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A Novel Hybrid Modified Binary Particle Swarm Optimization Algorithm for the Uncertain p-Median Location Problem

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Here, we investigate the classical p-median location problem on a network in which the vertex weights and the distances between vertices are uncertain. We propose a programming model for the uncertain p-median location problem with tail value at risk objective. Then, we show that it is NP-hard. Therefore, a novel hybrid modified binary particle swarm optimization algorithm is presented to obtain the approximate optimal solution of the proposed model. The algorithm contains the tail value at risk simulation and the expected value simulation. Finally, by computational experiments, the algorithm is illustrated to be efficient.

***Keywords**: Location problem, p-median, Uncertainty theory, Tail value at risk, Uncertain programming, Binary particle swarm optimization.*

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1. Introduction

Location problems are important optimization problems recently been studied by many researchers. A famous location problems is the p-median location problem (pMLP) introduced as follows: Assume that $N = (V, E)$ is an undirected connected network with vertex set V, $|V| = n$, and edge set E. The distance between two points on N is equal to the length of the shortest path connecting the two points. Each vertex is associated with a nonnegative weight the demand of the client at the vertex. In a pMLP, the aim is to find p locations for establishing facilities on edges or vertices of N such that the sum of the weighted distances from the clients to the closest facility becomes minimum. The pMLP model is probably the most widely used model and most extensively researched in location problems [1, 17, 18, 32].

Concerning the median location problems on networks, Hakimi [23, 24] proved that optimal locations of the facilities exist at the vertices of the network. Kariv and Hakimi [26] showed that pMLP on a general network is NP-hard. Also, they designed an $O(p^2n^2)$ time algorithm for the problem on tree networks. Goldman in 1971 presented simple one-pass solution algorithms for 1MLP on a tree network and a network which contained exactly one cycle [22]. The first algorithm is based on a reduction procedure for a useful simplification of problems involving general networks. Goldman showed that the 1-median on a tree is independent of edge lengths and based on this

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condition he presented a linear time algorithm for the problem. For 2MLP on trees, by using the properties relating to the 1-median and the 2-median, Gavish and Sridhar developed an improved algorithm based on edge-deletion method in $O(nlogn)$ time [21]. Tamir [45] improved the time complexity of pMLP on trees to $O(pn^2)$. In 2005, Benkoczi and Bhattacharya [6] designed an $O(n \log^{p+2} n)$ time algorithm to solve pMLP on trees. Drezner [15] proposed optimal solution procedures when p = 2. Also, problems, with $p = 2, 3$, were solved by Schobel and Scholz [42]. For 1MLP on a cactus network, a linear time algorithm was developed in [9].

In the context of the median location problems in the plane, the interested reader is referred to [7, 8, 10, 11, 12, 16, 33, 34, 41, 43].

In real life, we are usually faced with various types of uncertainty. For example, some parameters of the location problems, such as the vertex weights, the travel times between vertices, the establishment costs of facilities and the network modification costs may not be known certainly. The uncertainty theory introduced by Liu [29] is a suitable tool to deal with such parameters. Gao [19] modeled the single facility location problems with uncertain demands. Wen et al. [49] investigated the capacitated facility location-allocation problem with uncertain demands. Nguyen and Chi [37] studied I1-MLP on a tree with uncertain costs and showed that the inverse distribution function of the minimum cost can be obtained at $O(n^2 \log n)$ time. For a survey on uncertain location problems, see e.g., [20, 25, 31, 40, 46, 52].

Note that uncertainty leads to risk. Liu [30] introduced the risk concept in the uncertain environment. Measuring the risk is an important step in the decision-making process. The risk metrics contain techniques and data sets used to calculate the risk value of the problem under investigation. Tail value at risk ($TVaR$) metric [39] is a measure of risk that is widely acceptable among industry segments and market participants. For a survey on risk management in location problems with random and fuzzy variables, the reader is referred to [4, 5, 47, 48, 50].

Here, we concentrate on pMLP with uncertain vertex weights and uncertain distances. We propose a model for the uncertain p-median location problem (UpMLP) with tail value at risk objective. Then, we show that the problem is NP-hard. Considering the uncertain and NP-hardness of UpMLP, evolutionary and meta-heuristic algorithms can be efficiently used for successful generation of approximate optimal solutions. Hence, we present a hybrid modified binary particle swarm optimization (HMBPSO) algorithm to obtain the approximate optimal solution of the proposed model, containing the tail value at risk simulation and the expected value simulation.

To our knowledge, two papers considered the implementation of metaheuristic algorithms for location problems. Alizadeh and Bakhteh [2] studied the general inverse pMLPs on networks and presented a modified firefly algorithm for the problem under investigation. Mirzapolis Adeh et al. [36] investigated the general inverse ordered pMLP on crisp networks and designed a modified particle swarm optimization algorithm for it. There is no work on implementation of hybrid metaheuristic algorithms on inverse pMLPs in uncertain networks. However, many papers can be found in the literature for other classical location problems on uncertain networks; see, e.g., [3, 25, 44, 49, 51].

The rest of our work is organized as follows. In the next section, we first introduce uncertainty theory and the $TVaR$ metric in an uncertain environment. Then, we introduce the uncertain optimization model and present a new model with $TVaR$ objective. In Section 3, we propose a programming model for UpMLP and solve it with $TVaR$ objective. Then, we show that the problem under investigation is NP-hard. In Section 4, we present a novel HMBPSO algorithm to obtain an approximate optimal solution of the proposed model. The algorithm contains the tail value at risk simulation and the expected value simulation. To show the effectiveness of the proposed HMBPSO algorithm, we give a numerical example in Section 5. We conclude in Section 6.

2. Preliminaries

In this section, we first present some definitions and theorems of uncertainty theory and $TVaR$ metric. Then, we introduce an uncertain optimization model and present a new model with $TVaR$ objective [29].

2.1. Uncertainty Theory

Let Γ be a nonempty set and Θ be a σ-algebra over Γ. An uncertain measure is a set function $M:\Theta \to [0,1]$ that satisfies the normality, duality and subadditivity axioms. The triple (Γ, Θ, M) is called an uncertainty space.

Definition 2.1. Let (Γ, Θ, M) be an uncertainty space. A measurable function θ from (Γ, Θ, M) to the set of real numbers is called an uncertain variable.

Definition 2.2. Let θ be an uncertain variable. For any real number x, the function $\Upsilon(x) = M\{\theta \leq x\}$ is called an uncertainty distribution of θ.

Definition 2.3. Let $\theta_i,\ i = 1,\dots,n,$ be the uncertain variables. We call $\theta_i,\ i = 1,\dots,n$, independent if for any Borel sets $B_1, B_2, \dots, B_n$ of real numbers, we have

$$M\left\{\bigcap_{i=1}^{n}\{\theta_i \in B_i\}\right\} = \min_{i=1}^{n} M\{\theta_i \in B_i\}.$$

Theorem 2.4. Let $\theta_i,\ i = 1,\dots,n$, be the independent uncertain variables and $\Upsilon_i^{-1}, i = 1,\dots,n$, be the inverse uncertainty distributions of $\theta_i,\ i = 1,\dots,n$, respectively. Also, let $f(x_1, x_2, \dots, x_n)$ be a strictly increasing function with respect to $x_i,\ i = 1,2,\dots,m$, and a strictly decreasing function with respect to $x_i, i = m+1,\dots,n$. Then, the uncertain variable $\vartheta = f(\theta_1, \theta_2, \dots, \theta_n)$ has the following inverse uncertainty distribution:

$$\Upsilon^{-1}(\alpha) = f\left(\Upsilon_1{}^{-1}(\alpha), \dots, \Upsilon_m{}^{-1}(\alpha), \Upsilon_{m+1}{}^{-1}(1-\alpha), \Upsilon_{m+2}{}^{-1}(1-\alpha), \dots, \Upsilon_n{}^{-1}(1-\alpha)\right),$$

having the following expected value:

$$E(\vartheta) = \int_0^1 f(\Upsilon_1{}^{-1}(\alpha), \dots, \Upsilon_m{}^{-1}(\alpha), \Upsilon_{m+1}{}^{-1}(1-\alpha), \Upsilon_{m+2}{}^{-1}(1-\alpha), \dots, \Upsilon_n{}^{-1}(1-\alpha))\, d\,\Upsilon.$$

2.2. $TVaR$ Metric in an Uncertain Environment

Risk demonstrates a situation, in which there is a chance of loss or danger. The quantification of risk is a key step towards the management and mitigation of risk. In this section, we introduce the definition of the $TVaR$ metric to account for the probability of loss and the severity of loss in an uncertain environment [39].

In order to define the $TVaR$ metric, we first introduce the definition of loss function [30].

Definition 2.5. Consider $\theta_i,\ i = 1,\dots,n$, as the uncertain factors of a system. A function f is said to be a loss function if some specified loss occurs whenever

$$f(\theta_1,\theta_2,\dots,\theta_n) > 0.$$

In the uncertain environment, $TVaR$ of loss function is defined as follows [39].

Definition 2.6. Let $\theta_i,\ i = 1,\dots,n,$ be the uncertain factors and f be the loss function of a system. Then, $TVaR$ of f is defined to be

$$TVaR_\alpha(f) = \frac{1}{\alpha}\int_0^\alpha \sup\{\lambda | M\{f(\theta_1,\theta_2,\dots,\theta_n) \geq \lambda\} \geq \beta\}\, d\beta.$$

for each given risk confidence level $\alpha \in (0,1]$.

Theorem 2.7. Let $\theta_i,\ i = 1,\dots,n,$ be the uncertain factors of a system and $\Upsilon_i^{-1},\ i = 1,\dots,n,$ be the inverse uncertainty distributions of $\theta_i, i = 1,\dots,n$. Also assume that the loss function $f(x_1,x_2,\dots,x_n)$ is a strictly increasing function with respect to $x_i,\ i = 1,2,\dots,n$. Then, for each risk confidence level $\alpha \in (0,1]$, we have

$$TVaR_\alpha(f) = \frac{1}{\alpha}\int_0^\alpha f({\Upsilon_1}^{-1}(1-\beta), {\Upsilon_2}^{-1}(1-\beta),\dots,{\Upsilon_n}^{-1}(1-\beta))d\beta.$$

2.3. Uncertainty Optimization

Let $x = (x_1,x_2,\dots,x_n)$ be a decision vector, and $\theta = (\theta_1,\theta_2,\dots,\theta_n)$ be an uncertain vector. Consider the following optimization model:

$$\begin{aligned} \min\ & f(x,\theta) \\ \text{s.t.}\ & g_j(x,\theta) \leq 0 \quad j = 1,\dots,p, \\ & z_l(x) \leq 0 \quad l = 1,\dots,m, \\ & x \geq 0. \end{aligned} \tag{1}$$

where f and $g_j,\ j = 1,\dots,p$, are uncertain functions and $z_l,\ l = 1,\dots,m$, are crisp functions. Since the objective function of the model (1) involves uncertainty, it cannot be directly optimized. Therefore, by considering $f(x,\theta)$ as a loss function, we minimize its $TVaR$. In addition, we use the expected value of uncertain constraints to get a crisp feasible set. Thus, the model (1) can be reformulated as

$$
\begin{aligned}
&\min\ TVaR_{\alpha}\big(f(x,\theta)\big)\\
&\text{s.t.}\ E\left(g_j(x,\theta)\right) \le 0 \quad j=1,\dots,p,\\
&\qquad\qquad z_l(x) \le 0 \quad l=1,\dots,m,\\
&\qquad\qquad x \ge 0.
\end{aligned} \tag{2}
$$

According to theorems 2.4 and 2.7, we can rewrite problem (2) as follows:

$$
\begin{aligned}
&\min \frac{1}{\alpha}\int_0^{\alpha} f(x, \Upsilon_1{}^{-1}(1-\beta), \Upsilon_2{}^{-1}(1-\beta), \dots, \Upsilon_n{}^{-1}(1-\beta))d\beta\\
&\text{s.t.}\ \int_0^1 g_j\Big(\Upsilon_1{}^{-1}(\gamma), \dots, \Upsilon_m{}^{-1}(\gamma), \Upsilon_{m+1}{}^{-1}(1-\gamma), \Upsilon_{m+2}{}^{-1}(1-\gamma), \dots,\\
&\qquad \Upsilon_n{}^{-1}(1-\gamma)\Big)\, d\Upsilon \le 0 \quad j=1,\dots,p,\\
&\qquad\qquad z_l(x) \le 0 \quad l=1,\dots,m,\\
&\qquad\qquad x \ge 0.
\end{aligned} \tag{3}
$$

where $g_j(x, \theta_1, \theta_2, \dots, \theta_n)$ is strictly increasing with respect to $\theta_1, \theta_2, \dots, \theta_m$ and strictly decreasing with respect to $\theta_{m+1}, \theta_{m+2}, \dots, \theta_n$.

3. Problem Formulation

Let $N=(V,E)$ be an undirected connected network with vertex set V, $|V|=n$, edge set E and a constant $p \le n$. The length of each edge $e \in E$ is positive and each vertex $v_i \in V$ is associated with a nonnegative weight as the demand of the client at the vertex. Let $d(x,y)$ denote the distance between $x, y \in N$ which is equal to the distance of the shortest path connecting these two points. In the classical pMLP, the aim is to locate p pairwise different facilities $m_1, \dots, m_p$ on the network N (i.e., on edges or vertices) such that the sum of weighted distances from each vertex to its closest facility is minimized. In fact, the optimal solution of the following problem is a p-median:

$$
\min_{X_p \subset N, |X_p|=p} \sum_{v_i \in V} w_i d(v_i, x_p), \tag{4}
$$

where

$$
d(v_i, X_p) = \min_{j=1,2,\dots,p} d(v_i, m_j) \qquad\qquad X_p = \{m_1, \dots, m_p\}.
$$

Hakimi [26] proved the existence of an optimal solution among the set of vertices.

Let $d_{ij} = d(v_i, v_j)$ be the distance between vertices $v_i \in V$ and $v_j \in V$. Also, let $w = \{w_i | v_i \in V\}$ and $d = \{d_{ij} | v_i, v_j \in V\}$ denote the set of vertex weights and the set of distances between vertices, respectively. Then, the optimal objective value of pMLP is a function of w and d, which we denoted by $f_p(w,d)$ hereafter.

In the following, we are going to present a 0-1 linear programming formulation of the problem under investigation. Let y_{ij} be a variable that is equal to 1, if the demands of the vertex v_i are served by a facility at the vertex v_j, and 0, otherwise. Also, let the variable x_j be equal to 1, if there is an open facility at the vertex v_j, and 0, otherwise. Then, the 0-1 linear programming formulation of the classical discrete pMLP can be stated as follows:

$$\min \sum_{i=1}^{n}\sum_{j=1}^{n} w_i d_{ij} y_{ij} \tag{5}$$

$$\text{s.t.} \sum_{j=1}^{n} y_{ij} = 1, \quad \forall i = 1, \dots, n, \tag{5.1}$$

$$y_{ij} \le x_j, \quad \forall i, j = 1, \dots, n, \tag{5.2}$$

$$\sum_{j=1}^{n} x_j = p \tag{5.3}$$

$$y_{ij}, x_j \in \{0,1\} \quad \forall i, j = 1, \dots, n. \tag{5.4}$$

This objective is to minimize the total weighted distance between each demand vertex and the nearest facility. Constraints (5.1) require each demand vertex to be assigned to exactly one facility. Constraints (5.2) allow for the demand of vertex v_i to be assigned to a vertex v_j only if there is an open facility in this location. Constraints (5.3) state that exactly p facilities are to be located. Finally, the last constraints are the standard integrality conditions [13,35].

Let x_{ij} be the demand of customer in vertex v_i which is provided by facility in v_j. Then, the model can be rewritten as follows:

$$\begin{aligned} &\min \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij} x_{ij} \\ &\text{s.t.} \sum_{j=1}^{n} x_{ij} = w_i && \forall i = 1, \dots, n, \\ &\qquad x_{ij} \le w_i x_j && \forall i, j = 1, \dots, n, \\ &\qquad \sum_{j=1}^{n} x_j = p \\ &\qquad x_{ij} \ge 0, x_j \in \{0,1\} && \forall i, j = 1, \dots, n. \end{aligned} \tag{6}$$

Now, consider $N = (V, E)$ as an undirected connected network with uncertain vertex weights and uncertain distances. Some assumptions are listed as follows.

1. The weight of each vertex v_i is a positive uncertain variable η_i.
2. The distance between two vertices v_i and v_j is a positive uncertain variable θ_{ij}.
3. All the uncertain variables η_i and θ_{ij} are independent.

4. The uncertain variables η_i and θ_{ij} have regular uncertainty distributions Φ_i and Υ_{ij}, respectively.

Therefore, the classical discrete pMLP with uncertain data can be formulated as follows:

$$\begin{aligned} &\min \sum_{i=1}^{n}\sum_{j=1}^{n}\theta_{ij}x_{ij} \\ &\text{s.t.} \sum_{j=1}^{n} x_{ij} = \eta_i \qquad \forall i = 1,\dots,n, \\ &\qquad x_{ij} \le \eta_i x_j \qquad \forall i,j = 1,\dots,n, \\ &\qquad \sum_{j=1}^{n} x_j = p \\ &\qquad x_{ij} \ge 0, x_j \in \{0,1\} \quad \forall i,j = 1,\dots,n. \end{aligned} \tag{7}$$

Definition 3.1. Let $V_p \subseteq V, |V_p| = p$ and

$$x_j = \begin{cases} 1, & v_j \in V_p \\ 0, & \text{o.w.} \end{cases} \qquad x_{ij} = \begin{cases} E(\eta_i), & \text{if the location of the nearest facility} \\ & \text{to the vertex } v_i \text{ in } V_p \text{ is the vertex } v_j, \\ 0, & \text{o.w.} \end{cases}$$

Using the above notations, $V_p \subseteq V$ is called an expected p-facility location if and only if

$$\begin{cases} \sum_{j=1}^{n} x_{ij} - E(\eta_i) = 0, \ \ \forall i = 1,\dots,n, \\ x_{ij} - E(\eta_i)x_j \le 0, \ \ \forall i,j = 1,\dots,n, \\ \sum_{j=1}^{n} x_j = p, \\ x_{ij} \ge 0, x_j \in \{0,1\}, \ \ \forall i,j = 1,\dots,n. \end{cases}$$

Now, let V_p be an expected p-facility location. Define

$$S(V_p) = \sum_{i=1}^{n}\sum_{j=1}^{n}\theta_{\mathrm{ij}}x_{\mathrm{ij}}.$$

It is clear that $S(V_p)$ is also an uncertain variable.

Definition 3.2. For a risk confidence level $\alpha \in (0,1]$, an expected p-facility location V_p^* is called $TVaR_\alpha$-p-median if

$$TVaR_\alpha(S(V^*{}_p)) \le TVaR_\alpha\left(S(V_p)\right)$$

holds for any expected p-facility location V_p.

Therefore, for a risk confidence level $\alpha \in (0,1]$, the $TVaR_\alpha$-p-median of uncertain network N is the optimal expected p-facility location of the following model:

$$
\begin{aligned}
\min & \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{\alpha}\int_0^{\alpha}{\Upsilon_{ij}}^{-1}(1-\beta)d\beta\right)x_{ij} \\
\text{s.t.} & \sum_{j=1}^{n}x_{ij}=\int_0^1{\Phi_i}^{-1}(1-\gamma)d\gamma \qquad \forall i=1,\dots,n, \\
& x_{ij}\le\left(\int_0^1{\Phi_i}^{-1}(1-\gamma)d\gamma\right)x_j \qquad \forall i,j=1,\dots,n, \\
& \sum_{j=1}^{n}x_j=p \\
& x_{ij}\ge 0, x_j\in\{0,1\} \quad \forall i,j=1,\dots,n.
\end{aligned}
\tag{8}
$$

Thus, we can obtain the p-median with minimal risk measure and expected value of constraints by setting $d_{ij}^{\alpha}=\left(\frac{1}{\alpha}\right)\int_0^{\alpha}{\Upsilon_{ij}}^{-1}(1-\beta)d\beta,\ v_i,v_j\in V$, for each risk confidence level $\alpha \in (0,1]$, and $w_i=\int_0^1{\Phi_i}^{-1}(1-\gamma)d\gamma,\ v_i\in V$. Kariv and Hakimi [26] showed that pMLP on general networks is NP-hard. Therefore, we conclude the following proposition immediately.

Proposition 3.3. UpMLP with $TVaR$ criterion on general networks is NP-hard.

The above proposition implies that it is not possible to present exact polynomial time methods to solve UpMLP on general networks. Therefore, we propose an efficient HMBPSO algorithm for approximating the optimal solution of (8).

4. HMBPSO Algorithm

Kennedy and Eberhart [27] in 1995 developed the particle swarm optimization (PSO) algorithm as a nature-inspired evolutionary computation algorithm. Consider the following model:

$$
\begin{aligned}
\min\ & f(z) \\
\text{s.t.}\ & z\in Z,
\end{aligned}
$$

where Z is the continuous and restricted space. In a PSO algorithm, a potential solution is presented as a particle $z_l \in Z$ and a direction $v_l \in R$ in which the particle will move. A swarm of particle is defined to be a set $\{z_1, z_2, \dots, z_N\}$, in which N is the number of particles. Each particle z_l retains a record of the position of its previous best performance in a vector called $P_{best,l}$. The particle with best performance so far in the population is maintained in a vector G_{best}. An iteration involves evaluating each z_l, and then randomly setting v_l in the direction of particle z_l's best previous position $P_{best,l}$ and the best previous position G_{best} of any particle in the population.

Group particle optimization algorithm is inherently a continuous optimization algorithm. However, many optimization problems have a discrete space. Kennedy and Eberhart in [28] presented the PSO algorithm with discrete binary variables. They used the following transfer function, to calculate the particle position:

$$M(v_{lk}) = \frac{1}{1 + exp(-v_{lk})}, \tag{9}$$

where v_{lk} is the probability of changing the k-th component of z_l towards one. Therefore, after calculating v_l for each particle z_l, the new position of the particle is obtained as follows:

$$z_{lk} = \begin{cases} 1, & \text{rand} \leq M(v_{lk}) \\ 0, & \text{rand} > M(v_{lk}), \end{cases} \tag{10}$$

in which rand is a random value between zero and one.

Now, consider the model (8) of UpMLP. First, for each risk confidence level $\alpha \in (0,1]$, we obtain the shortest distance between vertices by applying Algorithm 1. Then, we consider a particle of UpMLP as $z_l = (x_1, x_2, \ldots, x_n)$ such that $\sum_1^n x_j = p$, $x_j \in \{0,1\}$. Using the distances of vertices obtained by Algorithm 1, we set

$$x_{ij} = \begin{cases} w_i, & \text{if } x_j = 1 \text{ and it is the location of} \\ & \text{the nearest facility to the client } i, \\ 0, & \text{o.w.} \end{cases} \tag{11}$$

Algorithm 1: Find shortest distance between two vertices v_i and v_j for risk confidence level $\alpha \in (0,1]$.

1: Set $d_{ij}^{\alpha} = 0$.
2: **for** $r = 1, \ldots, M$ **do**
3: compute $d_{ij}^{r} = \Upsilon_{ij}^{-1}(1 - \frac{r}{M}\alpha)$,
4: set $d_{ij}^{\alpha} = d_{ij}^{\alpha} + \frac{r}{M}\alpha d_{ij}^{r}$.
5: **end for**
6: Compute $d_{ij}^{\alpha} = \frac{1}{\alpha} d_{ij}^{\alpha}$.
7: Report d_{ij}^{α} as the shortest distance between two vertices v_i and v_j.

Note that we set $x_{ij} = w_i$, $(x_j = 1)$, only for one nearest facility to the client i. We calculate the weight of vertex v_i, i.e., w_i, by using Algorithm 2 [38].

Algorithm 2: Find the weight of vertex v_i.

1: Set $w_i = 0$.
2: **for** $k = 1, \ldots, 99$ **do**
3: compute $w_k = 0.01(\Phi_i^{-1}(1 - 0.0k))$,
4: set $w_i = w_i + w_k$.
5: **end for**
6: Report w_i as the weight of vertex v_i.

Therefore, we first randomly generate the particle z_l of UpMLP. Repeat this process N times and get N initial feasible particles $z_1, z_2, \ldots, z_N$. Then, we assume that the fitness of each z_l is $TVaR$, i.e., $Fit(z_l) = TVaR_\alpha(z_l)$. Thus, the particle with smallest objective value has the best fitness. The fitness of each particle z_l is obtained as follows:

$$TVaR_\alpha(z_l) = \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}^{\alpha} x_{ij}. \tag{12}$$

In the process of updating the $(i+1)$th iteration, we first denote $P_{best,l}(i)$ for each particle z_l and $G_{best}(i)$, and then obtain the new directions of the particles using the following equation:

$$v_l(i+1) = v_l(i) + C_1 r_1 \left[P_{best,l}(i) - z_l(i)\right] + C_2 r_2 [G_{best}(i) - z_l(i)], \tag{13}$$

where,

$$P_{best,l}(i) = \begin{cases} z_l(i), & Fit\big(z_l(i)\big) = Fit\big(z_l(i-1)\big) \\ P_{best,l}(i-1), & \text{o. w.}, \end{cases}$$

and $G_{best}(i) = P_{best,k}(i)$, with $k = \text{argmin}\{P_{best,l}(i) : l = 1, \ldots, N\}$.

In addition, r_1 and r_2 are uniformly distributed random numbers in the interval $[0,1]$ and C_1 and C_2 are learning rates, to well adjust the convergence of the particles. In the basic PSO algorithm, the values of C_1 and C_2 are fixed. But, if these coefficients depend on the repetition of the algorithm, we will get better results. Therefore, here, we propose modifications of C_1 and C_2 of the basic PSO. In other words, we consider

$$C_1 = C_2 = 2 + \left(\frac{t}{t+1}\right),$$

in which $t = 1, 2, \ldots, MaxIt$, and $MaxIt$ indicates the number of generations of the HMBPSO algorithm. It can easily be seen that $C_1, C_2 \in [2,3]$.

After obtaining v_l by (13), we use (10) to get a new particle of the next generation. We obtain a new generation of particles by repeating the above process N times.

Now, based on all the above explanations, we summarize the HMBPSO algorithm for solving the model (8) as follows.

Algorithm 3: Obtain an approximate optimal solution of the model (8)

1: Initialize the feasible particles $z_1, z_2, \ldots, z_N$.
2: Compute x_{ij}, $i, j = 1, \ldots, n$ by (11).
3: Compute the particles fitness by using (12) and evaluate each particle of UpMLP.
4: Update all the particles by using equations (13) and (10).
5: Repeat Steps 3 and 4 for $MaxIt$ times.
6: Return G_{best} as the optimal solution of the model (8), and $TVaR_\alpha(G_{best}) = Fit(G_{best})$ as the corresponding optimal value.

5. Computational Experiments

In this section, we give a numerical example to show the efficiency of the HMBPSO algorithm. The results of the numerical experiments are obtained on a PC with processor Intel(R) Core(TM) i3 CPU 2.27GHZ and 4GB of RAM under Windows 7.

Consider N as a network with 18 vertices and 19 edges (see Figure 1). Let N be a medicines distribution system. In this system, assume that

(1) the vertices indicate urban areas,
(2) at each area, there is a drugstore,
(3) warehouses of the distribution company and drugstores are considered as facilities and clients, respectively,
(4) the weight η_i of the vertex v_i is equal to the average monthly purchase of residents of this area from the drugstore located at vertex v_i,
(5) the vertex weights and the distances between vertices (the travel times between vertices) of N are uncertain variables,
(6) Our aim is to find three vertices on the network N to locate warehouses of the distribution company to minimize the sum of $TVaR$ of weighted distances from each urban area to its closest warehouse.

We applied the HBPSO and HMBPSO algorithms for different instances of UpMLPs and observed that the HMBPSO algorithm in comparison with the HBPSO algorithm can solve these problems more efficiently. In the following, among all the tested instances, we present the instance which the HMBPSO algorithm uses for solving U3MLP with $TVaR$ objective at a risk confidence level of $\alpha = 0.8$ on given network N.

Let the edge lengths $\theta_e, e \in E$, be linear uncertain variables (see Table 1). For $\alpha \in (0,1)$, the inverse uncertainty distribution of a linear uncertain variable $\mathcal{L}(a,b)$ is

$$\phi^{-1}(\alpha) = (1-\alpha)a + \alpha b.$$

Therefore, if $\theta_e = \mathcal{L}(a,b)$ is the uncertain variable of the edge $e \in E$, then

$$d_e^\alpha = \frac{1}{\alpha}\left[\int_0^\alpha (\beta a + (1-\beta)b)d\beta\right] = \frac{\alpha}{2}(a-b) + b.$$

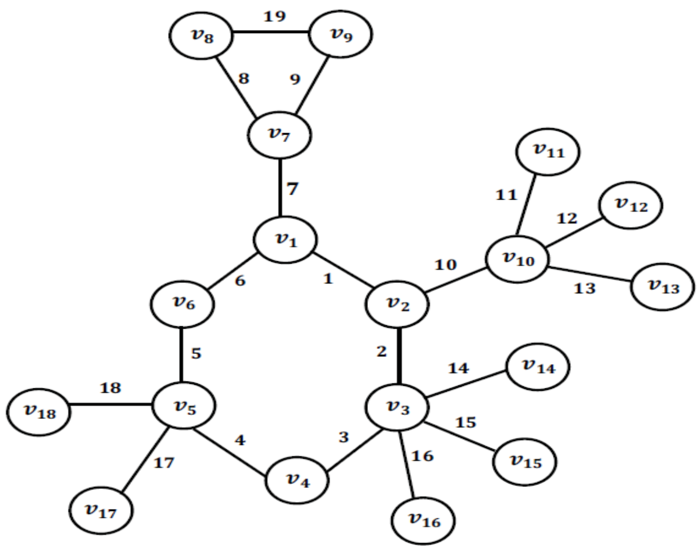

Figure 1. The network *N*

Table 1. The input data of the network *N*

θ_e	$(\mathcal{L}(8,10),\mathcal{L}(18,21),\mathcal{L}(19,21),\mathcal{L}(4,6),\mathcal{L}(3,4),\mathcal{L}(14,16),\mathcal{L}(28,30),\mathcal{L}(10,12),$ $\mathcal{L}(17,18),\mathcal{L}(6,8),\mathcal{L}(22,24),\mathcal{L}(7,9),\mathcal{L}(15,17),\mathcal{L}(18,21),\mathcal{L}(26,28),$ $\mathcal{L}(28,30),\mathcal{L}(16,18),\mathcal{L}(4,6),\mathcal{L}(4,6))$
η_i	$(\mathcal{L}(24,26),\mathcal{L}(27,28),\mathcal{L}(3,5),\mathcal{L}(27,28),\mathcal{L}(19,20),\mathcal{L}(1,4),\mathcal{L}(8,10),\mathcal{L}(16,18),$ $\mathcal{L}(29,30),\mathcal{L}(29,30),\mathcal{L}(4,6),\mathcal{L}(30,31),\mathcal{L}(29,30),\mathcal{L}(14,16),\mathcal{L}(24,26),$ $\mathcal{L}(4,6),\mathcal{L}(12,13),\mathcal{L}(27,28))$

Thus, with the obtained crisp length edges and using the Dijkstra's algorithm, we find the shortest distance between vertices (Dijkstra's algorithm is a successful algorithm for finding shortest path between two vertices [14]). Also, let the vertex weights η_i be linear uncertain variables (see Table 1). Therefore, we find the expected value of the vertex weight η_i as follows:

$$w_i = \int_0^1 (\gamma a + (1-\gamma)b)d\gamma = \frac{(a+b)}{2}.$$

We use the HMBPSO and HBPSO algorithms for approximating the optimal solution and compare the obtained results.

Table 2. Performance results of HBPSO and HMBPSO on U3MLP

Approach	N	Optimal solution	Obj.v.	R.t.
HMBPSO	50	(1, 2, 2, 18, 18, 2, 2, 2, 1, 2, 2, 2, 2, 18, 2, 1, 18, 18)	7341	33.679s
HBPSO	50	(2, 2, 2, 18, 18, 18, 2, 7, 7, 2, 7, 7, 2, 2, 2, 18, 7, 18)	7509	40.95s
HMBPSO	100	(1, 3, 3, 3, 1, 1, 1, 8, 8, 1, 1, 1, 3, 3, 3, 3, 1, 1)	6860.3	78.93s
HBPSO	100	(1, 1, 4, 4, 10, 4, 1, 10, 1, 10, 10, 10, 10, 1, 1, 4, 4, 4)	7409.8	81.3s

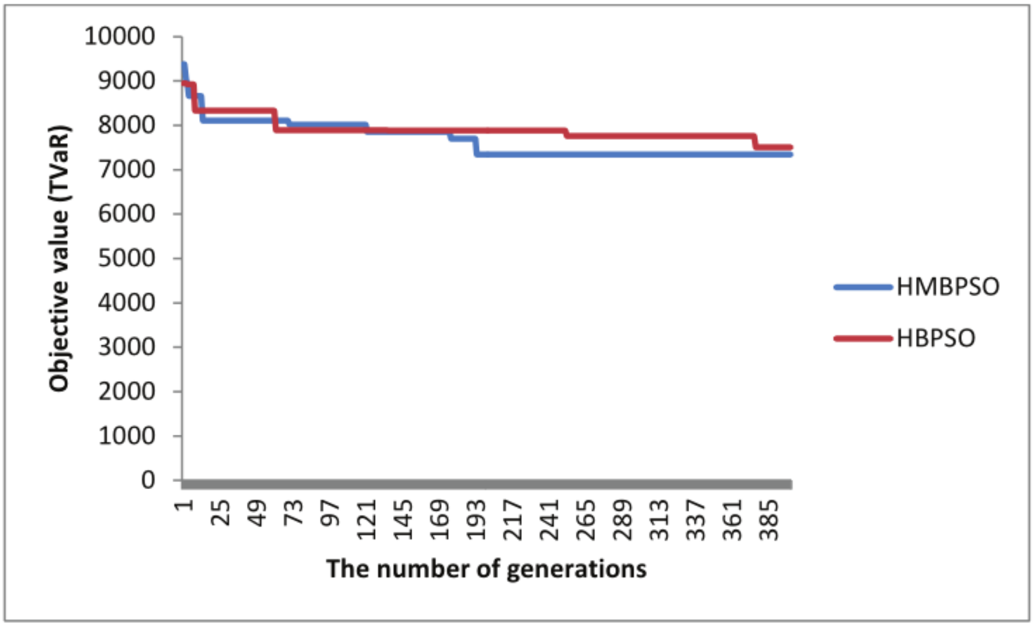

Figure 2. Convergence comparison with population of 50

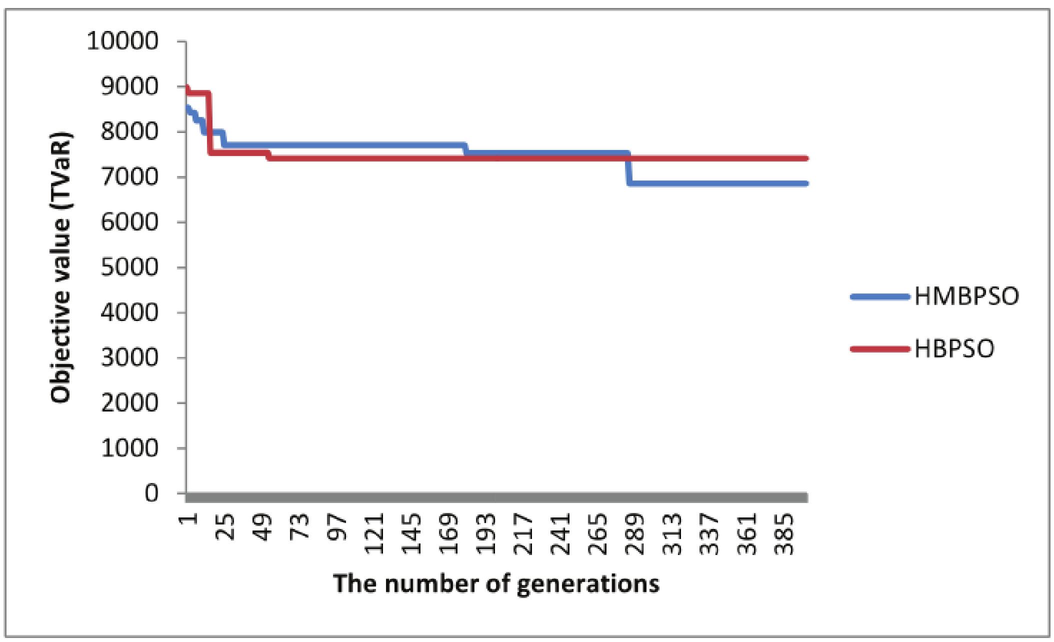

Figure 3. Convergence comparison with population of 100

Table 2 compares the best objective values in each population size calculated by the HBPSO and HMBPSO algorithms with $N = 50, 100$ and $MaxIt = 400$. Note that in Table 2, '$\mathrm{Obj.v.}$' is the objective value of the problem and '$\mathrm{R.t.}$' is the running time of the algorithm. From Table 2, we can conclude that the HMBPSO algorithm finds the solution of the problem under investigation more efficiently. The convergence comparisons are shown in figures 2 and 3, respectively. Based on the obtained results, it is obvious that the HMBPSO algorithm outperforms. Note that, there is no guarantee that the HMBPSO and HBPSO algorithms with the smaller initial population will find better optimal solution.

6. Conclusion

We investigated the pMLP model with uncertain vertex weights and uncertain distances. We showed that UpMLP with *TVaR* objective is NP-hard. Thus, we proposed a novel HMBPSO algorithm for approximating the optimal solution of the problem, containing the tail value at risk simulation and the expected value simulation. Finally, by computational experiments, the efficiency of the algorithm was illustrated.

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