Iranian Journal of Operations Research
Vol. 11, No. 1, 2020, pp. 25-42
DOI: 10.29252/iors.11.1.25

Evaluating Cost Efficiency Using Fuzzy Data Envelopment Analysis method

Jafar Pourmahmoud[1,*], Naser Bafekr Sharak[2]

Cost efficiency models evaluate the ability of decision-making units (DMUs) to produce current outputs at minimal cost. In real applications, the observed values of the input-output data and their corresponding input prices are imprecise and vague. This paper employs a fuzzy data envelopment analysis (Fuzzy DEA) method to study cost efficiency of DMUs. In previous studies on the cost efficiency, no attention has been paid to the issue of ranking problem in fuzzy environment. In addition, adequate accuracy is ignored in regards to appropriate range of fuzzy cost efficiency scores. In this study, the proposed method is applied to assess fuzzy cost efficiency in accordance with the α-level based approach. In this method, data information is considered as triangular fuzzy numbers. The main idea is to convert the fuzzy DEA model into a family of parametric crisp models to estimate the lower and upper bounds of the α-cut of the membership functions of the cost efficiency measures. Moreover, the problem of ranking DMUs is investigated based on the fuzzy cost efficiency, using a new method. Finally, the proposed method is illustrated applying a numerical example, and then comparisons between the proposed method and previous approaches are carried out.

Keywords: *Cost Efficiency; Data Envelopment Analysis (DEA); Fuzzy Sets; The α-level Based Approach.*

Manuscript was received on 11/13/2019, revised on 01/10/2020 and accepted for publication on 02/15/2020.

1. Introduction

Data envelopment analysis (DEA) has been widely used as a non-parametric method to evaluate the performance of organizations. It is a technique to measure the efficiency of a set of decision-making units (DMUs) with multiple inputs and outputs. In practice, when the price of input and output for the units are specified, units could be evaluated based on the costs. Cost is a measurement, in monetary terms, of a number of resources are used for the purpose of production of goods or rendering services. The ability of a DMU to produce the current outputs at minimal cost is evaluated with the cost efficiency. Organizations use cost efficiency to measure the worth, costs, or benefits of a project before they start it. The concept of the cost efficiency was first introduced by Farrell [1] and then developed by Färe et al. [2] applying a linear programming problem. Färe et al. [2] considered the cost efficiency of a DMU the ratio of minimum production cost to actually observed cost. Tone [3] developed the cost efficiency model introduced by Färe et al., and expanded

* Corresponding Author.
1 Department of Applied Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran, Email: pourmahmoud@azaruniv.ac.ir
2 Department of Applied Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran, Email: naser.bafekr@azaruniv.ac.ir

the evaluation space of the production possibility set to the cost space. In fact, tone evaluated DMUs in cost-based production possibility set instead of classic production possibility set.

In many organizations, financial performance of units is evaluated according to the cost efficiency. In some companies, costs are happened for departments of producing, purchasing, and transportation. Moreover, costs are existed for provision of services in quality companies such as hospital, bank, school etc. In these cases, the observed values of the data set are sometimes imprecise. However, conventional methods consider the cost efficiency models for situations where the input–output data and their corresponding input prices are known at each DMU. Fuzzy technique is one of the suitable methods applied when uncertain data exist in DEA. Some researchers have used the fuzzy set theory in a range of the mathematical models literature [4-9].

The applications of fuzzy set theory in DEA for evaluating DMUs are frequently categorized into the tolerance approach, the α-level based approach, the fuzzy ranking approach, and the defuzzification approach [10]. The α-level based approach is one of the most important approaches to address the fuzzy DEA problems. The α-level based approach was introduced by Meada et al. [11], and then developed by some researchers [12-15]. In this approach, the fuzzy model is solved by parametric programming problems based on the α-cuts. Kao and Liu [12] transformed the fuzzy model into a family of parametric crisp models to estimate the lower and upper bounds of the α-cut of the membership functions of the efficiency measures. Saati et al. [13] used a different α-cut approach to evaluate fuzzy efficiency score, based on the transforming the fuzzy DEA model into an interval programming problem. The obtained model is solved as a crisp parametric linear model applying some variable substitutions. Agarwal [16] applied a fuzzy CCR model to measure fuzzy technical efficiency of DMUs, according to the α-level based approach and Zadeh's extension principle. Besides, Agarwal applied the ranking method proposed by Chen and Klein [17] to rank fuzzy efficiency scores for each unit. Wanke et al. [18] proposed new Fuzzy-DEA models to evaluate performance of units, based on the α-level based approach. In fuzzy DEA literature, numerous applications based on the fuzzy method have been also studied, including sustainable supply chain management, healthcare systems, banking system, transportation system, and energy [18-24].

In recent years, some authors have studied cost efficiency models with imprecise data in the fuzzy environment that initially studied by Jahanshahloo et al. [25]. They proposed fuzzy DEA models to evaluate the cost efficiency and they considered input-output data as fuzzy number with precise input prices. Bagherzadeh Valami [26] extended the classic cost efficiency model to a fuzzy DEA model for the case input-output data are precise and input prices are fuzzy numbers. Paryab et al. [27] studied the fuzzy cost efficiency measurement in fuzzy input–output data and fuzzy input prices. They introduced two methods based on the convex DEA and non-convex free disposable hull model using fuzzy variables. Puri and Yadav [28] extended the conventional cost efficiency and revenue efficiency models to fully fuzzy environments where the input–output data with their corresponding prices are not fully recognized. In addition, they considered fuzzy data in triangular fuzzy numbers form and applied component-wise arithmetic operation on fuzzy numbers to explain their method. Moreover, Pourmahmoud and Bafekr Sharak [29] improved the cost efficiency evaluation in fuzzy environment. They suggested a new fuzzy cost-minimizing model considering a Possibility Linear Programming problem to evaluate the cost efficiency.

In this research, a new method is proposed for evaluating the fuzzy cost efficiency of DMUs with fuzzy input and output data. Based on the α-level based approach, a fuzzy DEA model is proposed considering input-output data as triangular fuzzy numbers. In the proposed approach, the fuzzy model is converted to a family of parametric crisp models to calculate the lower and upper bounds of the α-cut of the fuzzy cost efficiency measures. Furthermore, the problem of ranking DMUs in the fuzzy environment is studied based on the cost efficiency. At the end, this method is compared with previous approaches using numerical examples.

The structure of this paper is as follows. Section 2 includes a brief introduction to basic definitions of the fuzzy sets theory. Moreover, an overview of the cost efficiency model is provided in Section 2. Section 3 presents a new proposed fuzzy cost efficiency model. In Section 4, a numerical example illustrating the proposed approach is provided. Finally, the conclusion is given in Section 5.

2. Preliminaries

2.1. Preliminary Definitions

This subsection presents some basic definitions of the fuzzy theory [30-32].

Definition 2.1. The fuzzy number $\tilde{x}$ is called L-R fuzzy number, denoted by $(m,\alpha,\beta)_{LR}$, if there are L and R functions for left and right respectively. The membership function of $\tilde{x}$ is defined as follows:

$$\mu_{\tilde{x}}(z)=\begin{cases} L(\frac{m-z}{\alpha}) & ;z\le m \\ R(\frac{z-m}{\beta}) & ;z\ge m \end{cases}, \tag{1}$$

where scalars α and β are non-negative. Moreover, m is called the mean value of $\tilde{x}$, and α, β are left and right spreads, respectively.

A triangular fuzzy number may be demonstrated as $\tilde{x}=(x^L,x^M,x^U)$, that is defined with following membership function.

$$\mu_{\tilde{x}}(z)=\begin{cases} \dfrac{z-x^L}{x^M-x^L} & ;x^L\le z\le x^M \\ \dfrac{x^U-z}{x^U-x^M} & ;x^M\le z\le x^U \end{cases}, \tag{2}$$

where x^L, x^M and x^U are lower bound, mean value, and upper bound of fuzzy number $\tilde{x}$, respectively.

Definition 2.2. For any scalar $\alpha\in[0,1]$, α-cut of fuzzy number $\tilde{x}$, denoted by $(x)_\alpha$, is defined as follows.

$$(x)_\alpha=[(x)_\alpha^L,(x)_\alpha^U]=\{z\in X;\mu_{\tilde{x}}(z)\ge\alpha\}, \tag{3}$$

where $(x)_\alpha^L$ and $(x)_\alpha^U$ are the lower bound and upper bound of the α-cut of the fuzzy number $\tilde{x}$ respectively.

Thus, if $\tilde{x}$ be a triangular fuzzy number as form $\tilde{x}=(x^L,x^M,x^U)$, then α-cut of fuzzy number $\tilde{x}$ is defined as follows:

$$(x)_\alpha=[(x)^L_\alpha,(x)^U_\alpha]=[\alpha x^M+(1-\alpha)x^L,\ \alpha x^M+(1-\alpha)x^U] \tag{4}$$

2.2. Cost Efficiency Model

Assume a set of n observations on the DMUs are supposed to be evaluated, $DMU_j;(j=1,2,...,n)$ produce s outputs and consumes m inputs. Let $x^j=(x_{1j},x_{2j},...,x_{mj})$ be the input vector of DMU_j, and $y^j=(y_{1j},y_{2j},...,y_{sj})$ be its observed output vector. Furthermore, $w^j=(w_{1j},w_{2j},...,w_{mj})$ be a vector of input prices, paid by this DMU. Besides, in this paper, index j ; $j=1,2,...,\mathrm{n}$, index i ; $i=1,2,...,\mathrm{m}$, and index r ; $r=1,2,...,s$ are used for DMUs, inputs, and outputs, respectively.

Tone [3] introduced cost efficiency model in cost-based production possibility set for the DMU_k (the unit under consideration) as follows:

$$\begin{aligned}
CE_k=\min\quad & \frac{1}{C_k}\sum_{i=1}^{m}\bar{x}_i\\
s.t.\quad & \sum_{j=1}^{n}\lambda_j\bar{x}_{ij}\le\bar{x}_i && ;\forall i\\
& \sum_{j=1}^{n}\lambda_j y_{rj}\ge y_{rk} && ;\forall r\\
& \lambda_j\ge 0 && ;\forall j\\
& \bar{x}_i\ge 0 && ;\forall i,
\end{aligned} \tag{5}$$

where $\bar{x}_{ij}=w_{ij}\,x_{ij}$ is ith input cost of DMU_j. Furthermore, $C_k=\sum_{i=1}^{m}\bar{x}_{ik}$ is the actually observed cost of DMU_k, and CE_k is defined cost efficiency.

Moreover, the dual problem (multiplier form) of cost efficiency model (5) is obtained as follows:

$$\begin{aligned}
CE_k=\max\quad & \sum_{r=1}^{s}u_r y_{rk}\\
s.t.\quad & \sum_{r=1}^{s}u_r y_{rj}-\sum_{i=1}^{m}v_i\bar{x}_{ij}\le 0 && ;\forall j\\
& v_i\le\frac{1}{C_k} && ;\forall i\\
& u_r\ge 0 && ;\forall r\\
& v_i\ge 0 && ;\forall i,
\end{aligned} \tag{6}$$

where u_r and v_i are the relative weights of the ith input and rth output respectively.

3. Fuzzy Cost Efficiency Method

In this section, a new method is applied for evaluating the cost efficiency where input–output data and prices are triangular fuzzy numbers.

Now, suppose that the performance of a set of n units is evaluated, and each unit produces s fuzzy outputs from m fuzzy inputs. Let $\tilde{x}^j$ be the fuzzy input vector of DMU_j, and $\tilde{y}^j$ be its the observed fuzzy output vector. Furthermore, let $\tilde{w}^j$ be the vector of fuzzy input prices paid by DMU_j.

In addition, the used fuzzy data is considered in triangular fuzzy numbers form. Let $\tilde{x}_{ij} = (\mathrm{x}_{ij}^L, \mathrm{x}_{ij}^M, \mathrm{x}_{ij}^U)$ and $\tilde{y}_{rj} = (y_{rj}^L, y_{rj}^M, y_{rj}^U)$ be the ith fuzzy input and the rth fuzzy output of DMU_j. Furthermore, assume that $\tilde{w}_{ij} = (w_{ij}^L, w_{ij}^M, w_{ij}^U)$ be the vector of fuzzy ith input prices paid by DMU_j. Then, the ith fuzzy input cost of DMU_j, denoted by $\tilde{\bar{x}}_{ij} = (\bar{x}_{ij}^L, \bar{x}_{ij}^M, \bar{x}_{ij}^U)$, is defined as follows.

$$\tilde{\bar{x}}_{ij} = (\bar{x}_{ij}^L, \bar{x}_{ij}^M, \bar{x}_{ij}^U) = (x_{ij}^L\ w_{ij}^L, x_{ij}^M\ w_{ij}^M, x_{ij}^U\ w_{ij}^U) \tag{7}$$

The fuzzy model (8) is applied to obtain the fuzzy cost efficiency of the unit under consideration DMU_k that is considered as follows:

$$\begin{aligned} C\tilde{E}_k = \max \quad & \sum_{r=1}^{s} u_r \tilde{y}_{rk} \\ s.t. \quad & \sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{i=1}^{m} v_i \tilde{\bar{x}}_{ij} \le 0 \quad ;\forall j \\ & v_i \le \tilde{q}_k \quad ;\forall i \\ & u_r \ge 0 \quad ;\forall r \\ & v_i \ge 0 \quad ;\forall i, \end{aligned} \tag{8}$$

where notation '~' shows that, the data are fuzzy numbers and $\tilde{q}_k$ is associated with fuzzy value of $\frac{1}{C_k}$. Moreover, in the fuzzy model (8) value $\tilde{\bar{x}}_{ij}$ is the ith input fuzzy cost of DMU_j. In model (8), the fuzzy cost efficiency is measured by the optimal objective function value.

Model (8) is a fuzzy DEA model and several methods have been introduced to solve this model. It can be solved by using the following four approaches, proposed in the fuzzy DEA literature [10]: the fuzzy ranking approach, the possibility approach, the tolerance approach, and the α-level based approach. Fuzzy DEA models are transferred to crisp models and some important information is neglected applying the fuzzy ranking approach, the possibility approach, and the tolerance approach. In this study, the α-level based approach is used to consider all the fuzzy information in the performance evaluation. In addition, Kao and Liu [12] used the α-level based approach and Zadeh's extension principle [30] to extend the classic DEA model to a fuzzy situation for evaluating the efficiency of DMUs, with the given fuzzy input and output data. They converted fuzzy programming problems into a family of conventional parametric crisp problems by applying the α-level sets.

In this study, the presented method by Kao and Liu [12] is used to solve the proposed fuzzy cost efficiency model with regards to the concept of α-cut of fuzzy numbers. Besides, information data is assumed in triangular fuzzy numbers form and the fuzzy cost efficiency model (8) is transformed to a pair of two-level mathematical models. The lower bound and upper bound of the α-cut of the fuzzy cost efficiency $C\tilde{E}_k$, denoted by $(CE_k)^L_\alpha$ and $(CE_k)^U_\alpha$ respectively, can be obtained with following models:

$$(CE_k)^L_\alpha = \min_{\substack{(y_{rj})^L_\alpha \le y_{rj} \le (y_{rj})^U_\alpha \\ (\bar{x}_{ij})^L_\alpha \le \bar{x}_{ij} \le (\bar{x}_{ij})^U_\alpha \\ (q_r)^L_\alpha \le q_k \le (q_k)^U_\alpha \\ \forall i,r,j}} \left\{ \begin{array}{lll} CE_k = \max & \sum_{r=1}^{s} u_r y_{rk} & \\ \quad s.t. & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i \bar{x}_{ij} \le 0 & ;\forall j \\ & v_i \le q_k & ;\forall i \\ & u_r \ge 0 & ;\forall r \\ & v_i \ge 0 & ;\forall i, \end{array} \right. \tag{9}$$

$$(CE_k)^U_\alpha = \max_{\substack{(y_{rj})^L_\alpha \le y_{rj} \le (y_{rj})^U_\alpha \\ (\bar{x}_{ij})^L_\alpha \le \bar{x}_{ij} \le (\bar{x}_{ij})^U_\alpha \\ (q_r)^L_\alpha \le q_k \le (q_k)^U_\alpha \\ \forall i,r,j}} \left\{ \begin{array}{lll} CE_k = \max & \sum_{r=1}^{s} u_r y_{rk} & \\ \quad s.t. & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i \bar{x}_{ij} \le 0 & ;\forall j \\ & v_i \le q_k & ;\forall i \\ & u_r \ge 0 & ;\forall r \\ & v_i \ge 0 & ;\forall i, \end{array} \right. \tag{10}$$

Note that, models (9), (10) are two-level linear programming models which these models are difficult to solve and cannot be easily solved. To solve models (9), (10), these models are transformed into one-level models and can be obtained by substitution values of the lower bound and upper bound of the α-cut of the fuzzy data.

First, for the two-level model (9), the minimum value of cost efficiency CE_k is derived when following cases is happened:

I. The output vector of DMU_k and the inputs of other DMUs are set to their lower bounds of the α-cut of the fuzzy data.
II. The inputs of DMU_k and the outputs of other DMUs are set to their upper bounds of the α-cut of the fuzzy data.
III. The value q_k is set to its lower bound of the α-cut of the fuzzy value $\tilde{q}_k$.

Moreover, for the two-level model (10), the maximum value of cost efficiency CE_k is derived when following cases is happened:

I. The output vector of DMU_k and the inputs of other DMUs are set to their upper bounds of the α-cut of the fuzzy data.
II. The inputs of DMU_k and the outputs of other DMUs are set to their lower bounds of the α-cut of the fuzzy data.
III. The value q_k is set to its upper bound of the α-cut of the fuzzy value $\tilde{q}_k$.

Therefore, by replacing the new substitution values, models (9), (10) can be written as follows:

$$\begin{aligned}
(CE_k)_\alpha^L = \max \quad & \sum_{r=1}^{s} u_r (y_{rk})_\alpha^L \\
s.t. \quad & \sum_{r=1}^{s} u_r (y_{rk})_\alpha^L - \sum_{i=1}^{m} v_i (\overline{x}_{ik})_\alpha^U \le 0 \\
& \sum_{r=1}^{s} u_r (y_{rj})_\alpha^U - \sum_{i=1}^{m} v_i (\overline{x}_{ij})_\alpha^L \le 0 \qquad ;\forall j,\ j \ne k \\
& v_i \le (\mathrm{q}_k)_\alpha^L \qquad ;\forall i \\
& u_r \ge 0 \qquad ;\forall r \\
& v_i \ge 0 \qquad ;\forall i,
\end{aligned} \tag{11}$$

$$\begin{aligned}
(CE_k)_\alpha^U = \max \quad & \sum_{r=1}^{s} u_r (y_{rk})_\alpha^U \\
s.t. \quad & \sum_{r=1}^{s} u_r (y_{rk})_\alpha^U - \sum_{i=1}^{m} v_i (\overline{x}_{ik})_\alpha^L \le 0 \\
& \sum_{r=1}^{s} u_r (y_{rj})_\alpha^L - \sum_{i=1}^{m} v_i (\overline{x}_{ij})_\alpha^U \le 0 \qquad ;\forall j,\ j \ne k \\
& v_i \le (\mathrm{q}_k)_\alpha^U \qquad ;\forall i \\
& u_r \ge 0 \qquad ;\forall r \\
& v_i \ge 0 \qquad ;\forall i
\end{aligned} \tag{12}$$

In the proposed method, the lower bound and upper bound of the α-cut of the fuzzy cost efficiency is obtained by models (9) and (10), which are two-level models and these models are difficult to solve. Then, models (11) and (12) are introduced to obtain the corresponding solutions of models (9) and (10) respectively. Therefore, it is necessary that the solutions of models (11) and (12) correspond to the solutions of models (9) and (10) respectively. The following theorem shows the equivalence of these solutions.

Theorem 3.1. The models (9) and (10) are equivalent with the models (11) and (12) respectively.

Proof. First, equivalence of the model (10) with the model (12) is explained. To do this, it is proved that for each feasible solution to model (10) there is a corresponding feasible solution for model (12) and vice versa. Suppose $u_r^*;\forall r$, $v_i^*;\forall i$, $\overline{x}_{ij}^*;\forall i,j$ $y_{rj}^*;\forall r,j,$ and q_k^* are the feasible solution to model (10) in evaluating DMU_k. Then, we have following relations based on the constraints to model (10).

$$\begin{aligned}
&\sum_{r=1}^{s} u_r^* y_{rj}^* - \sum_{i=1}^{m} v_i^* \overline{x}_{ij}^* \le 0 \quad ;\forall j\\
&v_i^* \le q_k^*, u_r^* \ge 0, v_i^* \ge 0\\
&(y_{rj})_\alpha^L \le y_{rj}^* \le (y_{rj})_\alpha^U\\
&(\overline{x}_{ij})_\alpha^L \le \overline{x}_{ij}^* \le (\overline{x}_{ij})_\alpha^U\\
&(q_r)_\alpha^L \le q_k^* \le (q_k)_\alpha^U
\end{aligned} \tag{13}$$

The first constraint of relation (13) is written for j = k as follows:

$$\sum_{r=1}^{s} u_r^* y_{rk}^* - \sum_{i=1}^{m} v_i^* \overline{x}_{ik}^* \le 0$$

$$\Rightarrow \frac{\sum_{r=1}^{s} u_r^* y_{rk}^*}{\sum_{i=1}^{m} v_i^* \overline{x}_{ik}^*} \le 1$$

This fraction holds for all values of inputs and outputs, including the case that the output value will be set at the upper bound and the input value at the lower bound, so it can be written:

$$\frac{\sum_{r=1}^{s} u_r^* (y_{rk})_\alpha^U}{\sum_{i=1}^{m} v_i^* (\overline{x}_{ik})_\alpha^L} \le 1,$$

which, it makes that the first constraint to model (12) is satisfied. Moreover, from relations (13) we will have:

$$\sum_{r=1}^{s} u_r^* (y_{rj})_\alpha^L \le \sum_{r=1}^{s} u_r^* y_{rj}^*$$

$$-\sum_{i=1}^{m} v_i^* (\overline{x}_{ij})_\alpha^U \le -\sum_{i=1}^{m} v_i^* \overline{x}_{ij}^*$$

Therefore, the following inequality relation is implied.

$$\sum_{r=1}^{s} u_r^* (y_{rj})_\alpha^L - \sum_{i=1}^{m} v_i^* (\overline{x}_{ij})_\alpha^U \le \sum_{r=1}^{s} u_r^* y_{rj}^* - \sum_{i=1}^{m} v_i^* \overline{x}_{ij}^* \le 0 \tag{14}$$

Thus relation (14) satisfies to the second constraint to model (12). Besides, according to the inequalities $v_i^* \le q_k^*$ and $q_k^* \le (q_k)_\alpha^U$, it can be concluded $v_i^* \le (q_k)_\alpha^U$.

Consequently, every solution to model (10) is a solution to model (12) and vice versa is proved similarly. In addition, the maximum value is required in the value of the objective function $\sum_{r=1}^{s} u_r^* y_{rk}^*$, for the supposed solution. So, it is sufficient to consider the value of y_{rk}^* at the largest value $(y_{rk})_\alpha^U$. Then the objective function is writing as $\sum_{r=1}^{s} u_r^* (y_{rk})_\alpha^U$. Therefore, model (10) is equivalent with the model (12).

Equivalence of the models (9) with the model (11) is also proved similarly. This completes the proof. □

The models (11) and (12) are parametric linear programming problems that may be applied for data information in general fuzzy numbers form. In order to solve these models, data must be considered in a particular form. In general, fuzzy data can be displayed in several forms. The models (11) and (12) can be used for each type of fuzzy data. In this paper, fuzzy data is considered in triangular fuzzy number form due to the easy computation and the many applications of triangular fuzzy numbers form. The proposed method can be applied to other fuzzy numbers too. The α-cut of fuzzy input-outputs are specified as follows:

$$
\begin{aligned}
&(\bar{x}_{ij})_\alpha = [(\bar{x}_{ij})_\alpha^L, (\bar{x}_{ij})_\alpha^U] = [\alpha \bar{x}_{ij}^M + (1-\alpha)\bar{x}_{ij}^L,\ \alpha \bar{x}_{ij}^M + (1-\alpha)\bar{x}_{ij}^U] \quad ; \forall\, i, j \\
&(\mathrm{y}_{rj})_\alpha = [(\mathrm{y}_{rj})_\alpha^L, (\mathrm{y}_{rj})_\alpha^U] = [\alpha y_{rj}^M + (1-\alpha) y_{rj}^L,\ \alpha y_{rj}^M + (1-\alpha) y_{rj}^U] \quad ; \forall\, r, j \qquad (15) \\
&(q_j)_\alpha = [(q_j)_\alpha^L, (q_j)_\alpha^U] = [\alpha q_j^M + (1-\alpha) q_j^L,\ \alpha q_j^M + (1-\alpha) q_j^U] \quad ; \forall\, j
\end{aligned}
$$

where $(\bar{x}_{ij})_\alpha, (\mathrm{y}_{rj})_\alpha, (q_j)_\alpha$ denote the α-cuts of fuzzy input $\tilde{\bar{x}}_{ij}$, fuzzy output $\tilde{y}_{rj}$ and fuzzy value $\tilde{q}_j$ respectively.

According to the relation (15), models (11), (12) can be simplified as follows:

$$\begin{aligned}
(CE_k)^L_\alpha = \max \ & \sum_{r=1}^{s} u_r(\alpha y^M_{rk} + (1-\alpha) y^L_{rk}) \\
s.t. \quad & \sum_{r=1}^{s} u_r(\alpha y^M_{rk} + (1-\alpha) y^L_{rk}) - \sum_{i=1}^{m} v_i(\alpha \overline{x}^M_{ik} + (1-\alpha)\overline{x}^U_{ik}) \le 0 \\
& \sum_{r=1}^{s} u_r(\alpha y^M_{rj} + (1-\alpha) y^U_{rj}) - \sum_{i=1}^{m} v_i(\alpha \overline{x}^M_{ij} + (1-\alpha)\overline{x}^L_{ij}) \le 0 \quad ;\forall j,\ j \ne k \qquad (16) \\
& v_i \le (\alpha q^M_k + (1-\alpha) q^L_k) \quad ;\forall i \\
& u_r \ge 0 \quad ;\forall r \\
& v_i \ge 0 \quad ;\forall i,
\end{aligned}$$

$$\begin{aligned}
(CE_k)^U_\alpha = \max \ & \sum_{r=1}^{s} u_r(\alpha y^M_{rk} + (1-\alpha) y^U_{rk}) \\
s.t. \quad & \sum_{r=1}^{s} u_r(\alpha y^M_{rk} + (1-\alpha) y^U_{rk}) - \sum_{i=1}^{m} v_i(\alpha \overline{x}^M_{ik} + (1-\alpha)\overline{x}^L_{ik}) \le 0 \\
& \sum_{r=1}^{s} u_r(\alpha y^M_{rj} + (1-\alpha) y^L_{rj}) - \sum_{i=1}^{m} v_i(\alpha \overline{x}^M_{ij} + (1-\alpha)\overline{x}^U_{ij}) \le 0 \quad ;\forall j,\ j \ne k \qquad (17) \\
& v_i \le (\alpha q^M_k + (1-\alpha) q^U_k) \quad ;\forall i \\
& u_r \ge 0 \quad ;\forall r \\
& v_i \ge 0 \quad ;\forall i,
\end{aligned}$$

where α is a parameter value belonging to the interval [0,1].

Note that, models (16), (17) are conventional parametric linear models which can be solved to derive the optimum solutions for each given value of α. The objective function of models (16), (17) reflects the value of the lower bound and upper bound of the α-cut of the fuzzy cost efficiency of the DMU under consideration respectively. Then, the α-cut of the fuzzy cost efficiency $C\tilde{E}_k$, is constructed as $(CE_k)_\alpha = [(CE_k)^L_\alpha,\ (CE_k)^U_\alpha]$ and the membership function of $C\tilde{E}_k$ is constructed from $(CE_k)_\alpha$ at given different values of α.

In this paper, the cost efficiency of DMUs is derived as a fuzzy number and using cost efficiency, the DMUs are ranked. There are several methods for ranking fuzzy numbers [17, 33-35]. In this regard, the ranking fuzzy numbers method proposed by Chen and Klein [17] is applied to specify the performance of the DMUs in terms of cost efficiency. Based on the presented method by Chen and Klein [17], the ranking index of DMU_j is defined as follows:

$$I_j = \frac{\sum_{t=0}^{t=T} ((\mathrm{CE}_j)^U_{\alpha_t} - \mathrm{c})}{\sum_{t=0}^{t=T} ((\mathrm{CE}_j)^U_{\alpha_t} - \mathrm{c}) - \sum_{t=0}^{t=T} ((\mathrm{CE}_j)^L_{\alpha_t} - d)} \qquad (18)$$

where, $c = \min_{\forall t,j}\left\{(\mathrm{CE}_j)^L_{\alpha_t}\right\}$ and $d = \max_{\forall t,j}\left\{(\mathrm{CE}_j)^U_{\alpha_t}\right\}$. α_t is a parameter belonging to the interval [0,1] and T is the number of α-cut. In this paper, the problem of ranking DMUs is investigated using fuzzy cost efficiency while this issue has been ignored in previous studies. In this study, the ranking index I_j is introduced as a criterion for ranking DMUs from the viewpoint of cost efficiency. The value of the ranking index I_j is always belonged to the [0,1], which is proved by the following theorem.

Theorem 3.2. The ranking index of DMU_j is belonged to the interval [0,1].

Proof. Let $(CE_j)_{\alpha_t} = [(CE_j)^L_{\alpha_t},\ (CE_j)^U_{\alpha_t}]$ is the α-cut of the fuzzy cost efficiency of DMU_j at given values of parameter α_t. Then $(CE_j)^L_{\alpha_t} \le (CE_j)^U_{\alpha_t} \quad \forall j,$ t .

Since $c = \min_{\forall t,j}\left\{(\mathrm{CE}_j)^L_{\alpha_t}\right\}$, this indicates that

$$c \le (\mathrm{CE}_j)^L_{\alpha_t} \le (\mathrm{CE}_j)^U_{\alpha_t} \Rightarrow \left((\mathrm{CE}_j)^U_{\alpha_t} - c\right) \ge 0 \Rightarrow \sum_{t=0}^{T}\left((\mathrm{CE}_j)^U_{\alpha_t} - c\right) \ge 0$$

Moreover, $d = \max_{\forall t,j}\left\{(\mathrm{CE}_j)^U_{\alpha_t}\right\}$ implies that

$$(CE_j)^L_{\alpha_t} \le (CE_j)^U_{\alpha_t} \le d \Rightarrow \left((CE_j)^L_{\alpha_t} - d\right) \le 0 \Rightarrow \sum_{t=0}^{T}\left((CE_j)^L_{\alpha_t} - d\right) \le 0$$

$$\Rightarrow -\sum_{t=0}^{T}\left((CE_j)^L_{\alpha_t} - d\right) \ge 0$$

Therefore, we will have:

$$0 \le \sum_{t=0}^{T}\left((\mathrm{CE}_j)^U_{\alpha_t} - c\right) \le \sum_{t=0}^{T}\left((\mathrm{CE}_j)^U_{\alpha_t} - c\right) + \left[-\sum_{t=0}^{T}\left((CE_j)^L_{\alpha_t} - d\right)\right]$$

$$\Rightarrow 0 \le I_j = \frac{\sum_{t=0}^{t=T}((\mathrm{CE}_j)^U_{\alpha_t} - \mathrm{c})}{\sum_{t=0}^{t=T}((\mathrm{CE}_j)^U_{\alpha_t} - \mathrm{c}) - \sum_{t=0}^{t=T}((\mathrm{CE}_j)^L_{\alpha_t} - d)} \le 1$$

This completes the proof. □

4. Numerical Example

In this section, a numerical example is presented to explain the proposed approach. In this explanation, the same data that used by Puri and Yadav [28] is applied. In this example, there are 10 DMUs, with two fuzzy outputs and two fuzzy inputs. Also, data is triangular fuzzy numbers, denoted by (a^L, a^M, a^U), where a^L is lower bound, a^M is mean value, and a^U is an upper bound. Table 1 shows the fuzzy input-output data and fuzzy input prices for each of the 10 DMUs.

Table 1. The fuzzy input-output data and fuzzy input prices for DMUs

DMU	Fuzzy inputs		Fuzzy outputs		Fuzzy input prices	
	Input1	Input2	Output1	Output2	Input1	Input2
1	(49.4,53,56.6)	(40.5,45,49.5)	(70,77.5,85)	(28.8,35.4,42)	(4.7,5,5.3)	(4.5,5,5.5)
2	(20,25,30)	(41.6,46.5,51.4)	(41.5,49.6,57.7)	(28.9,34.7,40.5)	(3.55,6.5)	(4.5,6,7.5)
3	(11.5,18,24.5)	(11.9,15.7,19.5)	(21.6,26.5,31.4)	(29.4,37.6,45.8)	(7.28,8.8)	(6.3,7,7.7)
4	(12.1,18,23.9)	(20.1,25.5,30.9)	(34.2,37.5,40.8)	(40.9,47.5,54.1)	(7.2,9,10.8)	(5,5.5,6)
5	(29.1,32,34.9)	(20.3,25,29.7)	(59.2,64,68.8)	(72.9,76.4,79.9)	(2.8,3,3.2)	(1.75,2,2.25)
6	(50.8,56,61.2)	(41.6,45.1,48.6)	(32.6,35.3,38)	(3742.5,48)	(5.4,6,6.6)	(3.25,4,4.75)
7	(17.6,24,30.4)	(13.6,17.5,21.4)	(75.6,82.9,90.2)	(33.2,38.5,43.8)	(1.6,2,2.4)	(1.5,2,2.5)
8	(71.2,78,84.8)	(18.5,23.9,29.3)	(60,66,72)	(38.2,47.4,56.6)	(1.8,3,4.2)	(3,3.9,4.8)
9	(45.5,52,58.5)	(13.7,19.8,25.9)	(51.1,56.5,61.9)	(51.3,56,60.7)	(4.8,5,5.2)	(89.8,11.6)
10	(44.6,49,53.4)	(16.3,20.6,24.9)	(38,46.5,55)	(32.1,38,43.9)	(2,2.9,3.8)	(2,2.6,3.2)

In this study, the models are solved with GAMS software with CPLEX solver. Furthermore, the lower bound $(CE_k)^L_\alpha$ and upper bound $(CE_k)^U_\alpha$ of the α-cut of the fuzzy cost efficiency of DMUs are obtained by the optimal objective function value of the models (16) and (17) respectively. These models are parametric linear programming problems, which are solved for the given different values of α, and the results are shown in Table 2.

Table 2. The α-cut of the fuzzy cost efficiency measures

DMU		$\alpha=0$	$\alpha=0.1$	$\alpha=0.2$	$\alpha=0.3$	$\alpha=0.4$	$\alpha=0.5$	$\alpha=0.6$	$\alpha=0.7$	$\alpha=0.8$	$\alpha=0.9$	$\alpha=1$
1	$(CE_1)^L_\alpha$	0.063	0.071	0.079	0.087	0.096	0.105	0.114	0.124	0.135	0.146	0.158
	$(CE_1)^U_\alpha$	0.362	0.337	0.313	0.290	0.268	0.247	0.227	0.208	0.190	0.173	0.158

2	$(CE_2)_\alpha^L$	0.046	0.054	0.063	0.073	0.084	0.096	0.109	0.123	0.138	0.155	0.173
	$(CE_2)_\alpha^U$	0.475	0.435	0.398	0.363	0.330	0.299	0.270	0.243	0.218	0.195	0.173
3	$(CE_3)_\alpha^L$	0.070	0.083	0.099	0.116	0.135	0.155	0.178	0.203	0.230	0.259	0.281
	$(CE_3)_\alpha^U$	0.708	0.653	0.602	0.553	0.507	0.463	0.422	0.383	0.347	0.313	0.281
4	$(CE_4)_\alpha^L$	0.077	0.092	0.109	0.127	0.146	0.168	0.191	0.217	0.244	0.274	0.299
	$(CE_4)_\alpha^U$	0.704	0.654	0.606	0.560	0.517	0.476	0.436	0.399	0.364	0.331	0.299
5	$(CE_5)_\alpha^L$	0.407	0.458	0.512	0.570	0.631	0.695	0.763	0.836	0.912	0.993	1.000
	$(CE_5)_\alpha^U$	1.000	1.004	1.008	1.010	1.012	1.012	1.012	1.010	1.008	1.004	1.000
6	$(CE_6)_\alpha^L$	0.043	0.050	0.058	0.067	0.077	0.087	0.098	0.109	0.122	0.135	0.149
	$(CE_6)_\alpha^U$	0.287	0.271	0.256	0.242	0.228	0.215	0.202	0.190	0.179	0.168	0.149
7	$(CE_7)_\alpha^L$	0.989	1.016	1.029	1.038	1.043	1.045	1.043	1.038	1.029	1.016	1.000
	$(CE_7)_\alpha^U$	1.000	1.026	1.047	1.062	1.071	1.074	1.071	1.062	1.047	1.026	1.000
8	$(CE_8)_\alpha^L$	0.076	0.089	0.104	0.121	0.140	0.160	0.182	0.207	0.233	0.263	0.292
	$(CE_8)_\alpha^U$	0.886	0.808	0.735	0.666	0.601	0.540	0.483	0.430	0.381	0.334	0.292
9	$(CE_9)_\alpha^L$	0.067	0.079	0.091	0.105	0.120	0.136	0.153	0.171	0.191	0.212	0.234
	$(CE_9)_\alpha^U$	0.489	0.458	0.429	0.401	0.374	0.349	0.325	0.302	0.280	0.259	0.234
10	$(CE_{10})_\alpha^L$	0.112	0.131	0.152	0.174	0.199	0.227	0.256	0.288	0.323	0.355	0.385
	$(CE_{10})_\alpha^U$	1.000	0.945	0.865	0.790	0.720	0.654	0.592	0.535	0.481	0.431	0.385

The α-cut of the fuzzy cost efficiency $C\tilde{E}_k$, is constructed as $(CE_k)_\alpha = [(CE_k)_\alpha^L,\ (CE_k)_\alpha^U]$ at given different values of α. The α-cut of the fuzzy cost efficiency $C\tilde{E}_k$, the lower bound

$(CE_k)_\alpha^L$ strictly increases and the upper bound $(CE_k)_\alpha^U$ strictly decreases when α in interval [0,1] is increased, for each DMU, as shown in Table 2. The values of the lower bound and the upper bound of the α-cut $(CE_k)_\alpha$ of all DMUs are equal for parameter $\alpha = 1$, presented in the last column of the Table 2.

Furthermore, it should be noted that, the values of the lower and the upper bounds of the fuzzy cost efficiency of DMU_7 are almost equal to one for all the values of $\alpha \in [0,1]$. This indicates that DMU_7 is a cost efficient unit. However, the values of the upper bounds of the fuzzy cost efficiency of DMU_5 are almost equal to one for all parameters.

Table 3. The results of ranking

DMU	1	2	3	4	5	6	7	8	9	10
Index Ij	0.178	0.215	0.322	0.331	0.725	0.151	0.955	0.364	0.251	0.429
Rank	9	8	6	5	2	10	1	4	7	3

In this numerical example, the lower bound and upper bound of the α-cut of the fuzzy cost efficiency scores are obtained at eleven α values, as shown in Table 2. Moreover, based on the proposed method by Chen-Klein [17] and using relation (12), the ranking index is calculated for each DMU, which is shown in row 2 in Table 3. Therefore, an order of ranking of DMUs is provided as follows: DMU_7, DMU_5, DMU_{10}, DMU_8, DMU_4, DMU_3, DMU_9, DMU_2, DMU_1, and DMU_6.

In this section the proposed method is compared with previous existing methods to show the accuracy of the proposed method. The obtained results of cost efficiency for the considered DMUs by using previous methods are shown in Table 4. In this Table, CE^L and CE^U are the minimum value of lower bounds and the maximum value of upper bounds of the α-cut of the fuzzy cost efficiency respectively. Columns 2 show the results of fuzzy cost efficiency obtained using Puri and Yadav [28] method, which are obtained in triangular fuzzy numbers form for each DMUs. The values of CE^L and CE^U have been provided in columns 3-6, based on the previous methods by Puri and Yadav [28], and Pourmahmoud and Bafekr Sharak [29]. Moreover, columns 7-8 show the obtained results of this paper. Puri and Yadav [28] introduced DMU_5 and DMU_7 as cost efficient units. Besides, in Pourmahmoud and Bafekr Sharak method [29], DMU_7 are introduced as fuzzy cost efficient units and DMU_5 is cost efficient unit for values of parameter $\alpha \geq 0.5$. However, in the proposed method, DMU_7 is shown as fuzzy cost efficient unit, and as well, DMU_5 is identified as fuzzy cost inefficient unit. Therefore, the proposed method in this study can discriminate between cost efficient DMUs while, the others cannot.

The results of fuzzy cost efficiency obtained using previous methods for some DMUs exceeds one, whereas by using the proposed method the obtained values for cost efficiency are in the intervals [0,1]. In this example, it can be seen that the fuzzy cost efficiency values obtained by the proposed method are in the appropriate interval in comparison to the previous methods. Also, in the proposed method, DMU_5 are ranked based on the fuzzy cost efficiency, while the problem of ranking is ignored in previous methods.

Table 4. The results of cost efficiency by previous approaches

DMU	**Puri and Yadav method**			**Pourmahmoud and Bafekr method**		**The proposed method**	
	Fuzzy cost efficiency	CE^L	CE^U	CE^L	CE^U	CE^L	CE^U
1	(0.0786,0.1584,0.2909)	0.079	0.291	0.082	0.219	0.063	0.362
2	(0.0766,0.1736,0.4002)	0.077	0.4	0.069	0.317	0.046	0.475
3	(0.129,0.283,0.6486)	0.129	0.649	0.112	0.528	0.070	0.708
4	(0.1478,0.3010,0.6473)	0.148	0.647	0.121	0.545	0.077	0.704
5	(0.6555,1,1.5256)	0.655	1.526	0.499	1.346	0.407	1
6	(0.0932,0.1582,0.2655)	0.093	0.265	0.078	0.226	0.043	0.287
7	(0.3840,1,2.6042)	0.384	2.604	0.397	1.980	0.989	1
8	(0.1177,0.2944,0.7813)	0.118	0.781	0.102	0.582	0.076	0.886
9	(0.1347,0.2397,0.4304)	0.135	0.43	0.108	0.366	0.067	0.489
10	(0.1775,0.3851,0.8822)	0.177	0.882	0.155	0.712	0.112	1

5. Conclusions

Cost efficiency of the decision-making units, is one of the most important information which may be given by DEA models. Cost efficiency models evaluate the ability of a set of DMUs to produce current outputs at the minimal cost. However, in real application, the observed data are frequently imprecise and vague. In this study, fuzzy DEA is employed for assessing the cost efficiency. Furthermore, a method is proposed for assessing the cost efficiency using the α-level based approach. In this method, fuzzy data is supposed in triangular fuzzy numbers form. The main idea is to transform the fuzzy model into a family of parametric crisp models to obtain the lower and upper bounds of the α-cut of the fuzzy cost efficiency measures. Moreover, the problem of ranking DMUs in the fuzzy environment using proposed method is studied. This study shows that the proposed approach is more reliable and informative than other studied approaches.

References

[1] Farrell, M.J. (1957), The measurement of productive efficiency, Journal of the Royal Statistical Society, Series A (General), 120, 253-290.

[2] Färe, R., Grosskopf, S. and Lovell, C.K. (1985), The measurement of efficiency of production, (Kluwer-Nijhoff).

[3] Tone, K. (2002), A strange case of the cost and allocative efficiencies in DEA, Journal of the Operational Research Society, 53, 1225-1231.

[4] Allahviranloo, T., Nuraei, R., Ghanbari, M., Haghi, E. and Hosseinzadeh, A. (2012), A new metric for L–R fuzzy numbers and its application in fuzzy linear systems, Soft Computing, 16, 1743-1754.

[5] Ramík, J. and Vlach, M. (2012), Aggregation functions and generalized convexity in fuzzy optimization and decision making, Annals of Operations Research, 195, 261-276.

[6] Saati, S., Hatami-Marbini, A., Tavana, M. and Agrell, P.J. (2013), A fuzzy data envelopment analysis for clustering operating units with imprecise data, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 21, 29-54.

[7] Liu, Y. and Lin, Y. (2015), Intuitionistic fuzzy rough set model based on conflict distance and applications, Applied Soft Computing, 31, 266-273.

[8] Fakhrzad, M. and Emami, L. (2015), A fuzzy multi-objective linear programming approach for solving a new multi-objective job shop scheduling with sequence-dependent setup times, Iranian Journal of Operations Research, 6, 1-14.

[9] Hamidi, H. (2015), A Fuzzy Decision Making Approach to Enterprise Resource Planning System Selection, Iranian Journal of Operations Research, 6, 34-52.

[10] Lertworasirikul, S., Fang, S.-C., Joines, J.A. and Nuttle, H.L. (2003), Fuzzy data envelopment analysis (DEA): a possibility approach, Fuzzy sets and Systems, 139, 379-394.

[11] Meada, Y., Entani, T. and Tanaka, H. (1998), Fuzzy DEA with interval efficiency, In, ed^eds. (European Congress on Intelligent Techniques and Soft Computing), p^pp.

[12] Kao, C. and Liu, S.-T. (2000), Fuzzy efficiency measures in data envelopment analysis, Fuzzy sets and systems, 113, 427-437.

[13] Saati, S.M., Memariani, A. and Jahanshahloo, G.R. (2002), Efficiency analysis and ranking of DMUs with fuzzy data, Fuzzy Optimization and Decision Making, 1, 255-267.

[14] Kao, C. and Liu, S.-T. (2003), A mathematical programming approach to fuzzy efficiency ranking, International Journal of Production Economics, 86, 145-154.

[15] Chiang, T.-A. and Che, Z. (2010), A fuzzy robust evaluation model for selecting and ranking NPD projects using Bayesian belief network and weight-restricted DEA, Expert Systems with Applications, 37, 7408-7418.

[16] Agarwal, S. (2014), Efficiency measure by fuzzy data envelopment analysis model, Fuzzy Information and Engineering, 6, 59-70.

[17] Chen, C.-B. and Klein, C.M. (1997), A simple approach to ranking a group of aggregated fuzzy utilities, IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics), 27, 26-35.

[18] Wanke, P., Barros, C.P. and Emrouznejad, A. (2016), Assessing productive efficiency of banks using integrated Fuzzy-DEA and bootstrapping: A case of Mozambican banks, European Journal of Operational Research, 249, 378-389.

[19] Egilmez, G., Gumus, S., Kucukvar, M. and Tatari, O. (2016), A fuzzy data envelopment analysis framework for dealing with uncertainty impacts of input–output life cycle assessment models on eco-efficiency assessment, Journal of cleaner production, 129, 622-636.

[20] Izadikhah, M. and Khoshroo, A. (2018), Energy management in crop production using a novel Fuzzy Data Envelopment Analysis model, RAIRO-Operations Research, 52, 595-617.

[21] Yen, B.T. and Chiou, Y.-C. (2019), Dynamic fuzzy data envelopment analysis models: case of bus transport performance assessment, RAIRO-Operations Research, 53, 991-1005.

[22] Khalili-Damghani, K., Tavana, M. and Haji-Saami, E. (2015), A data envelopment analysis model with interval data and undesirable output for combined cycle power plant performance assessment, Expert Systems with Applications, 42, 760-773.

[23] Mashayekhi, Z. and Omrani, H. (2016), An integrated multi-objective Markowitz–DEA cross-efficiency model with fuzzy returns for portfolio selection problem, Applied Soft Computing, 38, 1-9.

[24] Soltanzadeh, E. and Omrani, H. (2018), Dynamic network data envelopment analysis model with fuzzy inputs and outputs: An application for Iranian Airlines, Applied Soft Computing, 63, 268-288.

[25] Jahanshahloo, G., Hosseinzadeh Lotfi, F., Alimardani Jondabeh, M., Banihashemi, S. and Lakzaie, L. (2008), Cost efficiency measurement with certain price on fuzzy data and application in insurance organization, Applied Mathematical Sciences, 2, 1-18.

[26] Bagherzadeh Valami, H. (2009), Cost efficiency with triangular fuzzy number input prices: An application of DEA, Chaos, Solitons and Fractals, 42, 1631-1637.

[27] Paryab, K., Tavana, M. and Shiraz, R.K. (2015), Convex and non-convex approaches for cost efficiency models with fuzzy data, International Journal of Data Mining, Modelling and Management, 7, 213-238.

[28] Puri, J. and Yadav, S.P. (2016), A fully fuzzy DEA approach for cost and revenue efficiency measurements in the presence of undesirable outputs and its application to the banking sector in India, International Journal of Fuzzy Systems, 18, 212-226.

[29] Pourmahmoud, J. and Bafekr Sharak, N. (2018), Measuring Cost Efficiency with New Fuzzy DEA Models, International Journal of Fuzzy Systems, 20, 155-162.

[30] Yager, R.R. (1986), A characterization of the extension principle, Fuzzy sets and systems, 18, 205-217.

[31] Zadeh, L.A. (1965), Fuzzy sets, Inform. and Control, 8, 338-353.

[32] Zimmerman, H. (1996), Fuzzy sets and its applications, In, ed^eds. (Kluwer, Aacdemic Publisher), p^pp.

[33] Chen, S.-H. (1985), Ranking fuzzy numbers with maximizing set and minimizing set, Fuzzy sets and Systems, 17, 113-129.

[34] Liou, T.-S. and Wang, M.-J.J. (1992), Ranking fuzzy numbers with integral value, Fuzzy sets and systems, 50, 247-255.

[35] Tseng, T.Y. and Klein, C.M. (1989), New algorithm for the ranking procedure in fuzzy decision-making, IEEE Transactions on Systems, Man, and Cybernetics, 19, 1289-1296.