Iranian Journal of Operations Research
Vol. 1, No. 2, 2009, pp.68-84

Fuzzy Primal Simplex Algorithms for Solving Fuzzy Linear Programming Problems

Nezam Mahdavi-Amiri[1]
Seyed Hadi Nasseri[2]
Alahbakhsh Yazdani[3]

Fuzzy set theory has been applied to many fields, such as operations research, control theory, and management sciences. We consider two classes of fuzzy linear programming (FLP) problems: Fuzzy number linear programming and linear programming with trapezoidal fuzzy variables problems. We state our recently established results and develop fuzzy primal simplex algorithms for solving these problems. Finally, we give illustrative examples.

Keywords: *Fuzzy linear programming; Trapezoidal fuzzy number; Ranking function; Fuzzy simplex method.*

1. Introduction

Many application problems, modeled as mathematical programming problems, may be formulated with uncertainty. The concept of fuzzy mathematical programming at general level was first proposed by Tanaka et al. [13] in the framework of the fuzzy decision of Bellman and Zadeh [1]. The first formulation of fuzzy linear programming (FLP) was proposed by Zimmermann [17]. A review of the literature concerning fuzzy mathematical programming as well as comparison of fuzzy numbers can be seen in Klir and Yuan [6] and also Lai and Hwang [7]. Several authors considered various types of the FLP problems and proposed several approaches for solving them [3, 4, 5, 8, 9, 10,

[1] Corresponding author, Department of Mathematical Sciences, Sharif University of Technology, Tehran, Iran, Email: **nezamm@ sharif.edu**
[2] Department of Mathematical Sciences, Mazandaran University, Email: **nasseri@umz.ac.ir**.
[3] Email: Department of Mathematical Sciences, Mazandaran University,**yazdani@umz.ac.ir**

[Downloaded from iors.ir on 2025-12-13]

14]. One convenient method is based on the concept of comparison of fuzzy numbers by use of ranking functions [3, 8, 9, 10, 11, 12].
Here, we first review some necessary concepts of fuzzy set theory in Section 2. In Section 3, we explain the notion of comparison of fuzzy numbers by use of a linear ranking function. After defining fuzzy linear programming problems in Section 4, we describe fuzzy number linear programming in Section 5 and give a corresponding primal simplex algorithm in Section 6. Section 7 discusses linear programming with trapezoidal fuzzy variables and a corresponding primal simplex algorithm is given in Section 8. Section 9 illustrates the working of the two algorithms through two examples. We conclude in Section 10.

2. Definitions and Notations

Here, we give some necessary definitions and results of fuzzy set theory given by Bellman and Zadeh [1] (taken from Bezdek [2] and Lai and Hwang [7]).

A. Fuzzy sets

Definition 2.1. *Fuzzy sets and membership functions*. If X is a collection of objects denoted generically by x , then a **fuzzy set** A in X is defined to be a set of ordered pairs $A = \{(x, \mu_A(x)) \mid x \in X\}$, where $\mu_A(x)$ is called the **membership function** for the fuzzy set. The membership function maps each element of X to a membership value between 0 and 1.

Remark 2.1. We assume that X is the real line $\mathcal{R}$.

Definition 2.2. *Support.* The **support** of a fuzzy set A is the set of points x in X with $\mu_A(x) > 0$.

Definition 2.3. *Core.* The **core** of a fuzzy set is the set of points x in X with $\mu_A(x) = 1$.

Definition 2.4. *Normality.* A fuzzy set A is called **normal** if its core is nonempty. In other words, there is at least one point $x \in X$ with $\mu_A(x) = 1$.

Definition 2.5. $\alpha - cut$ and **strong** $\alpha - cut$ .The $\alpha - cut$ or $\alpha - level$ set of a fuzzy set A is a crisp set defined by $A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}$. The **strong** $\alpha - cut$ is defined to be $\overline{A}_\alpha = \{x \in X \mid \mu_A(x) > \alpha\}$.

Definition 2.6. *Convexity.* A fuzzy set A on X is **convex** if for any $x, y \in X$ and any $\lambda \in [0,1]$, we have,

$$\mu_A(\lambda x + (1-\lambda)y) \geq \min\{\mu_A(x), \mu_A(y)\}.$$

[Downloaded from iors.ir on 2025-12-13]

Remark 2.2. A fuzzy set is convex if and only if all its $\alpha - cuts$ are convex.

Definition 2.7. *Fuzzy number*. A **fuzzy number** A is a fuzzy set on the real line that satisfies the conditions of normality and convexity.
In fact, any fuzzy number is defined by its corresponding membership function. Assume that the membership function of any fuzzy number $\tilde{a}$ is:

$$\mu_{\tilde{a}}(x) = \begin{cases} 1-\frac{a^L - x}{\alpha}, & a^L - \alpha \le x < a^L \\ 1, & a^L \le x \le a^U \\ 1-\frac{x-a^U}{\beta}, & a^U < x \le a^U + \beta \\ 0, & \textit{otherwise.} \end{cases}$$

The fuzzy number with the above membership function is shown in Fig. 1, and is called **trapezoidal fuzzy number**.

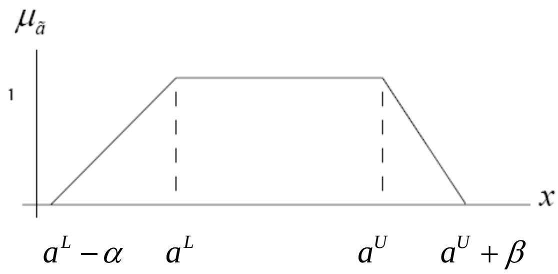


Figure 1. A trapezoidal fuzzy number**.**

A trapezoidal fuzzy number can be shown by $\tilde{a} = (a^L, a^U, \alpha, \beta)$. The support of $\tilde{a}$ is $(a^L - \alpha, a^U + \beta)$, and the core of $\tilde{a}$ is $[a^L, a^U]$. Let $F(\mathcal{R})$ denote the set of all trapezoidal fuzzy numbers.

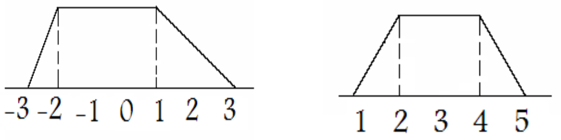
B. Arithmetic on trapezoidal fuzzy numbers

Let $\tilde{a} = (a^L, a^U, \alpha, \beta)$ and $\tilde{b} = (b^L, b^U, \gamma, \theta)$ be two trapezoidal fuzzy numbers and $x \in R$. Define [6]:

$$x \ge 0, x\tilde{a} = (xa^L, xa^U, x\alpha, x\beta),$$
$$x < 0, x\tilde{a} = (xa^U, xa^L, -x\beta, -x\alpha),$$
$$\tilde{a} + \tilde{b} = (a^L + b^L, a^U + b^U, \alpha + \gamma, \beta + \theta).$$

As an illustration of the above arithmetic, consider two trapezoidal fuzzy numbers as given in Fig. 2.

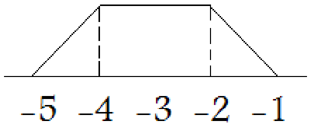
[Downloaded from iors.ir on 2025-12-13]



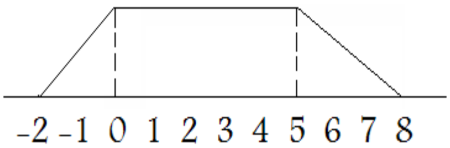
$\tilde{a} = (-2,1,1,2)$ $\tilde{b} = (2,4,1,1)$

Figure 2. Two trapezoidal fuzzy numbers.

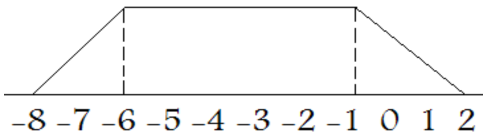
The results of negation, addition and subtraction are shown in Fig. 3.



$-\tilde{b} = (-4,-2,1,1)$



$\tilde{a} + \tilde{b} = (0,5,2,3)$



$\tilde{a} - \tilde{b} = (-6,-1,2,3)$

Figure 3. Results of negation, addition and subtraction of trapezoidal fuzzy numbers.

3. Ranking Functions

Ranking is a viable approach for ordering fuzzy numbers. Various types of ranking functions have been introduced and some have been used for solving linear programming problems with fuzzy parameters [3, 7, 8, 9, 10]. A review of some common methods for ranking fuzzy subsets of the unit interval can be seen in [15]. Here, we deal with ranking the elements of $F(\mathcal{R})$. In fact, an effective approach for ordering the elements of $F(\mathcal{R})$ is to define a ranking function $\mathcal{R}: F(\mathcal{R}) \rightarrow (\mathcal{R})$ mapping trapezoidal fuzzy numbers into $\mathcal{R}$. Consider $\tilde{a}$ and $\tilde{b}$ in $F(\mathcal{R})$. Define order on $F(\mathcal{R})$ as follows [8, 9]:

$$\tilde{a} \underset{\Re}{\geq} \tilde{b} \text{ if } \Re(\tilde{a}) \geq \Re(\tilde{b}), \qquad (1)$$

[Downloaded from iors.ir on 2025-12-13]

$$\tilde{a} \underset{\Re}{>} \tilde{b} \text{ if } \Re(\tilde{a}) > \Re(\tilde{b}), \qquad (2)$$

$$\tilde{a} \underset{\Re}{=} \tilde{b} \text{ if } \Re(\tilde{a}) = \Re(\tilde{b}), \qquad (3)$$

where $\tilde{a}$ and $\tilde{b}$ are in $F(\mathcal{R})$. Also we write $\tilde{a} \underset{\Re}{\leq} \tilde{b}$ if $\tilde{b} \underset{\Re}{\geq} \tilde{a}$. Then, for any linear ranking function $\Re$ we may obtain: $\tilde{a} \underset{\Re}{\geq} \tilde{b}$ if and only if $\tilde{a} - \tilde{b} \underset{\Re}{\geq} \tilde{0}$, or if and only if $-\tilde{b} \underset{\Re}{\geq} -\tilde{a}$. Also, if $\tilde{a} \underset{\Re}{\geq} \tilde{b}$ and $\tilde{c} \underset{\Re}{\geq} \tilde{d}$, then $\tilde{a} + \tilde{c} \underset{\Re}{\geq} \tilde{b} + \tilde{d}$.

Remark 3.1: For a trapezoidal fuzzy number $\tilde{a}$, the relation $\tilde{a} \underset{\Re}{\geq} \tilde{0}$ holds, if there exist $\varepsilon \geq 0$ and $\alpha \geq 0$ such that $\tilde{a} \underset{\Re}{\geq} (-\varepsilon, \varepsilon, \alpha, \alpha)$. We realize that $\Re(-\varepsilon, \varepsilon, \alpha, \alpha) = 0$ (we also consider $\tilde{a} \underset{\Re}{=} \tilde{0}$ if and only if $\Re(\tilde{a}) = 0$). Thus, without loss of generality, throughout the paper we let $\tilde{0} = (0,0,0,0)$ as the zero trapezoidal fuzzy number.

We consider a linear ranking function on $\tilde{a} = (a^L, a^U, \alpha, \beta) \in F(\mathcal{R})$ as:

$$\Re(\tilde{a}) = c_L a^L + c_U a^U + c_\alpha \alpha + c_\beta \beta,$$

where c_L, c_U, c_α and c_β are constants, at least one of which is nonzero. A special version of the above linear ranking function was first proposed by Yager [16]:

$$\Re(\tilde{a}) = \frac{1}{2}(a^L + a^U + \frac{1}{2}(\beta - \alpha)). \qquad (4)$$

Thus, using (4), for trapezoidal fuzzy numbers $\tilde{a} = (a^L, a^U, \alpha, \beta)$ and $\tilde{b} = (b^L, b^U, \gamma, \theta)$, we have:

$$\tilde{a} \underset{\Re}{\geq} \tilde{b} \text{ if and only if } a^L + a^U + \frac{1}{2}(\beta - \alpha) \geq b^L + b^U + \frac{1}{2}(\theta - \gamma).$$

4. Fuzzy Linear Programming

An application of fuzzy set theory to decision making is fuzzy linear programming, first introduced by Zimmermann [17]. Here, we introduce fuzzy linear programming (FLP) problems and divide them into two main subclasses: Fuzzy number linear programming (FNLP) and linear programming with trapezoidal fuzzy variables (FVLP) problems.

A crisp linear programming (LP) problem in a standard form is defined as:

$$\begin{aligned} \text{Max} \quad & z = cx \\ \text{s.t.} \quad & Ax = b \qquad (5) \\ & x \geq 0, \end{aligned}$$

where the parameters $c = (c_1, \ldots, c_n)$, $b = (b_1, \ldots, b_m)^T$, $m \leq n$, and $A = [a_{ij}]_{m \times n}$ are given

[Downloaded from iors.ir on 2025-12-13]

with crisp components (members of $\mathcal{R}$) and $x \in \mathcal{R}^n$ is an unknown vector of variables to be found. If some parameters are considered to be fuzzy numbers, then we obtain a fuzzy linear programming (FLP) problem. We consider the FLP problems in two main classes: (1) Fuzzy number linear programming (FNLP), and (2) linear programming with trapezoidal fuzzy variables (FVLP) problems.

5. Fuzzy Number Linear Programming

A. The definition of FNLP problem

A fuzzy number linear programming (FNLP) problem is defined as:

$$\begin{aligned} \text{Max } & \tilde{z} \underset{\Re}{=} \tilde{c}x \\ \text{s.t. } & Ax = b \qquad (6) \\ & x \geq 0\,, \end{aligned}$$

where $b \in \Re^n, x \in \Re^n, A \in \Re^{m \times n}, \tilde{c}^T \in (F(\Re))^n$, and is a linear ranking function.

Definition 5.1. Any x satisfying the set of constraints (6) of the FNLP problem is called a feasible solution. Let Q_N be the set of all feasible solutions of the FNLP problem. Then, we say that $x_0 \in Q_N$ is an optimal feasible solution for the FNLP problem if $\tilde{c}x_0 \underset{\Re}{\geq} \tilde{c}x$ for all $x \in Q_N$.

One approach for solving these problems is to use the same ranking function for every equality and inequality. For example, we may use the ranking function (4). Applying the ranking function, we obtain a crisp model, equivalent to the FNLP problem, the optimal solution of which is the optimal solution of the FNLP problem.
In fact, one can show that problem (5) and (6) are equivalent (see Mahdavi-Amiri and Nasseri [9]), in the sense that the feasible solution sets of the two problems are the same. Then, the two problems either are infeasible and hence have no solution, or are feasible, and hence have the same optimal solutions, or are unbounded.

B. Basic feasible solution

Consider the system $Ax = b$ and $x \geq 0$, where A is an $m \times n$ matrix and b is an m vector. Suppose that $rank(A,b) = rank(A) = m$. Partition A, after possibly rearranging the columns of A, as $[B \quad N]$ where B, $m \times m$, is nonsingular. It is apparent that $x_B = (x_{B_1}, ..., x_{B_m})^T = B^{-1}b, x_N = 0$ is a solution of $Ax = b$. The point $x = (x_B{}^T, x_N{}^T)^T$ where $x_N = 0$ is called a *basic solution* of the system. If $x_B \geq 0$, then x is called a *basic feasible solution* (BFS) of the system and the corresponding fuzzy objective value is $\tilde{z} \underset{\Re}{=} \tilde{c}_B x_B$, where $c_B = (c_{B_1}, ..., c_{B_m})$. Now, corresponding to every index

[Downloaded from iors.ir on 2025-12-13]

$j,\ \ 1 \le j \le n,$ define: $y_j = B^{-1}a_j$ and $\tilde{z}_j \underset{\Re}{=} \tilde{c}_B y_j$. Observe that for any basic index $j = B_i, 1 \le i \le m$, we have:

$$\tilde{z}_j - \tilde{c}_j \underset{\Re}{=} \tilde{c}_B B^{-1} a_j - \tilde{c}_j \underset{\Re}{=} \tilde{c}_B e_i - \tilde{c}_j \underset{\Re}{=} \tilde{c}_j - \tilde{c}_j \underset{\Re}{=} \tilde{0},$$

where e_i is the *ith* unit vector. Note that B *is* called the *basic matrix* and N is called the *nonbasic matrix*. The components of x_B are called *basic variables*, and the components of x_N are called *nonbasic variables.*

The following theorem concerns the so-called nondegenerate FNLP problems, where every basic variable corresponding to every feasible basis B is positive [9].

Theorem 5.2. Let the FNLP problem be nondegenerate. A basic feasible solution $x_B = B^{-1}b, x_N = 0$ is optimal to (6) if and only if $\tilde{z}_j \underset{\Re}{\ge} \tilde{c}_j$, for all $j,\ 1 \le j \le n$.

Proof: Suppose that $x_* = (x_B{}^T, x_N{}^T)^T$ is a basic feasible solution to (6), where $x_B = B^{-1}b, x_N = 0$. Then, $\tilde{z} \underset{\Re}{=} \tilde{c}_B x_B \underset{\Re}{=} \tilde{c}_B B^{-1} b$. On the other hand, for every feasible solution x, we have $b = Ax = Bx_B + Nx_N$. Hence, we obtain:

$$\tilde{z} \underset{\Re}{=} \tilde{c}x \underset{\Re}{=} \tilde{c}_B x_B + \tilde{c}_N x_N \underset{\Re}{=} \tilde{c}_B B^{-1} b - \sum_{j \ne B_i} (\tilde{c}_B B^{-1} a_j - \tilde{c}_j) x_j .$$

Then,

$$\tilde{z} \underset{\Re}{=} \tilde{z}_* - \sum_{j \ne B_i} (\tilde{z}_j - \tilde{c}_j) x_j . \quad (7)$$

The proof can now be completed using (7) and Theorem 6.2 given in Section 6. □

In the next section, we devise a fuzzy primal simplex algorithm for solving the FNLP problems.

6. Simplex Method for the FNLP Problems

A. FNLP simplex method in tableau format

Consider the FNLP problem as in (6). We rewrite the FNLP problem as:

$$\begin{aligned} \text{Max}\quad & \tilde{z} \underset{\Re}{=} \tilde{c}_B x_B + \tilde{c}_N x_N \\ \text{s.t.}\quad & Bx_B + Nx_N = b \\ & x_B \ge 0,\ x_N \ge 0. \end{aligned}$$

[Downloaded from iors.ir on 2025-12-13]

Hence, we have $x_B + B^{-1}Nx_N = B^{-1}b.$ Therefore, $\tilde{z} + (\tilde{c}_B B^{-1}N - \tilde{c}_N)x_N \underset{\Re}{=} \tilde{c}_B B^{-1}b.$ With $x_N = 0,$ we have $x_B = B^{-1}b = y_0$, and $\tilde{z} \underset{\Re}{=} \tilde{c}_B B^{-1}b$. Thus, we rewrite the above FNLP problem as in Table 1.

Table 1. The FNLP simplex tableau.

Basis	x_B	x_N	R.H.S.
$\tilde{z}$	$\tilde{0}$	$\tilde{z}_N - \tilde{c}_N \underset{\Re}{=} \tilde{c}_B B^{-1}N - \tilde{c}_N$	$\tilde{y}_{00} \underset{\Re}{=} \tilde{c}_B B^{-1}b$
x_B	I	$Y = B^{-1}N$	$y_0 = B^{-1}b$

Remark 6.1: Table 1 gives all the information needed to proceed with the simplex method. The fuzzy cost row in Table 1 is $\tilde{y}_0^T \underset{\Re}{=} \tilde{c}_B B^{-1}A - \tilde{c},$ where $\tilde{y}_{0j} \underset{\Re}{=} \tilde{c}_B B^{-1}a_j - \tilde{c}_j \underset{\Re}{=} \tilde{z}_j - \tilde{c}_j$, $1 \le j \le n$, with $\tilde{y}_{0j} \underset{\Re}{=} \tilde{0}$ for $j = B_i$, $1 \le i \le m$. According to the optimality conditions (Theorem 5.2), we are at the optimal solution if $\tilde{y}_{0j} \underset{\Re}{\ge} 0$ for all $j \ne B_i$, $1 \le i \le m$. On the other hand, if $\tilde{y}_{0k} \underset{\Re}{<} 0$, for some $k \ne B_i$, $1 \le i \le m$, then the problem is either unbounded or an exchange of a basic variable x_{B_r} and the nonbasic variable x_k can be made to increase the rank of the objective value (under nondegeneracy assumption). The following results established in [9] help us to devise the fuzzy primal simplex algorithm.

Theorem 6.1. If in an FNLP simplex tableau, there is a column k (not in basis) so that $\tilde{y}_{0k} \underset{\Re}{=} \tilde{z}_k - \tilde{c}_k \underset{\Re}{<} \tilde{0}$ and $y_{ik} \le 0$, $i = 1,\ldots,m$, then the FNLP problem is unbounded.

Theorem 6.2. If in an FNLP simplex tableau, a nonbasic index k exists such that $\tilde{y}_{0k} \underset{\Re}{=} \tilde{z}_k - \tilde{c}_k \underset{\Re}{<} \tilde{0}$ and there exists a basic index B_i such that $y_{ik} > 0$, then a pivoting row r can be found so that pivoting on y_{rk} yields a feasible tableau with a corresponding nondecreasing (increasing under nondegeneracy assumption) fuzzy objective value.

Remark 6.2 (see [9])**:** If k exists such that $y_{0k} = \Re(\tilde{y}_{0k}) < 0$ and the problem is not unbounded, then r can be chosen so that

$$\frac{y_{r0}}{y_{rk}} = \min_{1 \le i \le m} \left\{ \frac{y_{i0}}{y_{ik}} \middle| y_{ik} > 0 \right\},$$

[Downloaded from iors.ir on 2025-12-13]

in order to replace x_{B_r} in the basis by x_k, resulting in a new basis $\hat{B}=\left(a_{B_1},\ldots,a_{B_{r-1}},a_k,a_{B_{r+1}},\ldots,a_m\right)$. The new basis is primal feasible and the corresponding fuzzy objective value is nondecreasing (increasing under nondegeneracy assumption). It can be shown that the new simplex tableau is obtained by pivoting on y_{rk}, that is, doing Gaussian elimination on the k th column using the pivot row r, with the pivot y_{rk}, to transform the k th column to the unit vector e_r. It is easily seen that the new fuzzy objective value is: $\hat{y}_{00}=y_{00}-y_{0k}\dfrac{y_{r0}}{y_{rk}}\geq y_{00}$, where $y_{0j}=\Re(\tilde{y}_{0j})$, for all j, since $y_{0k}<0$ and $\dfrac{y_{r0}}{y_{rk}}$ (if the problem is nondegenerate, then $\dfrac{y_{r0}}{y_{rk}}>0$ and hence $\hat{y}_{00}>y_{00}$).

We now describe the pivoting strategy.

B. Pivoting and change of basis

If x_k enters the basis and x_{B_r} leaves the basis, then pivoting on y_{rk} in the simplex tableau is carried out, as follows:

1) Divide row r by y_{rk}.

2) For $i=0,1,\ldots,m$ and $i\neq r$, update the ith row by adding to it $-y_{ik}$ times the new r th row.

We now present the simplex algorithm for the FNLP problem.

C. The main steps of FNLP simplex algorithm

Algorithm 1: The fuzzy simplex method for the FNLP problem.

Assumption: A basic feasible solution with basis B and the corresponding simplex tableau is at hand.

1. The basic feasible solution is given by $x_B=y_0$ and $x_N=0$. The fuzzy objective value is: $\tilde{z}\underset{\Re}{=}\tilde{y}_{00}$.
2. Calculate $y_{0j}=\Re(\tilde{z}_j-\tilde{c}_j)$, $j=1,\ldots,n$, $j\neq B_i$, $i=1,\ldots,m$.
   Let $y_{0k}=\min\limits_{j=1,\ldots,n}\{y_{0j}\}$. If $y_{0k}\geq 0$, then stop; the current solution is optimal.
3. If $y_k\leq 0$, then stop; the problem is unbounded. Otherwise, determine an index r corresponding to a variable x_{B_r} leaving the basis as follows:
$$\frac{y_{r0}}{y_{rk}}=\min_{1\leq i\leq m}\{\frac{y_{i0}}{y_{ik}}\mid y_{ik}>0\}.$$

[Downloaded from iors.ir on 2025-12-13]

4. Pivot on y_{rk} and update the simplex tableau. Go to step 2.

7. Linear Programming with Trapezoidal Fuzzy Variables

A. The definition of FVLP problem

A linear programming with trapezoidal fuzzy variables (FVLP) problem is:

$$\begin{aligned} \text{Max} \quad & \tilde{z} \underset{\Re}{=} c\tilde{x} \\ \text{s.t.} \quad & A\tilde{x} \underset{\Re}{=} \tilde{b} \qquad (8) \\ & \tilde{x} \underset{\Re}{\geq} \tilde{0}, \end{aligned}$$

where $\tilde{b} \in (F(\Re))^m$, $\tilde{x} \in (F(\Re))^n$, $A \in \Re^{m\times n}$, and $c^T \in \Re^n$.

Note that an FVLP problem is a linear programming problem in fuzzy environment with the decision making variables being fuzzy numbers.

Definition 7.1. We say that a fuzzy vector $\tilde{x} \in (F(\Re))^n$ is a fuzzy feasible solution for (8) if $\tilde{x}$ satisfies the constraints $A\tilde{x} \underset{\Re}{=} \tilde{b}$ and $\tilde{x} \underset{\Re}{\geq} \tilde{0}$.

Definition 7.2. A fuzzy feasible solution $\tilde{x}_*$ is a fuzzy optimal solution for (8), if for every fuzzy feasible solution $\tilde{x}$ for (8), we have $c\tilde{x}_* \underset{\Re}{\geq} c\tilde{x}$.

B. The fuzzy basic feasible solution

Here, we describe fuzzy basic feasible solution (FBFS) for the FVLP problem (8) as established by Mahdavi-Amiri and Nasseri [8]. Let $A = [a_{ij}]_{m\times n}$.

Assume $rank(A) = m$. Partition A, rearranging columns of A, if needed, as $[B \; N]$, where B, $m\times m$, is a nonsingular matrix. Let y_j be the solution to $By = a_j$. It is apparent that the basic solution,

$$\tilde{x}_B = (\tilde{x}_{B_1}, \ldots, \tilde{x}_{B_m})^T \underset{\Re}{=} B^{-1}\tilde{b} \underset{\Re}{=} \tilde{y}_0, \tilde{x}_N \underset{\Re}{=} \tilde{0}, \qquad (9)$$

is a solution of $A\tilde{x} \underset{\Re}{=} \tilde{b}$. We call $\tilde{x}$, accordingly partitioned as $(\tilde{x}_B{}^T, \tilde{x}_N{}^T)^T$, a fuzzy basic solution corresponding to the basis B. If $\tilde{x}_B \underset{\Re}{\geq} \tilde{0}$, then the fuzzy basic solution is feasible. Now, corresponding to every fuzzy nonbasic variable $\tilde{x}_j, 1 \leq j \leq n, j \neq B_i, i = 1, \ldots, m,$ define:

$$z_j = c_B y_j = c_B B^{-1} a_j. \qquad (10)$$

The following result concerns the nondegenerate problems, where every fuzzy basic variable corresponding to every feasible basis B is positive [8].

[Downloaded from iors.ir on 2025-12-13]

Theorem 7.1. Assume the FVLP problem is nondegenerate. A fuzzy basic feasible solution $\tilde{x}_B \underset{\Re}{=} B^{-1}\tilde{b}, \tilde{x}_N \underset{\Re}{=} \tilde{0}$ is optimal to (8) if and only if $z_j \geq c_j$, for every j, $1 \leq j \leq n$.

Proof: See [8].

Maleki et al. [10] proposed a method for solving FVLP problems by use of an auxiliary problem. They discussed some relations between the FVLP problem and the auxiliary problem and used the results for solving the FVLP problem by an algorithm based on the solution of the auxiliary problem. Recently Mahdavi-Amiri and Nasseri [8] developed the duality results for the FVLP problem. They showed that the auxiliary problem in [10] is indeed the dual of the FVLP problem. Based on the results obtained, they presented a dual simplex algorithm for solving the FVLP problems directly using the primal simplex tableau. Here, we discuss and develop the fuzzy primal simplex algorithm for solving the FVLP problems.

8. Simplex Method for the FVLP Problems

A. FVLP simplex method in tableau format

Consider the FVLP problem (8), rewritten in the following form:

$$\begin{aligned} \text{Max} \quad & \tilde{z} \underset{\Re}{=} c_B \tilde{x}_B + c_N \tilde{x}_N \\ \text{s.t.} \quad & B\tilde{x}_B + N\tilde{x}_N \underset{\Re}{=} \tilde{b} \\ & \tilde{x}_B \underset{\Re}{\geq} \tilde{0} \\ & \tilde{x}_N \underset{\Re}{\geq} \tilde{0}. \end{aligned} \tag{11}$$

We can write $\tilde{x}_B \underset{\Re}{=} B^{-1}\tilde{b} - B^{-1}N\tilde{x}_N$, $\tilde{z} \underset{\Re}{=} c_B(B^{-1}\tilde{b} - B^{-1}N\tilde{x}_N) + c_N\tilde{x}_N$, and hence $\tilde{x}_B + B^{-1}N\tilde{x}_N \underset{\Re}{=} B^{-1}\tilde{b}$, and $\tilde{z} + (c_B B^{-1}N - c_N)\tilde{x}_N \underset{\Re}{=} c_B B^{-1}\tilde{b}.$

Letting $\tilde{x}_N \underset{\Re}{=} \tilde{0}$, we have $\tilde{x}_B \underset{\Re}{=} \tilde{y}_0 \underset{\Re}{=} B^{-1}\tilde{b}$ and $\tilde{z} \underset{\Re}{=} c_B \tilde{y}_0$. Thus, we write the above FVLP problem in the following tableau format (Table 2).

Table 2. The FVLP simplex tableau.

Basis	$\tilde{x}_B$	$\tilde{x}_N$	R.H.S.
$\tilde{z}$	0	$z_N - c_N = c_B B^{-1}N - c_N$	$\tilde{y}_{00} \underset{\Re}{=} c_B B^{-1}\tilde{b}$
$\tilde{x}_B$	I	$Y = B^{-1}N$	$\tilde{y}_0 \underset{\Re}{=} B^{-1}\tilde{b}$

Table 2 gives us all the information needed to proceed with the fuzzy primal simplex method. The cost row in the above tableau is:

[Downloaded from iors.ir on 2025-12-13]

$$y_{0j} = 0,\ j = B_i,\ 1 \le i \le m,$$
$$y_{0j} = c_B y_j - c_j = z_j - c_j,$$
$$1 \le j \le n,\ \ j \ne B_i,\ 1 \le i \le m.$$

According to the optimality conditions for these problems, we are at the optimal solution if $y_{0j} \ge 0$, for all $j \ne B_i$, $1 \le i \le m$. On the other hand, if $y_{0k} < 0$, for some $k \ne B_i$, $1 \le i \le m$, then the problem is either unbounded or an exchange of a basic variable $\tilde{x}_{B_r}$, for some r, and the nonbasic variable $\tilde{x}_k$ can be made to increase the rank of the objective value (under nondegeneracy assumption).

The following theorems taken from [8] state the conditions for unboundedness of the FVLP problem and the conditions permitting the update of the tableau to a new tableau having a nondecreasing (increasing under nondegeneracy assumption) rank of the objective value.

Theorem 8.1. If in an FVLP simplex tableau, there is a column k (not in basis) for which $z_k - c_k < 0$ and $y_{ik} \le 0$, $i = 1, \ldots, m$, then the FVLP problem is unbounded.

Theorem 8.2. If in an FVLP simplex tableau, a nonbasic index k exists such that $z_k - c_k < 0$ and there exists a basic index B_i such that $y_{ik} > 0$, then a pivoting row r can be found so that pivoting on y_{rk} yields a fuzzy feasible tableau with a corresponding nondecreasing (increasing under nondegeneracy assumption) objective value.

B. Pivoting and change of basis

If $\tilde{x}_k$ enters the basis and $\tilde{x}_{B_r}$ leaves the basis, then pivoting on y_{rk} in the simplex tableau is carried out, as follows:

1) Divide row r by y_{rk}.

2) For $i = 0, 1, \ldots, m$ and $i \ne r$, update the ith row by adding to it $-y_{ik}$ times the new rth row.

Note: The pivoting results in the simplex tableau corresponding to the new basis.

C. The main steps of FVLP simplex algorithm

Algorithm 2: The fuzzy simplex method for the FVLP problem

Assumption: A basic feasible solution with basis B and the corresponding simplex tableau is at hand.

[Downloaded from iors.ir on 2025-12-13]

1. The basic feasible solution is given by $\tilde{x}_B \underset{\Re}{=} \tilde{y}_0$ and $\tilde{x}_N \underset{\Re}{=} 0$. The fuzzy objective value is: $\tilde{z} \underset{\Re}{=} \tilde{y}_{00} \underset{\Re}{=} c_B \tilde{y}_0$.
2. Let $y_{0k} = \min_j \{y_{0j}\},\ j = 1,\ldots,n,\ j \neq B_i, i = 1,\ldots,m$. If $y_{0k} \geq 0$, then stop; the current solution is optimal.

3. If $y_k \leq 0$, then stop; the problem is unbounded. Otherwise, determine the index B_r of the variable $\tilde{x}_{B_r}$ leaving the basis as follows:

$$\frac{y_{r0}}{y_{rk}} = \min_{1 \leq i \leq m} \{ \frac{y_{i0}}{y_{ik}} \mid y_{ik} > 0 \},$$

where $y_{i0} = \Re(\tilde{y}_{i0}),\ i = 1,\ldots,m$.
4. Pivot on y_{rk} and update the simplex tableau. Go to Step 2.

9. Numerical Illustrations

For illustrations of the FNLP and FVLP simplex methods, we solve an FNLP problem and an FVLP problem by use of FNLP and FVLP simplex algorithms, respectively.

Example 9.1. Consider the FNLP problem,

$$\begin{aligned} \text{Max} \quad & \tilde{z} \underset{\Re}{=} (5,8,2,5)x_1 + (6,10,2,6)x_2 \\ \text{s.t.} \quad & 2x_1 + 3x_2 \leq 6 \\ & 5x_1 + 4x_2 \leq 10 \\ & x_1, x_2 \geq 0 . \end{aligned}$$

Adding the slack variables, we have the new constraints:

$$\begin{aligned} & 2x_1 + 3x_2 + x_3 = 6 \\ & 5x_1 + 4x_2 + x_4 = 10 \\ & x_1, x_2, x_3, x_4 \geq 0 . \end{aligned}$$

The FNLP simplex tableau corresponding to $B = I,\ B_1 = 3,\ B_2 = 4$ is given in Table 3.

[Downloaded from iors.ir on 2025-12-13]

Table 3. The first simplex tableau of the FNLP problem.

basis	x_1	x_2	x_3	x_4	R.H.S.
$\tilde{z}$	$(-8,-5,5,2)$	$(-10,-6,6,2)$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$
x_3	2	3	1	0	6
x_4	5	4	0	1	10

Since $(\tilde{y}_{01},\tilde{y}_{02}) \underset{\Re}{=} ((-8,-5,5,2),(-10,-6,6,2))$, and $(\Re(\tilde{y}_{01}),\Re(\tilde{y}_{02})) = (-7.25,-9)$, then x_2 enters the basis and the leaving variable is x_3. Pivoting on $y_{32} = 3$ results in the next tableau as in Table 4.

Table 4. The next feasible simplex tableau.

basis	x_1	x_2	x_3	x_4	R.H.S.
$\tilde{z}$	$(-4,\frac{5}{3},\frac{19}{3},6)$	$\tilde{0}$	$(2,\frac{10}{3},\frac{2}{3},2)$	$\tilde{0}$	$(12,20,4,12)$
x_2	$\frac{2}{3}$	1	$\frac{1}{3}$	0	2
x_4	$\frac{7}{3}$	0	$\frac{-4}{3}$	1	2

From $(\tilde{y}_{01},\tilde{y}_{03}) \underset{\Re}{=} ((-4,\frac{5}{3},\frac{19}{3},6),(2,\frac{10}{3},\frac{2}{3},2))$, $(y_{01},y_{03}) = (\Re(\tilde{y}_{01}),\Re(\tilde{y}_{03})) = (-1.25,3)$, it follows that x_1 is an entering and x_4 is a leaving variable. The last tableau is shown in Table 5.

Table 5. The optimal simplex tableau of the FNLP problem.

basis	x_1	x_2	x_3	x_4	$R.H.S.$
$\tilde{z}$	$\tilde{0}$	$\tilde{0}$	$(\frac{-2}{7},\frac{30}{7},\frac{30}{7},\frac{38}{7})$	$(\frac{-5}{7},\frac{12}{7},\frac{18}{7},\frac{19}{7})$	$(\frac{90}{7},\frac{148}{7},\frac{32}{7},\frac{90}{7})$
x_2	0	1	$\frac{5}{7}$	$\frac{-2}{7}$	$\frac{10}{7}$
x_1	1	0	$\frac{-4}{7}$	$\frac{3}{7}$	$\frac{6}{7}$

Note that Table 5 is optimal, because $\tilde{y}_{03}, \tilde{y}_{04} \underset{\Re}{>} \tilde{0}$, as shown below:

$$\tilde{c}_B B^{-1} b \underset{\Re}{=} (\tfrac{90}{7},\tfrac{148}{7},\tfrac{32}{7},\tfrac{90}{7}),$$

$$(\tilde{y}_{03},\tilde{y}_{04}) = \tilde{c}_B B^{-1} N - \tilde{c}_N \underset{\Re}{=} ((\tfrac{-2}{7},\tfrac{30}{7},\tfrac{30}{7},\tfrac{38}{7}),(\tfrac{-5}{7},\tfrac{12}{7},\tfrac{18}{7},\tfrac{19}{7})).$$

[Downloaded from iors.ir on 2025-12-13]

Example 9.2. Consider the FVLP problem,

$$\begin{aligned} &\text{Max } \tilde{z} \underset{\Re}{=} 3\tilde{x}_1 + 4\tilde{x}_2 \\ &\text{s.t. } \ 3\tilde{x}_1 + \tilde{x}_2 \underset{\Re}{\leq} (2,4,1,3) \\ &\qquad 2\tilde{x}_1 + \tilde{x}_2 \underset{\Re}{\leq} (3,\tfrac{9}{2},3,\tfrac{1}{2}) \\ &\qquad \tilde{x}_1, \tilde{x}_2 \underset{\Re}{\geq} \tilde{0}. \end{aligned}$$

Adding the slack variables, we rewrite the constraints of the problem as:

$$\begin{aligned} &3\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 \underset{\Re}{=} (2,4,1,3) \\ &2\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_4 \underset{\Re}{=} (3,\tfrac{9}{2},3,\tfrac{1}{2}) \\ &\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4 \underset{\Re}{\geq} \tilde{0}. \end{aligned}$$

Now, we rewrite the above problem as the FVLP simplex tableau (Table 6).

Table 6. The first simplex tableau of the FVLP problem.

Basis	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	R.H.S.
$\tilde{z}$	-3	-4	0	0	$(0,0,0,0)$
$\tilde{x}_3$	3	1	1	0	$(2,4,1,3)$
$\tilde{x}_4$	2	1	0	1	$(3,\frac{9}{2},3,\frac{1}{2})$

Since $(y_{01}, y_{02}) = (-3,-4)$, then $\tilde{x}_2$ enters the basis and the leaving variable is $\tilde{x}_4$, by the fact that $y_{20} = \min\{\Re(\tilde{y}_{10}) = \Re(\frac{-5}{2},1,\frac{3}{2},6), \Re(\tilde{y}_{20}) = \Re(3,\frac{9}{2},3,\frac{1}{2})\}$. Then, pivoting on $y_{22} = 1$, we obtain the next tableau given as Table 7.

Table 7. The optimal simplex tableau of the FVLP problem.

Basis	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	R.H.S.
$\tilde{z}$	5	0	0	4	$(12,18,12,2)$
$\tilde{x}_3$	1	0	1	-1	$(\frac{-5}{2},1,\frac{3}{2},6)$
$\tilde{x}_2$	2	1	0	1	$(3,\frac{9}{2},3,\frac{1}{2})$

Since $y_{0j} \geq 0$, for all $j \neq B_i$, $1 \leq i \leq 2$, then the basis is optimal and the optimal fuzzy objective value is: $\tilde{z} \underset{\Re}{=} (12,18,12,2)$.

[Downloaded from iors.ir on 2025-12-13]

10. Conclusions

We considered two classes of fuzzy linear programming problems: (1) Fuzzy number linear programming (FNLP), and (2) linear programming with trapezoidal fuzzy variables (FVLP) problems. We made use of trapezoidal fuzzy numbers and a linear ranking function to describe a fuzzy concept of the basic feasible solutions for both problems. We then used the optimality conditions for the FNLP and the FVLP problems and developed fuzzy primal simplex algorithms for solving these problems. Finally, we solved illustrative examples using the proposed simplex algorithms.

Acknowledgements

The first author thanks the Research Council of Sharif University of Technology for its support and the second author thanks the Research Center of Algebraic Hyperstructures and Fuzzy Mathematics for its partial support.

References

[1] Bellman, R.E. and Zadeh, L.A. (1970), Decision making in a fuzzy environment, *Management Science*, 17, 141-164.
[2] Bezdek, J.C. (1993), Fuzzy models – What are they, and why?, *IEEE Transactions on Fuzzy Systems*, 1, 1-9.
[3] Campos, L. and Verdegay, J.L. (1989), Linear programming problems and ranking of fuzzy numbers, *Fuzzy Sets and Systems*, 32, 1-11.
[4] Delgado, M., Verdegay, J.L. and Vila, M.A. (1989), A general model for fuzzy linear programming, *Fuzzy Sets and Systems*, 29, 21-29.
[5] Fang, S.C. and Hu, C.F. (1999), Linear programming with fuzzy coefficients in constraint, *Comp. Math. App*., 37, 63-76.
[6] Klir, G.J. and Yuan, B. (1995), *Fuzzy Sets and Fuzzy Logic: Theory and Applications*, Prentice-Hall, PTR, New Jersey.
[7] Lai, Y.J. and Hwang, C.L. (1992), *Fuzzy Mathematical Programming Methods and Applications*, Springer, *Berlin.*
[8] Mahdavi-Amiri, N. and Nasseri, S.H. (2007), Duality results and a dual simplex method for linear programming problems with trapezoidal fuzzy variables, *Fuzzy Sets and Systems*, 158, 1961-1978.
[9] Mahdavi-Amiri, N. and Nasseri, S.H. (2006), Duality in fuzzy number linear programming by use of a certain linear ranking function, *Applied Mathematics and Computation*, 180, 206-216.
[10] Maleki, H.R., Tata, M. and Mashinchi, M. (2000), Linear programming with fuzzy variables, *Fuzzy Sets and Systems*, 109, 21-33.
[11] Ozelkan, E. and Duckstein, L. (1999), Optimal fuzzy counterparts of scheduling rules, *European Journal of Operational Research*, 113, 593-609.
[12] Tanaka, H., Ichihashi, H. and Asai, K. (1984), A formulation of fuzzy linear programming problem based on comparison of fuzzy numbers, *Control and Cybernetics*, 13, 185-194.

[Downloaded from iors.ir on 2025-12-13]

[13] Tanaka, H., Okuda, T. and Asai, K. (1974), On fuzzy mathematical programming, *The Journal of Cybernetics*, 3(4), 37-46.
[14] Verdegay, J.L. (1984), A dual approach to solve the fuzzy linear programming problem, *Fuzzy Sets and Systems*, 14, 131-141.
[15] Wang, X. and Kerre, E. (2001), Reasonable properties for the ordering of fuzzy quantities (2 parts), *Fuzzy Sets and Systems*, 118, 375-405.
[16] Yager, R.R. (1981), A procedure for ordering fuzzy subsets of the unit interval, *Inform. Science*, 24, 143-161.
[17] Zimmermann, H.J. (1978), Fuzzy programming and linear programming with several objective functions, *Fuzzy Sets and Systems*, 1, 45-55.

[Downloaded from iors.ir on 2025-12-13]