that $0 \in \partial^+ \theta(x^0)$. Compute $\partial^+ \theta(x^0)$,

$$\begin{aligned} \partial^+ \theta(x^0) &= \underline{\partial} \theta(x^0) \dot{-} (-\bar{\partial} \theta(x^0)) \\ &= \text{co}[\bigcup_{k \in R(x^0)} (C_k^1 + \sum_{i \in R(x) \setminus \{k\}} C_i^2)] \dot{-} (\sum_{i \in R(x^0)} C_i^2) \\ &\subseteq \text{co}\{(C_k^1 + \sum_{i \in R(x) \setminus \{k\}} C_i^2) \dot{-} \sum_{i \in R(x^0)} C_i^2 \mid k \in R(x^0)\} \quad (\text{Lemma 2.3}) \\ &= \text{co}\{C_k^1 \dot{-} C_k^2 \mid k \in R(x^0)\} \quad (\text{Lemma 2.2}) \end{aligned}$$

while,

$$\begin{aligned} C_k^1 \dot{-} C_k^2 &= \text{co}(\bigcup_{l=1}^{\beta_k} \sum_{\nu \neq l} B_\nu^l) \dot{-} \sum_{l=1}^{\beta_k} B_k^l \\ &\subseteq \text{co} \bigcup_{l=1}^{\beta_k} (\sum_{\nu \neq l} B_\nu^l \dot{-} \sum_{l=1}^{\beta_k} B_k^l) \quad (\text{Lemma 2.3}) \\ &= \text{co} \bigcup_{l=1}^{\beta_k} (0 \dot{-} B_k^l) \quad (\text{Lemma 2.2}) \\ &= -\text{co} \bigcup_{l=1}^{\beta_k} B_k^l. \end{aligned}$$

As a consequence, one has:

$$\begin{aligned} 0 &\in \text{co}\{-\text{co} \bigcup_{l=1}^{\beta_k} B_k^l \mid k \in R(x^0)\} \\ &= \text{co}\{\bigcup_{l=1}^{\beta_k} (\bigcup_{\alpha \in A(x^0, y_k^l)} \{\nabla_x L_k(x^0, y_k^l, \alpha)\}) \mid k \in R(x^0)\} \\ &= \text{co}\{\nabla_x L_k(x^0, y_k^l, \alpha) \mid \alpha \in A(x^0, y_k^l), l = 1, \dots, \beta_k, k \in R(x^0)\}, \end{aligned}$$

that is, there exists a finite number of $\lambda(\alpha, l, k) \geq 0$, such that

$$\sum_{\alpha \in A_0(x^0, y_k^l), l=1, \dots, \beta_k, k \in R(x^0)} \lambda(\alpha, l, k) \nabla_x L_k(x^0, y_k^l, \alpha) = 0,$$

where, $A_0(x^0, y_k^l)$ is a finite subset of $A(x^0, y_k^l)$. \square

For (P_1) , if $\varphi_i, i = 1, \dots, m$, are strictly concave, $g_{ij}, i = 1, \dots, m, j = 1, \dots, p_i$, are convex, $h_{ik}, i = 1, \dots, m, k = p_i + 1, \dots, q_i$, are affine, and *Slate* constraint qualification holds, then (1) holds and the solution of lower level problem (1) is unique, that is, $Y_i(x^0) = \{y_i^0\}$; see [9]. Therefore, one has:

$$v'_i(x^0; d) = \inf_{\alpha \in A_i(x^0, y_i^0)} d^T \nabla_x L_i(x^0, y_i^0, \alpha). \quad (4)$$

In other words, v_i is superdifferentiable at x^0 , and $\bar{\partial}v_i(x^0) = \text{co}\{\nabla_x L(x^0, y_i^0, \alpha) \mid \alpha \in A_i(x^0, y_i^0)\}$. By the quasidifferential calculus of composition functions, the quasidifferential of $\theta(\cdot)$ at x^0 is formulated as:

$$\begin{aligned} \underline{\partial}\theta(x^0) &= \{w \mid w = (\underline{v}^{(1)}, \dots, \underline{v}^{(n)}) + \sum_{i=n+1}^{n+m} (v^{(i)} - v''^{(i)}) \mu_i, \\ &\quad (v^{(1)}, \dots, v^{(n+m)}) \in \underline{\partial}f(y^0), \mu_i \in B_{i-n}\}, \\ \bar{\partial}\theta(x^0) &= \{w \mid w = (\bar{v}^{(1)}, \dots, \bar{v}^{(n)}) + \sum_{i=n+1}^{n+m} (u^{(i)} + v''^{(i)}) \mu_i, \\ &\quad (u^{(1)}, \dots, u^{(n+m)}) \in \bar{\partial}f(y^0), \mu_i \in B_{i-n}\}, \end{aligned}$$

where,

$$\begin{aligned} v' &\leq v \leq v'', v' \leq 0, v'' \geq 0, y^0 = (x^0, v_1(x^0), \dots, v_m(x^0))^T, \\ \underline{v}^{(i)} &= v^{(i)} - v'^{(i)}, \bar{v}^{(i)} = u^{(i)} + v'^{(i)}, \\ B_{i-n} &= \text{co}\{\nabla_x L_{i-n}(x^0, y_{i-n}^0, \alpha) \mid \alpha \in A_{i-n}(x^0, y_{i-n}^0)\}. \end{aligned}$$

Similarly, if f is a maximal function, then we have the following theorem.

Theorem 2.4. *Assume that $\varphi_i, i = 1, \dots, m$, are strictly concave, $g_{ij}, i = 1, \dots, m, j = 1, \dots, p_i$, are convex, $h_{ik}, i = 1, \dots, m, k = p_i + 1, \dots, q_i$, are affine, and *Slate* constraint qualification holds. If x^0 is a minimizer of (3), then there exists a finite number of $\lambda(\alpha, k) \geq 0$, such that*

$$\sum_{\alpha \in A_0(x^0, y_k^0), k \in R(x^0)} \lambda(\alpha, k) \nabla_x L_k(x^0, y_k^0, \alpha) = 0,$$

where, $A_0(x^0, y_0^l)$ is a finite subset of $A(x^0, y_0^l)$, and $Y_k(x^0) = \{y_k^0\}$.

Proof: By Theorem 2.2 one has that $0 \in \partial^+ \theta(x^0)$. We only need to compute $\partial^+ \theta(x^0)$,

$$\begin{aligned} \partial^+ \theta(x^0) &= \underline{\partial}\theta(x^0) \dot{+} (-\bar{\partial}\theta(x^0)) \\ &= \text{co}[\bigcup_{k \in R(x^0)} (\underline{\partial}v_k(x^0) - \sum_{i \in R(x) \setminus \{k\}} \bar{\partial}v_i(x^0))] \dot{+} (-\sum_{k \in R(x^0)} \bar{\partial}v_k(x^0)) \\ &\subseteq \text{co}\{(\underline{\partial}v_k(x^0) - \sum_{i \in R(x) \setminus \{k\}} \bar{\partial}v_i(x^0)) \dot{+} \\ &\quad (-\sum_{k \in R(x^0)} \bar{\partial}v_k(x^0)) \mid k \in R(x^0)\} \quad (\text{Lemma 2.3}) \\ &= \text{co}\{\underline{\partial}v_k(x^0) \dot{+} (-\bar{\partial}v_k(x^0)) \mid k \in R(x^0)\} \quad (\text{Lemma 2.2}) \\ &= \text{co}\{\nabla_x L(x^0, y_k^0, \alpha) \mid k \in R(x^0), \alpha \in A_k(x^0, y_k^0)\}. \end{aligned}$$

Hence, it follows from the definition of convex hull that there exists a finite number of $\lambda(\alpha, k) \geq 0$, such that

$$\sum_{\alpha \in A_0(x^0, y_k^0), k \in R(x^0)} \lambda(\alpha, k) \nabla_x L_k(x^0, y_k^0, \alpha) = 0,$$

where, $A_0(x^0, y_k^0)$ is a finite subset of $A(x^0, y_k^0)$, and $Y_k(x^0) = \{y_k^0\}$. \square

3 – The Case of (P_2)

Here, we consider problem (P_2) , which is equivalent to the following problem,

$$\begin{aligned} \min \quad & f(x, y) \\ \text{s. t.} \quad & z_j(x, y) \leq 0, \quad j = 1, \dots, r, \\ & v_i(x) \leq \varphi_i(x, y_i), \quad i = 1, \dots, m, \\ & G_i(x, y_i) \leq 0, \quad i = 1, \dots, m, \\ & H_i(x, y_i) = 0, \quad i = 1, \dots, m, \end{aligned} \tag{1}$$

where, $v_i(x) = \max\{\varphi_i(x, y_i) \mid G_i(x, y_i) \leq 0, H_i(x, y_i) = 0\}$.

Lemma 3.1. *Let $A \subseteq R^n, B \subseteq R^m, d = (d_1, d_2) \in R^n \times R^m$. Then, $G_d(A \times B) = G_{d_1}(A) \times G_{d_2}(B)$, where $G_d(A)$ denotes the maximal face of A determined by d .*

Proof: According to the definition of maximal face one has that $(\bar{x}, \bar{y}) \in G_d(A \times B)$ if and only if

$$\langle(\bar{x}, \bar{y}), (d_1, d_2)\rangle = \max_{(x, y) \in A \times B} \langle(x, y), (d_1, d_2)\rangle,$$

that is,

$$\langle\bar{x}, d_1\rangle + \langle\bar{y}, d_2\rangle = \max_{x \in A} \langle x, d_1\rangle + \max_{y \in B} \langle y, d_2\rangle.$$

Therefore,

$$\max_{x \in A} \langle x - \bar{x}, d_1\rangle + \max_{y \in B} \langle y - \bar{y}, d_2\rangle = 0. \tag{2}$$

Since $(\bar{x}, \bar{y}) \in G_d(A \times B)$, we have $(\bar{x}, \bar{y}) \in A \times B$, that is, $\bar{x} \in A$ and $\bar{y} \in B$. Hence,

$$\max_{x \in A} \langle x - \bar{x}, d_1\rangle \geq 0$$

and

$$\max_{y \in B} \langle y - \bar{y}, d_2\rangle \geq 0.$$

Using the last two inequalities in (2), one has:

$$\max_{x \in A} \langle x - \bar{x}, d_1\rangle = 0$$

and

$$\max_{y \in B} \langle y - \bar{y}, d_2 \rangle = 0,$$

that is,

$$\langle \bar{x}, d_1 \rangle = \max_{x \in A} \langle x, d_1 \rangle, \quad \langle \bar{y}, d_2 \rangle = \max_{y \in B} \langle y, d_2 \rangle.$$

This leads to $(\bar{x}, \bar{y}) \in G_{d_1}(A) \times G_{d_2}(B)$. □

Corollary 3.1. *Let $A \subseteq R^n, B \subseteq R^m$ and $d = (d_1, d_2) \in R^n \times R^m$. If $\delta^*(\cdot | A \times B)$ is differentiable at d , then $\delta^*(\cdot | A)$ is differentiable at d_1 and $\delta^*(\cdot | B)$ is differentiable at d_2 . Moreover,*

$$\nabla \delta^*(d | A \times B) = \nabla \delta^*(d_1 | A) \times \nabla \delta^*(d_2 | B).$$

□

Lemma 3.2. *Let $T \subseteq R^n \times R^m$ be a full measure. Then, $P_{R^n}(T)$ and $P_{R^m}(T)$ are full measures with respect to R^n and R^m , respectively, where $P_X(T)$ denotes the projection of T onto X .*

Proof: By contradiction, assume that $P_{R^n}(T)$ is not a full measure subset of R^n . Then, there exists $A \subseteq R^n$, not a zero measure, and $A \subseteq R^n \setminus P_{R^n}(T)$. Therefore, $A \times R^m \subseteq R^n \times R^m$ is not a zero measure, which contradicts the fact that $A \times R^m \subseteq (R^n \times R^m) \setminus T$ and T is a full measure with respect to $R^n \times R^m$. Hence, $P_{R^n}(T)$ is a full measure with respect to R^n . In a similar way, we can prove that $P_{R^m}(T)$ is a full measure with respect to R^m . □

Lemma 3.3. *Let $A, C \subseteq R^n, B, D \subseteq R^m$ be convex compact. Then, $(A \times B) \dot{-} (C \times D) \subseteq (A \dot{-} C) \times (B \dot{-} D)$.*

Proof: Let $T = \{d \in R^n \times R^m \mid \delta^*(\cdot | A \times B) \text{ and } \delta^*(\cdot | C \times D) \text{ are differentiable at } d\}$. One has from the definition of Demyanov difference, Corollary 3.1 and Lemma 3.2,

$$\begin{aligned} & (A \times B) \dot{-} (C \times D) \\ &= \text{clco} \{ \nabla \delta^*(d | A \times B) - \nabla \delta^*(d | C \times D) \mid d \in T \} \\ &= \text{clco} \{ \nabla \delta^*(d_1 | A) \times \nabla \delta^*(d_2 | B) - \nabla \delta^*(d_1 | C) \times \nabla \delta^*(d_2 | D) \mid d \in T \} \\ &\subseteq \text{clco} \left\{ \nabla \delta^*(d_1 | A) \times \nabla \delta^*(d_2 | B) - \nabla \delta^*(d_1 | C) \times \nabla \delta^*(d_2 | D) \mid \begin{array}{l} d_1 \in P_{R^n}(T) \\ d_2 \in P_{R^m}(T) \end{array} \right\} \\ &= \text{clco} \{ \nabla \delta^*(d_1 | A) - \nabla \delta^*(d_1 | C) \mid d_1 \in P_{R^n}(T) \} \times \\ &\quad \text{clco} \{ \nabla \delta^*(d_2 | B) - \nabla \delta^*(d_2 | D) \mid d_2 \in P_{R^m}(T) \} \\ &= (A \dot{-} C) \times (B \dot{-} D). \end{aligned}$$

□

Theorem 3.1. Suppose Assumptions 1, 2 and 3 hold, and for i , $i = 1, \dots, m$, $Y_i(x^0)$ is finite. If x^0 is a minimizer of (P_2) , then there exist $\lambda_j \geq 0$, $j = 0, \dots, r$, $\mu_i \geq 0$, $i = 1, \dots, m$, $\nu_{ij} \geq 0$, $i = 1, \dots, m$, $j = 1, \dots, p_i$ and ω_{ik} , $i = 1, \dots, m$, $k = p_i + 1, \dots, q_i$, not all zero, such that

$$0_n \in \lambda_0 \partial^+ f_x(x^0, y^0) + \sum_{j=1}^r \lambda_j \partial^+ z_{jx}(x^0, y^0) + \sum_{i=1}^m \mu_i (\text{co} \bigcup \{\nabla_x L_i(x^0, y_i^l, \alpha) \mid \alpha \in A(x^0, y_i^l), l = 1, \dots, \beta_i\} - \nabla \varphi_{ix}(x^0, y_i)) + \sum_{i=1}^m \sum_{j=1}^{p_i} \nu_{ij} \nabla_x g_{ij}(x^0, y_i^0) + \sum_{i=1}^m \sum_{k=p_i+1}^{q_i} \omega_{ik} \nabla_x h_{ik}(x^0, y_i^0),$$

$$0_{s_i} \in \lambda_0 \partial^+ f_{y_i}(x^0, y^0) + \sum_{j=1}^r \lambda_j \partial^+ z_{jy_i}(x^0, y^0) + \sum_{i=1}^m \mu_i (-\nabla_{y_i} \varphi_i(x^0, y_i^0)) + \sum_{i=1}^m \sum_{j=1}^{p_i} \nu_{ij} \nabla_{y_i} g_{ij}(x^0, y_i^0) + \sum_{i=1}^m \sum_{k=p_i+1}^{q_i} \omega_{ik} \nabla_{y_i} h_{ik}(x^0, y_i^0),$$

where, $y^0 = (y_1^0, \dots, y_m^0)^T$.

Proof: Problem (1) is equivalent to:

$$\begin{aligned} & \min \quad f(x, y) \\ \text{s. t.} \quad & z_j(x, y) \leq 0, \quad j = 1, \dots, r, \\ & v_i(x) - \varphi_i(x, y_i) \leq 0, \quad i = 1, \dots, m, \\ & G_i(x, y_i) \leq 0, \quad i = 1, \dots, m, \\ & H_i(x, y_i) = 0, \quad i = 1, \dots, m. \end{aligned} \tag{3}$$

If x^0 is a minimizer of (P_2) , then there exists $y^0 \in R^m$ such that (x^0, y^0) is a minimizer of (3). Consider v_i as a function of (x, y) . We have from Theorem 2.1 that

$$\underline{\partial} v_i(x^0, y^0) = C_i^1 \times 0_m,$$

$$\overline{\partial} v_i(x^0, y^0) = -C_i^2 \times 0_m.$$

Similarly, we have that $v_i - \varphi_i$ is quasidifferentiable at (x^0, y^0) and

$$\begin{aligned} \underline{\partial}(v_i - \varphi_i)(x^0, y^0) &= (C_i^1 - \nabla \varphi_{ix}(x^0, y_i^0)) \times \\ & \quad \underbrace{0 \times \dots \times 0 \times (-\nabla \varphi_{iy_i}(x^0, y_i^0)) \times 0 \times \dots \times 0}_m, \\ \overline{\partial}(v_i - \varphi_i)(x^0, y^0) &= -C_i^2 \times 0_m. \end{aligned}$$

By virtue of the necessary conditions of constrained quasidifferentiable optimization due to Gao [5], there exist $\lambda_j \geq 0$, $j = 0, \dots, r$, $\mu_i \geq 0$, $i = 1, \dots, m$, $\nu_{ij} \geq 0$, $i = 1, \dots, m$, $j = 1, \dots, p_i$ and ω_{ik} , $i = 1, \dots, m$, $k = p_i + 1, \dots, q_i$, not all zero, such that

$$\begin{aligned} 0 \in & \lambda_0 \partial^+ f(x^0, y^0) + \sum_{j=1}^r \lambda_j \partial^+ z_j(x^0, y^0) \\ & + \sum_{i=1}^m \mu_i \{[(C_i^1 - \nabla \varphi_{ix}(x^0, y_i^0)) \dot{-} C_i^2] \times 0 \times \dots \times \\ & \quad (-\nabla \varphi_{iy_i}(x^0, y_i^0)) \times \dots \times 0\} + \sum_{i=1}^m \sum_{j=1}^{p_i} \nu_{ij} \nabla_{(x,y)} g_{ij}(x^0, y_i^0) \\ & + \sum_{i=1}^m \sum_{k=p_i+1}^{q_i} \omega_{ik} \nabla_{(x,y)} h_{ik}(x^0, y_i^0). \end{aligned} \tag{4}$$

Computing $(C_i^1 - \nabla \varphi_{ix}(x^0, y_i^0)) \dot{-} C_i^2$, one has:

$$\begin{aligned}
 (C_i^1 - \nabla \varphi_{ix}(x^0, y_i^0)) \dot{-} C_i^2 &= -\nabla \varphi_{ix}(x^0, y_i^0) + (C_i^1 \dot{-} C_i^2) \\
 &= -\nabla \varphi_{ix}(x^0, y_i^0) - \text{co} \bigcup_{l=1}^{\beta_i} B_i^l \\
 &= -\nabla \varphi_{ix}(x^0, y_i^0) - \text{co} \bigcup_{l=1}^{\beta_i} (-\text{co} \bigcup \{\nabla_x L_i(x^0, y_i^l, \alpha) \mid \alpha \in A(x^0, y_i^l)\}) \\
 &= -\nabla \varphi_{ix}(x^0, y_i^0) + \text{co} \bigcup \{\nabla_x L_i(x^0, y_i^l, \alpha) \mid \alpha \in A(x^0, y_i^l), l = 1, \dots, \beta_i\}.
 \end{aligned}$$

Combining the above formula with (4), the conclusion is obtained from Lemma 3.3. \square

If in (P_2) , $f, z_j, j = 1, \dots, r$, are differentiable, then the necessary conditions given in Theorem 3.1 turns to qualities.

Corollary 3.2. Suppose that $f, z_j, j = 1, \dots, r$, are differentiable in (P_2) , Assumptions 1, 2 and 3 hold, and for i , $i = 1, \dots, m$, $Y_i(x^0)$ is finite. If x^0 is a minimizer of (P_2) , then there exist $\lambda_j \geq 0, j = 0, \dots, r$, $\mu_i \geq 0, i = 1, \dots, m$, $\nu_{ij} \geq 0, i = 1, \dots, m, j = 1, \dots, p_i$ and $\omega_{ik}, i = 1, \dots, m, k = p_i + 1, \dots, q_i$, not all zero, such that

$$\begin{aligned}
 0_n &= \lambda_0 \nabla f_x(x^0, y^0) + \sum_{j=1}^r \lambda_j \nabla z_{jx}(x^0, y^0) + \\
 &\quad \sum_{i=1}^m \mu_i (\text{co} \bigcup \{\nabla_x L_i(x^0, y_i^l, \alpha) \mid \alpha \in A(x^0, y_i^l), l = 1, \dots, \beta_i\} - \nabla \varphi_{ix}(x^0, y_i^0)) + \\
 &\quad \sum_{i=1}^m \sum_{j=1}^{p_i} \nu_{ij} \nabla_x g_{ij}(x^0, y_i^0) + \sum_{i=1}^m \sum_{k=p_i+1}^{q_i} \omega_{ik} \nabla_x h_{ik}(x^0, y_i^0),
 \end{aligned}$$

$$\begin{aligned}
 0_{s_i} &= \lambda_0 \nabla f_{y_i}(x^0, y^0) + \sum_{j=1}^r \lambda_j \nabla z_{jy_i}(x^0, y^0) + \sum_{i=1}^m \mu_i (-\nabla_{y_i} \varphi_i(x^0, y_i^0)) + \\
 &\quad \sum_{i=1}^m \sum_{j=1}^{p_i} \nu_{ij} \nabla_{y_i} g_{ij}(x^0, y_i^0) + \sum_{i=1}^m \sum_{k=p_i+1}^{q_i} \omega_{ik} \nabla_{y_i} h_{ik}(x^0, y_i^0),
 \end{aligned}$$

where, $y^0 = (y_1^0, \dots, y_m^0)^T$. \square

Acknowledgements

This work was supported by the Foundation of Ph.D. Units of the Ministry of Education of China (20020141013), National Science Foundations of China (10471015) and (10771026).

The authors are also greatly indebted to the anonymous referee for his valuable comments.

References

[1] Bonnans, J.F., Ioffe, A. D. and Shapiro, A. (1992), Expansion of exact and approximate solutions in nonlinear programming, In: Proceedings of the French-German Conference on Optimization, Pallaschke, D. (Editor), *Lecture Notes in Economics and Mathematical Systems*, Springer, Berlin, 103-117.

- [2] Demyanov, V.F. and Rubinov, A.M. (1980), On quasidifferentiable functions, *Doklady of USSR Acad. of Sci.*, 250, 1, 21-25.
- [3] Demyanov, V.F. and Rubinov, A.M. (1986), *Quasidifferential Calculus*, Optimization Software Inc., Publication Division, New York.
- [4] Diamond, P., Kloeden, P., Rubinov, A. and Vladimirov, A. (1997), Comparative properties of three metrics in the space of compact convex sets, *Set-Valued Analysis*, 5, 267-289.
- [5] Gao, Y. (2000), Demyanov difference of two sets and optimality conditions of Lagrange multiplier type for constrained quasidifferentiable optimization, *Journal of Optimization Theory and Applications*, 104, 377-394.
- [6] Gauvin, J. (1977), A necessary and sufficient regularity condition to have bounded multipliers in nonconvex programming, *Mathematical Programming*, 12, 136-138.
- [7] Gauvin, J. and Dubeau, F. (1982), Differential properties of the marginal function in mathematical programming, *Mathematical Programming Study*, 19, 101-119.
- [8] Gauvin, J. and Tolle, J.W. (1977), Differential stability in nonlinear programming, *SIAM Journal on Control and Optimization*, 15, 294-311.
- [9] Golshtain, E.G. (1972), *Theory of Convex Programming*, Translations of Mathematical Monographs, American Mathematical Society, Providence, Rhode Island, 26.
- [10] Kurcyusz, S. (1976), On the existence and nonexistence of Lagrange multipliers in Banach spaces, *Journal of Optimization Theory and Applications*, 20, 81-110.
- [11] Lempio, F. and Maurer, H. (1980), Differential stability in infinite-dimensional nonlinear programming, *Applied Mathematics and Optimization*, 6, 139-152.
- [12] Levitin, E.S. (1975), On differential properties of the optimal value of parametric problems of mathematical programming, *Doklady Akademii Nauk SSSR*, 224, 1354-1358.
- [13] Ruckmann, J.J. and Shapiro, A. (1999), First-Order optimality conditions in generalized semi-infinite programming, *Journal of Optimization Theory and Applications*, 101(3), 677-691.