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A novel type I and II fuzzy approach for solving single allocation ordered median hub location problem

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Hubs are facilities that can decrease the cost of many-to-many distribution systems by acting as an interconnector between the demand and supply nodes. This type of facility can reduce the number of direct links needed in a logistics network. Hub location problems (HLP) have been discussed by many authors for more than four decades, and different approaches have been developed for modeling and solving this problem. We propose a fuzzy type I and II programming approach for a new model presented in the literature, i.e., the single allocation ordered median problem. The level of flow among the nodes will be considered as a fuzzy parameter. In the fuzzy type I approach, a linear programming problem with fuzzy parameters is used, while for the fuzzy type II approach, the rules of interval arithmetic are developed to simplify the problem to the fuzzy type I case. Finally, we apply our method on Kalleh Dairy Co. data of transportation as a case study and compare crisp and fuzzy situations. We show that the results of the fuzzy approach could be 2% better than the crisp approach and also discuss the pros and cons of fuzzy type I and type II approaches.

***Keywords**: Type I and II Fuzzy Systems, Hub Location Problem, Single Allocation Ordered Median Problem*

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1. Introduction

Hubs are facilities that can improve transshipment in many-to-many distribution systems. Instead of having a direct path between each origin-destination pair to serve the demand, hub facilities play an intermediate role. In other words, the flows of different nodes are directed to the hub, and then each demand goes from the hub to its relevant supplier. This routine can efficiently utilize the economy of scale. Figure 1 shows a network of hub and non-hub nodes. Figure 2 shows the network of American airlines, where New York, Miami, and Boston are depicted as hubs. In general, the number of nodes is denoted by *N*. It seems that Goldman [13] is the first paper to address the network hub location problem. During recent decades, there have been numerous studies focusing on the hub location problem with a great level of variation. These variations involve modeling concepts, the type of the objective function, and the constraints. The main focus in these studies has been on minimizing the overall cost in the system, which is the sum of the transportation cost of each origin-destination path (see e.g. Campbell [2]).

Among the studies so far carried out on this problem, the one performed by Alumur and Kara [1] can be considered as the most influential. From the structural point of view of a network, two types of problems can be considered, i.e., *single allocation* and *multiple allocation.* In the single allocation problem, each non-hub node is assigned to just one hub node (see e.g., Momayezi et al. [21] and Sangsawang and Chanta [32]). However, in the multiple allocation problem, each demand

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center is assigned to more than one hub, and thus it can receive and send flows through more than one hub (see e.g., Ghaffarinasab et al. [12] and Monemi et al. [22]).

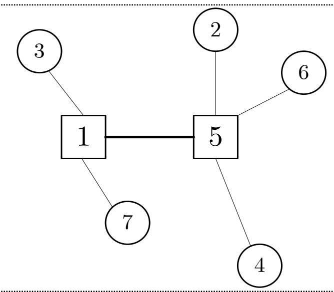

Figure 1. A network, in which nodes 1 and 5 are hub nodes and other vertices are non-hubs

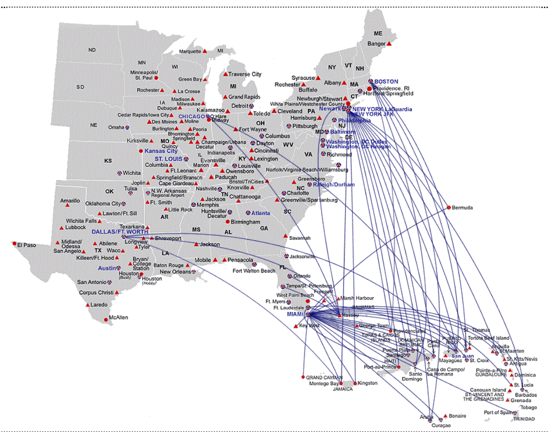

Figure 2. Part of American airlines that shows New York, Miami, and Boston as hub nodes of the network

The vast scope of the hub location problem is made up of four major branches. The first and the most-studied branch is the *p*-hub *median* problem, whose aim is to minimize the total transportation cost of *n* demand nodes with a fixed number of hubs (*p*). Some of the studies focusing on this type of objective function include Rouzpeykar et al. [31], Fernández and Sgalambro [11], and Mokhtar et al. [20]. Another group of problems deals with a *fixed cost* for establishing a hub. In these problems, in addition to the transportation cost, there is a cost for choosing a node as a hub; hence, the number of hubs is not fixed to a value of *p*, and it is identified in the problem (see e.g., Taherkhani and Alumur [35], Monemi et al. [22], Özgün-Kibiroğlu et al. [27] and Khodemani-Yazdi et al. [16]). The third branch involves the *p*-hub *center* problem, which deals with an objective function of the minimax type. One possible objective is minimizing the maximum cost for each origin-destination pair. Campbell [3] has considered three different types of *p*-hub center problems and has formulated all three. Other relevant studies include Shahparvari et al. [33] and Ernst et al. [9]. The fourth and final major type of hub location problem is the *hub set-covering* problem. The goal of the hub set-covering problem is to place the hubs in such a way to cover all demand while minimizing the cost of opening the hub facilities (see e.g., Nickel et al. [24]). On the other hand, the maximal hub-covering problem maximizes the demand covered by a fixed number of hubs, and both of these problems have been modeled by Campbell [3].

Recently, some other studies have considered identifying a reliable hub location as a new field (see e.g., Shen et al. [34]). Moreover, some authors have recently worked on different types of discount factors, including Cunha and Silva [4]. The discount factor $\alpha, 0 \leq \alpha < 1$ is a parameter that results in a discount for the transportation cost of the inter-hub connections. In addition, O'Kelly and Bryan [26] indicated that the assumption of flow-independent costs would not only erroneously select the optimal hub locations and the allocations, but it would also miscalculate the total network cost. Furthermore, they proposed a non-linear cost function, allowing costs to increase at a *decreasing* rate as the flows increase.

On the other hand, an interesting problem in the area of *p*-hub *median* problems is the *ordered median* hub location problem Puerto et al. [29]. In this problem, a new parameter is defined, called the *rank-dependent compensation factor* $(\lambda_i, i \in \{1, \ldots, N\})$, which incorporates flexibility into the model. To put it more simply, the parameter $\lambda_i (0 \leq \lambda_i \leq 1)$ is a coefficient for the $i-th$ largest transportation cost that is a member of a special set of transportation costs, which will be defined later. Parameter λ_i acts as a scaling factor that will be assigned to the origin nodes depending on the order of the sequence of the transportation costs of the commodity with the same origin node as the first hub. Solution methods for hub location problems vary from exact methods, such as the B&B, to meta-heuristics, such as SA and TS. O'Kelly [25] made use of heuristics to solve his quadratic integer programming. Klincewicz [18] proposed an exchange heuristic to solve the problem. Moreover, the same author used Tabu Search (TS) and GRASP heuristics in Klincewicz [17]. The most effective heuristic is the Lagrangian relaxation-based heuristic presented in Pirkul and Schilling [28]. It should be noted that among the best meta-heuristics are the Tabu search heuristic presented in Ghaffarinasab et al. [12], and the simulated annealing heuristic presented in Zarandi et al. [41]. The most efficient exact solution procedure is the shortest-path-based branch-and-bound algorithm presented in Ernst and Krishnamoorthy [10]. So far, the largest set of problems that have been optimally solved has 100 nodes.

A new approach for solving the hub location problem involves fuzzy programming and there have been a limited number of studies carried out in this area, all published from 2010 to 2013, mostly by Iranian authors. Among these studies, Davari et al. [7] deals with the reliable fuzzy hub location problem, Davari and Fazel Zarandi [6] and Davari and Fazel Zarandi [5] took advantage of fuzzy parameters in the modeling process to obtain a more realistic model, Mirakhorli [19] utilized chance-constrained programming with a fuzzy cover radius in a hub covering problem, and Mostafa et al. [23] used a hybrid algorithm for solving a *p*-hub median problem. Moreover, among recent studies, there are distinguished articles that discuss supply chain and network problems with a fuzzy solving approach. For example, Tirkolaee et al. [37] uses fuzzy decision making for sustainable-reliable supplier selection in two-echelon supply chain design, Tirkolaee et al. [36] takes advantage of fuzzy approach in a multi-trip location-routing problem for medical waste management during the COVID-19 outbreak, and finally Rokhsari and Sadeghi-Niaraki [30] suggests fuzzy-AHP and TOPSIS in GIS environment to assess risk in an urban network.

The novelty of this paper involves utilizing fuzzy type I and II mathematical programming for an ordered p-hub median problem. In the next section, some notations and basic definitions will be presented. Section 3 discusses the formulation of the model. Section 4 deals with the solution method. In section 5 we apply our method on Kalleh Dairy Co. data of transportation as a case study and compare crisp and fuzzy situations. It will be shown that the results of the fuzzy approach could be better than the crisp approach and also the pros and cons of fuzzy type I and type II approaches will be discussed. Finally, we discuss the conclusion and some future points of research in Section 6.

2. Notation and Basic Definitions

2.1. Ordered p-Hub Median Problem Notations

Let N denote a given number of clients or nodes in a network, and $i, j, k, l, m \in \{1,2,\dots,N\}$ denote indices for identifying a specific node. Each node i sends a particular amount of commodity to another node j, denoted by w_{ij}. Some routine notations for the problems are as follows:

2.1.1. Parameters

p: Number of hubs
$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$: Vector of compensation parameters
β: Discount factor for inter-hub links ($0 \le \beta < 1$)
δ: Discount factor for hub and final destination links ($\beta < \delta \le 1$)
c_{ij}: Unit cost of traveling from node *i* to node *j*
w_{ij}: Amount of flow from node *i* to node *j*
$W_i = \sum_j w_{ij}$: All commodity sends from node *i*

2.1.2. Decision Variables

y_k: 1 if a hub locates at node *k*; 0 o.w.
r_{jk}^{i}: if the flow from origin site *j* goes first to hub *k* and $W_j c_{jk}$ is the *i*th lowest value of the transportation costs from each origin to its first hub; 0 o.w.
x_{klm}: flow that goes through a first hub *k* and a second hub *l* with destination *m*

Parameter λ_i is a type of rank-dependent weighting factor. The goal of these weights is to compensate for unfair situations. For example, the reader may note that we are simultaneously making decisions on locating the hubs that define the intermediate distribution system, and establishing the delivery paths from the origin nodes to the final destination. Thus, a solution that is good for the system (i.e., the entire supply chain) might not be acceptable for individual nodes if their costs for reaching the system in that solution are too high relative to similar costs for the other nodes. In this case, some compensation for unhappy nodes may be needed to prevent them from not using the system. For instance, if a solution places a set of hubs in a way that the accessibility cost for the origin node *i* is greater than the corresponding cost for the origin node *j*, the model tries to favor *i* over *j* when assigning the weights $\lambda_i \le \lambda_j$. (Note that these weights do not penalize node *j*; rather, they compensate node *i* because these lambdas reduce the dispersion of the costs). These scaling factors (i.e., lambdas) will be assigned to the origin nodes depending on the order of the sequence of the transportation costs of the commodity with the same origin node as the first hub.

Notice that depending on different choices of the λ vector, different criteria will be considered for the objective function. For instance, if $\lambda = (0,\dots,0,\overbrace{1,\dots,1}^{k})$, the objective function will be the sum of k biggest costs of transportation (k-centrum in the literature), and for $\lambda = (0,\dots,0,1)$, this will become a p-center problem.

2.2. Fuzzy Notations and Definitions

A fuzzy set $\tilde{A}$ of a universe Ω is characterized by its membership function (MF) (Zadeh [39]):

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in \Omega\} \tag{1}$$

Where $\mu_{\tilde{A}}(x)$ is the membership degree of x in $\tilde{A}$.
Moreover, the α-cut of a fuzzy set $\tilde{A}$ is defined as follows:

$$a_\alpha = \{x \in \Omega | \mu_{\tilde{A}}(x) \geq \alpha\} \tag{2}$$

In other words, the α-cut of a fuzzy set $\tilde{A}$ is a subset of the elements in $\tilde{A}$, whose membership degree is higher than or at least equal to α. Figure 3 shows the α-cut of the fuzzy set.

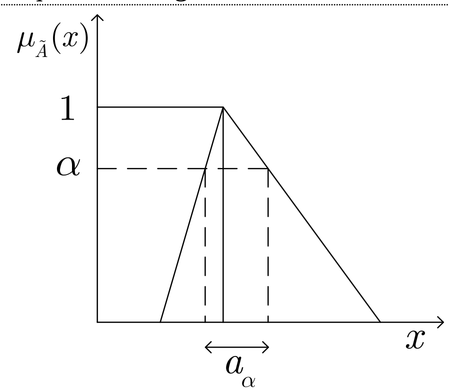

Figure 3. α-cut of a fuzzy set $\tilde{A}$

A fuzzy number is a fuzzy set $\tilde{a}$ on the real numbers line R, whose membership function $\mu_{\tilde{a}}(x)$ is upper semi-continuous such that:

$$r = \mu_{\tilde{a}}(x) = \begin{cases} 0 & \forall x \in (-\infty, a_1] \\ f_a(x) & \text{increasing on } [a_1, a_2] \\ 1 & \forall x \in [a_2, a_3] \\ g_a(x) & \text{decreasing on } [a_3, a_4] \\ 0 & \forall x \in [a_4, +\infty) \end{cases} \tag{3}$$

Therefore, we can show the α-cut of the fuzzy number $\tilde{a}$ as follows:

$$a_\alpha = [f_a^{-1}(\alpha), g_a^{-1}(\alpha)] \tag{4}$$

The expected interval of a fuzzy number $\tilde{a}$ is denoted by $EI(\tilde{a})$, which is defined as follows (Heilpern [14]):

$$EI(\tilde{a}) = [E_1^a, E_2^a] = \left[\int_0^1 f_a^{-1}(r)\ dr, \int_0^1 g_a^{-1}(r)\ dr\right] \tag{5}$$

The expected value of a fuzzy number $\tilde{a}$ is denoted by $EV(\tilde{a})$, which is defined as follows (Heilpern [14]):

$$EV(\tilde{a}) = \frac{E_1^a + E_2^a}{2} \tag{6}$$

Given two fuzzy numbers $\tilde{a}, \tilde{b}$, any arithmetic operation $\tilde{a} * \tilde{b}$ can be aggregated to a fuzzy number based on Zadeh's minimum extension principle (Zadeh [40]):

$$\mu_{\tilde{a}*\tilde{b}}(z) = \sup_{z=x*y} \min\{\mu_{\tilde{a}(x)}, \mu_{\tilde{b}(x)}\} \tag{7}$$

When the extended minimum principle is used to aggregate fuzzy numbers, Dubois and Prade [8] show the following relationship:

$$[f_{\lambda a+\gamma b}^{-1}(r), g_{\lambda a+\gamma b}^{-1}(r)] = [\lambda f_a^{-1}(r) + \gamma f_b^{-1}(r), \lambda g_a^{-1}(r) + \gamma g_b^{-1}(r)] \tag{8}$$

Where $\tilde{a}, \tilde{b}$ are fuzzy numbers, and λ, γ are non-negative real numbers.
Therefore, it can easily be deduced that:

$$EI(\lambda\tilde{a} + \gamma\tilde{b}) = \lambda EI(\tilde{a}) + \gamma EI(\tilde{b}) \tag{9}$$

$$EV(\lambda\tilde{a} + \gamma\tilde{b}) = \lambda EV(\tilde{a}) + \gamma EV(\tilde{b}) \tag{10}$$

3. Model Formulation

First, the deterministic model of the ordered p-Hub Median problem will be shown, and each constraint will be explained. Afterward, the paper discusses the fuzzy form of the problem. The deterministic model is as follows:

$$\min \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} \lambda_i c_{jk} r_{jk}^{i} W_j + \sum_{k=1}^{N}\sum_{l=1}^{N}\sum_{m=1}^{N} x_{klm}(\beta c_{kl} + \delta c_{lm}) \tag{11}$$

$$\text{s.t.} \sum_{i}\sum_{k} r_{jk}^{i} = 1, \forall j \tag{12}$$

$$\sum_{j}\sum_{k} r_{jk}^{i} \leq 1, \forall i \tag{13}$$

$$\sum_{i}\sum_{j} r_{jk}^{i} \leq N y_k, \forall k \tag{14}$$

$$\sum_{i} r_{jj}^{i} = y_j, \forall j \tag{15}$$

$$\sum_{l} x_{klm} = \sum_{i}\sum_{j} r_{jk}^{i}\, w_{jm}, \forall k, m \tag{16}$$

$$\sum_{j}\sum_{k} r_{jk}^{i}\, c_{jk} W_j \leq \sum_{j}\sum_{k} r_{jk}^{i+1}\, c_{jk} W_j, \forall i = 1, \ldots, N-1 \tag{17}$$

$$x_{klm} \leq (1 - y_m)\sum_{j} w_{jm}, \forall k, l, m, l \neq m \tag{18}$$

$$\sum_{l}\sum_{m} x_{klm} \leq y_k \sum_{j} W_j, \forall k \tag{19}$$

$$\sum_{k}\sum_{m} x_{klm} \leq y_l \sum_{j} W_j, \forall l \tag{20}$$

$$\sum_{k} y_k = p \tag{21}$$

$$r_{jk}^{i} \in \{0,1\},\ \ x_{klm}, y_k \geq 0,\ \ \forall i, j, k, l, m = 1, \ldots, N \tag{22}$$

In this model, the flows (w_{jm}) between different nodes can be considered as fuzzy numbers because of the uncertain nature of this parameter. Thus, the fuzzy form of the model above can be obtained by replacing w_{jm} with $\widetilde{w}_{jm}$ and W_j with $\widetilde{W}_j$, where $\widetilde{w}_{jm}$ and $\widetilde{W}_j$ are fuzzy numbers. We consider a triangular membership function for these parameters in the solution method and the numerical result of this paper. However, any other membership function is allowed. Note that $W_i = \sum_j w_{ij}$, so $\widetilde{W}_i = \sum_j \widetilde{w}_{ij}$.

4. The Solution Method

4.1. The Fuzzy type I Approach

The general form of our model is presented below, which is a linear programming problem with fuzzy parameters.

$$\min \quad z = \tilde{c}^t x \tag{23}$$

$$\text{s.t.} \quad x \in \aleph(\tilde{A}, \tilde{b}) = \{x \in R^n | \tilde{a}_i x \geq \tilde{b}_i, i = 1, \dots, m, x \geq 0\} \tag{24}$$

There are two challenges to solving this model. The first challenge involves the method for determining the feasibility of a decision vector *x* when the constraints contain fuzzy parameters. The second challenge involves the method that can be used for defining the optimality for an objective function with fuzzy coefficients.

We need a criterion for comparing the two fuzzy numbers and deciding on which number is larger. Based on JIMÉNEZ [15], for any pair of fuzzy numbers $\tilde{a}, \tilde{b}$, *the degree based on which* $\tilde{a}$ *is larger than* $\tilde{b}$ is defined as follows:

$$\mu_M(\tilde{a}, \tilde{b}) = \begin{cases} 0; \; if \; E_2^a - E_1^b < 0 \\ \dfrac{E_2^a - E_2^b}{E_2^a - E_1^b - (E_1^a - E_2^b)}; if \;\; 0 \in \left[E_1^a - E_2^b, E_2^a - E_1^b\right] \\ 1; if \;\; E_1^a - E_2^b > 0 \end{cases} \tag{25}$$

It can be observed that when $\mu_M(\tilde{a}, \tilde{b}) = 0.5$, $\tilde{a}, \tilde{b}$ will be the same. Based on the last definition, when $\mu_M(\tilde{a}, \tilde{b}) \geq \alpha$, we can say that $\tilde{a}$ is larger than or equal to $\tilde{b}$ at least at a degree of α, which is also shown by $\tilde{a} \geq_\alpha \tilde{b}$. This leads us to the next definition:

Suppose $x \in R^n$ is a decision vector. This vector is feasible at degree α if:

$$\min_{i=1,\dots,m} \{\mu_M(\tilde{a}_i x, \tilde{b}_i)\} = \alpha \tag{26}$$

In which, $\tilde{a}_i = (\tilde{a}_{i1}, \tilde{a}_{i2}, \dots, \tilde{a}_{in})$. Another form for (26) is:

$$\tilde{a}_i x \geq_\alpha \tilde{b}_i \quad i = 1, \dots, m \tag{27}$$

Based on (9) and (25), the following key relationship can be inferred:

$$\left[(1-\alpha)E_2^{a_i} + \alpha E_1^{a_i}\right] x \geq \alpha E_2^{b_i} + (1-\alpha)E_1^{b_i} \tag{28}$$

Based on the previous equation, the first challenge is mitigated. With regard to the second challenge, the following definition solves the problem:

Vector $x^0 \in R^n$ is an acceptable optimal solution for models (23) and (24) if it is an optimal solution of the following problem:

$$\min \quad EV(\tilde{c})x \tag{29}$$

$$\text{s.t.} \quad x \in \aleph_\alpha(\tilde{A}, \tilde{b}) = \{x \in R^n | \tilde{a}_i x \geq_\alpha \tilde{b}_i, i = 1, \dots, m, x \geq 0\} \tag{30}$$

In which, $EV(\tilde{c}) = (EV(\tilde{c}_1), EV(\tilde{c}_2), \dots, EV(\tilde{c}_n))$. The above-mentioned model is a crisp α-parametric model, which is very difficult to solve if α is considered as a variable. To solve this problem, it is common to consider a specific set as potential values for α. We solve the model for discrete values of α, i.e., when $\alpha_k \in M$, where generally:

$$M = \{\alpha_k = \alpha_0 + mk | k = 0,1, \dots, \frac{1-\alpha_0}{m}\} \subset [0,1] \tag{31}$$

In which, α_0 is the minimum constraint, the feasibility degree depends on the decision-maker (DM) of the system, and m is a step for incrementing α. In the specific model of this paper, there is equality (16) that does not allow α to be more than 0.5. Moreover, we suppose that α_0=0.1, m=0.05. Therefore, in our specific model, the set M is:

$$M = \{\alpha_k = 0.1 + 0.05k | k = 0,1,\ldots,\frac{0.5 - 0.1}{0.05} = 8\} \subset [0.1,0.5] \quad (32)$$

A set of α_k-acceptable optimal solutions will be obtained as $O = \{x^0(\alpha_k), \alpha_k \in M\}$, and based on each $x^0(\alpha_k)$, the value of $\tilde{z}^0(\alpha_k)$ is calculated as follows:

$$\tilde{z}^0(\alpha_k) = \tilde{c}x^0(\alpha_k) \quad (33)$$

When $\tilde{c}$ has a triangular MF, $\tilde{z}^0(\alpha_k)$will also be a triangular fuzzy number because $x^0(\alpha_k)$ is a crisp vector.

After observing different values of $\tilde{z}^0(\alpha_k)$, the DM should consider a tradeoff between the feasibility degree of the problem, denoted by α, and the possibility of reaching an acceptable value for the objective function. In order to handle the second issue, it is common to define a goal function, which is used as a comparison tool for identifying the degree of satisfaction for each $\tilde{z}^0(\alpha_k)$. This satisfaction degree is defined as follows:

$$\mu_{\tilde{G}}(z) = \begin{cases} 1; \ if \ \ z \leq \underline{G} \\ \lambda \in \{0,1\} \ \ \text{decreasing on} \ \ \underline{G} \leq z \leq \bar{G} \\ 0; \ if \ \ z \geq \bar{G} \end{cases} \quad (34)$$

Based on this, when $z \leq \underline{G}$, it is completely satisfactory; however, when $z \geq \underline{G}$, it is completely unsatisfactory. Now, an index, proposed by Yager [38], will be used to compute the degree of satisfaction for the fuzzy goal $\tilde{G}$ by each α-acceptable optimal solution (see Figure 4). The Yager's index is:

$$K_{\tilde{G}}(z^0(\alpha)) = \frac{\int_{-\infty}^{+\infty} \mu_{\tilde{z}^0(\alpha)}(z).\mu_{\tilde{G}}(z)dz}{\int_{-\infty}^{+\infty} \mu_{\tilde{z}^0(\alpha)}(z)dz} \quad (35)$$

The final step of the solution procedure is to balance the feasibility and the optimality of the solution. We define two fuzzy sets $\tilde{F}, \tilde{S}$. Let

$$\mu_{\tilde{F}}(x^0(\alpha_k)) = \alpha_k \quad (36)$$

$$\mu_{\tilde{S}}(x^0(\alpha_k)) = K_{\tilde{G}}(\tilde{z}^0(\alpha_k)) \quad (37)$$

Now, we are ready to define a fuzzy decision $\tilde{D} = \tilde{F} \cap \tilde{S}$, i.e.,:

$$\mu_{\tilde{D}}(x^0(\alpha_k)) = \alpha_k * K_{\tilde{G}}(\tilde{z}^0(\alpha_k)) \quad (38)$$

Where * is an arbitrary t-norm operator, such as the minimum, the drastic product, the algebraic product, and so on.

Therefore, $x^* \in O$ is the final solution with the highest membership degree in the fuzzy set decision ($\tilde{D}$) if:

$$\mu_{\tilde{D}}(x^*) = \max_{\alpha_k \in M}\{\alpha_k * K_{\tilde{G}}(\tilde{z}^0(\alpha_k))\} \quad (39)$$

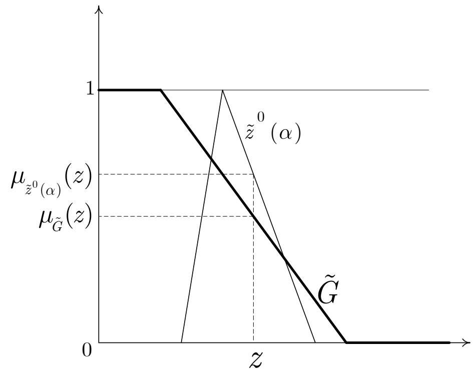

Figure 4. Possibility of occurrence for a crisp objective value z and its goal satisfaction degree.

4.2. The Type II fuzzy approach

In this section, we assume a fuzzy type II membership function for the flow parameter in the network. There are two general types of type II membership functions, i.e., Interval Type II, where the MF value for each x in the universe of discourse is an interval and not a single number (similar to the type I system), and Total Type II, where the MF value for each x in the universe of discourse has an MF itself, which is neither a single number (similar to the type I system) nor an interval (similar to the type II interval system).

In this paper, the interval type II fuzzy system is considered for the parameters, and the required relations will be explained.

For each interval type II fuzzy number (IT2FN, Figure 5), which is triangular in this case, define:

$$\tilde{A} = ([A_{11}, A_{12}], A_m, [A_{21}, A_{22}]) \tag{40}$$

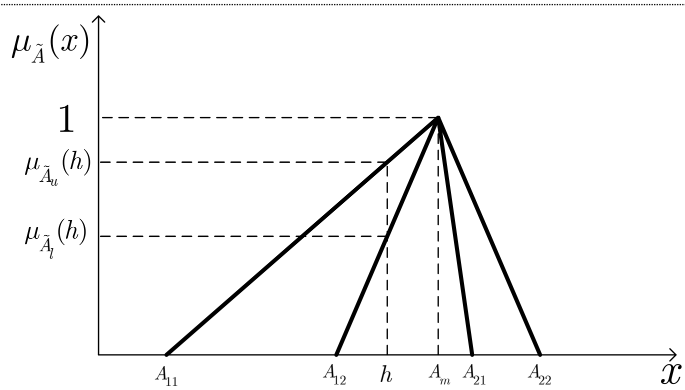

Figure 5. An Interval type II fuzzy number

Now, taking advantage of the interval arithmetic, shown below, we can develop the previous relations of type I system for an IT2 system:

$$[a, b] + [c, d] = [a + c, b + d] \tag{41}$$

$$[a, b] - [c, d] = [a - c, b - d] \tag{42}$$

$$[a, b].[c, d] = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)] \tag{43}$$

$$[a, b]/[c, d] = [a, b].[1/c, 1/d] \tag{44}$$

To develop the relations, the expected interval can be calculated as:

$$EI(\tilde{a}) = [E_1^a, E_2^a] = [[E_{11}^a, E_{12}^a], [E_{21}^a, E_{22}^a]] = \left[[\frac{1}{2}(A_{11} + A_m), \frac{1}{2}(A_{12} + A_m)], [\frac{1}{2}(A_{21} + A_m), \frac{1}{2}(A_{22} + A_m)]\right] \quad (45)$$

Thus, the expected value results from:

$$EV(\tilde{a}) = \frac{E_1^a + E_2^a}{2} = \frac{[E_{11}^a, E_{12}^a] + [E_{21}^a, E_{22}^a]}{2} = [\frac{E_{11}^a + E_{12}^a}{2}, \frac{E_{21}^a + E_{22}^a}{2}] \quad (46)$$

Based on the previous equation, we can start modeling the constraints in a mathematical model with IT2FN as parameters.

$$\left[(1-\alpha)[E_{21}^{a_i}, E_{22}^{a_i}] + \alpha[E_{11}^{a_i}, E_{12}^{a_i}]\right]x \geq \alpha[E_{21}^{b_i}, E_{22}^{b_i}] + (1-\alpha)[E_{11}^{b_i}, E_{12}^{b_i}] \quad (47)$$

After simplification, we have:

$$\left[(1-\alpha)E_{21}^{a_i} + \alpha E_{11}^{a_i}, (1-\alpha)E_{22}^{a_i} + \alpha E_{12}^{a_i}\right]x \geq \left[\alpha E_{21}^{b_i} + (1-\alpha)E_{11}^{b_i}, \alpha E_{22}^{b_i} + (1-\alpha)E_{12}^{b_i}\right] \quad (48)$$

Again, we use the feasibility level of α for this equation, thus:

$$((1-\alpha)^2 E_{22}^{a_i} + \alpha(1-\alpha)E_{12}^{a_i} + \alpha(1-\alpha)E_{21}^{a_i} + \alpha^2 E_{11}^{a_i})x \geq \alpha^2 E_{22}^{b_i} + \alpha(1-\alpha)E_{12}^{b_i} + \alpha(1-\alpha)E_{21}^{b_i} + (1-\alpha)^2 E_{11}^{b_i} \quad (49)$$

The fuzzy set $\tilde{G}$ is also defined as an IT2FN. Let:

$$\mu_{\tilde{G}}(z) = [\mu_{\tilde{G}}^1(z), \mu_{\tilde{G}}^2(z)] = [f(z), g(z)] \quad (50)$$

$$\mu_{\tilde{z}^0(\alpha)}(z) = [\mu_{\tilde{z}^0(\alpha)}^1(z), \mu_{\tilde{z}^0(\alpha)}^2(z)] = [h(z), i(z)] \quad (51)$$

$$j(z) = \min(f(z)h(z), f(z)i(z), g(z)h(z), g(z)i(z)) \quad (52)$$

$$k(z) = \max(f(z)h(z), f(z)i(z), g(z)h(z), g(z)i(z)) \quad (53)$$

Then:

$$K_{\tilde{G}}(z^0(\alpha)) = \frac{\int_{-\infty}^{+\infty}[\mu_{\tilde{z}^0(\alpha)}^1(z), \mu_{\tilde{z}^0(\alpha)}^2(z)].[\mu_{\tilde{G}}^1(z), \mu_{\tilde{G}}^2(z)]dz}{\int_{-\infty}^{+\infty}[\mu_{\tilde{G}}^1(z), \mu_{\tilde{G}}^2(z)]dz} = \frac{\int_{-\infty}^{+\infty}[j(z), k(z)]dz}{\int_{-\infty}^{+\infty}[\mu_{\tilde{G}}^1(z), \mu_{\tilde{G}}^2(z)]dz} = \frac{[\int_{-\infty}^{+\infty} j(z)dz, \int_{-\infty}^{+\infty} k(z)dz]}{[\int_{-\infty}^{+\infty} \mu_{\tilde{G}}^1(z)dz, \int_{-\infty}^{+\infty} \mu_{\tilde{G}}^2(z)dz]} = [\int_{-\infty}^{+\infty} j(z)dz, \int_{-\infty}^{+\infty} k(z)dz].[1/\int_{-\infty}^{+\infty} \mu_{\tilde{G}}^1(z)dz, 1/\int_{-\infty}^{+\infty} \mu_{\tilde{G}}^2(z)dz] = [K_{\tilde{G}}^1(\tilde{z}^0(\alpha_k)), K_{\tilde{G}}^2(\tilde{z}^0(\alpha_k))] \quad (54)$$

We have to develop the IT2F relations for the decision-making index as well. i.e.:

$$[\mu_{\tilde{D}}^1(x^0(\alpha_k)), \mu_{\tilde{D}}^2(x^0(\alpha_k))] = \alpha_k * [K_{\tilde{G}}^1(\tilde{z}^0(\alpha_k)), K_{\tilde{G}}^2(\tilde{z}^0(\alpha_k))] \quad (55)$$

$$[\mu_{\tilde{D}}^1(x^*), \mu_{\tilde{D}}^2(x^*)] = \max_{\alpha_k \in M}\{\alpha_k * [K_{\tilde{G}}^1(\tilde{z}^0(\alpha_k)), K_{\tilde{G}}^2(\tilde{z}^0(\alpha_k))]\} \quad (56)$$

5. The Case Study

5.1. The Crisp and Type I fuzzy approach

Kalleh Dairy Co. was established in 1991 with a vast range of products from cheese to dessert and ice cream. The main factory absorbs 2500 tons of raw milk and the result is 1800 tons of products each day. In this process, more than 4000 employees involved directly. The company has 42 branches around the country. Before running the result of this research, the products were sent directly from the factory in Amol city to these branches and then distributed to stores in nearby cities. (Figure 6)

Figure 6. The primal network of sending products from factory to branches

For this primal network the total cost of transportation was as follows: (w_j is the amount of products sent from factory to the branch j per kg and c_j is the average cost of transportation per kg)

$$\sum_{j=1}^{42} c_j w_j = 98{,}520(\$/day)$$

In the first phase, we applied the model of section 3 with crisp amount of parameters on the network. Using experts opinion we considered the following amounts $p = 10$, $\lambda = (0,\dots,0,\overbrace{1,\dots,1}^{10})$, $\beta_{ij} = 0.3\ \forall i,j$ and $(\delta_{ij} = 0.5) \le 1\ \forall i,j$. The amount of objective function with exact method using GAMS 23.5 software is as follows:

$$\sum_{i=1}^{42}\sum_{j=1}^{42}\sum_{k=1}^{42} \lambda_i c_{jk} r_{jk}^i W_j + \sum_{k=1}^{42}\sum_{l=1}^{42}\sum_{m=1}^{42} x_{klm}(\beta c_{kl} + \delta c_{lm}) = 89{,}320\ (\$/day)$$

In the second phase, we consider w_{ij} as a fuzzy parameter, and each w_{ij} has a triangular MF. We apply the model of section 3 using the procedure of section 4 to obtain a solution with the same p, λ, $\beta$and δ as phase I. The problem is executed for all $\alpha_k \in M$.

$$M = \{\alpha_k = 0.1 + 0.05k | k = 0,1,\dots,\frac{0.5-0.1}{0.05} = 8\} \subset [0.1,0.5].$$

To execute the solution procedure, we linked GAMS 23.5 and MATLAB R2010a. In this procedure, firstly, the *EV* and *EI* for each fuzzy parameter are built in MATLAB. Then, these parameters are entered into GAMS as input. Afterward, GAMS will solve the crisp α-parametric MIP model, and the results are returned to MATLAB. Next, MATLAB builds $\mu_{\tilde{G}}$ and $K_{\tilde{G}}$ functions, and finally, it uses the $\mu_{\tilde{D}}$ function to compare $x^0(\alpha_k)$, and it returns x^*. The results are presented in the following tables:

Table 1. Feasibility, optimality, decision making indices and MF of objective value for p=10

k	α_k	$K_{\tilde{G}}(\tilde{z}^0(\alpha_k))$	$\mu_{\tilde{D}}(x^0(\alpha_k))$	$z_l(\alpha_k)$	$z_m(\alpha_k)$	$z_u(\alpha_k)$
٠	٠.١	١	٠.١	٨٣,٧٣٩	٨٥,٧٨٦	٨٧,٤٩٧
١	٠.١٥	١	٠.١٥	٨٤,٠٤٨	٨٦,٠٩٥	٨٧,٨٠٦
٢	٠.٢	١	٠.٢	٨٤,٣٥٧	٨٦,٤٠٤	٨٨,١١٥
٣	٠.٢٥	٠.٩٨٨٢	٠.٢٤٧٠٥	٨٤,٨٢٩	٨٦,٦٩٨	٨٨,٣٦٣
٤	٠.٣	٠.٩٥٩٢	٠.٢٨٧٧٦	٨٥,٢٤٦	٨٧,١١٢	٨٨,٦٧٧
٥	٠.٣٥	٠.٩٢٠٦	٠.٣٢٢٢١	٨٥,٥٥٥	٨٧,٤٢١	٨٨,٩٨٦
٦	٠.٤	٠.٨٣٤٨	٠.٣٣٣٩٢	٨٥,٨٦٤	٨٧,٧٣٠	٨٩,٢٩٥
٧	٠.٤٥	٠.٦٨٣٢	٠.٣٠٧٤٤	٨٦,٣٣٦	٨٨,٠٢٤	٨٩,٥٤٣
٨	٠.٥	٠.٥٢٢٥	٠.٢٦١٢٥	٨٦,٦٤٥	٨٨,٣٣٣	٨٩,٨٥٢

It can clearly be seen that the best value for $\mu_{\tilde{D}}(x^0(\alpha_k))$=0.3339 occurs at k=6, which in the feasibility level is 0.4 and the optimality level is 0.8348. The value of $z_m(\alpha_k)$ is 87,730 which shows 1.8% improvement in comparison with phase I (crisp approach). The structure of the network for the solution and the position of hubs are shown below. (Figure 7)

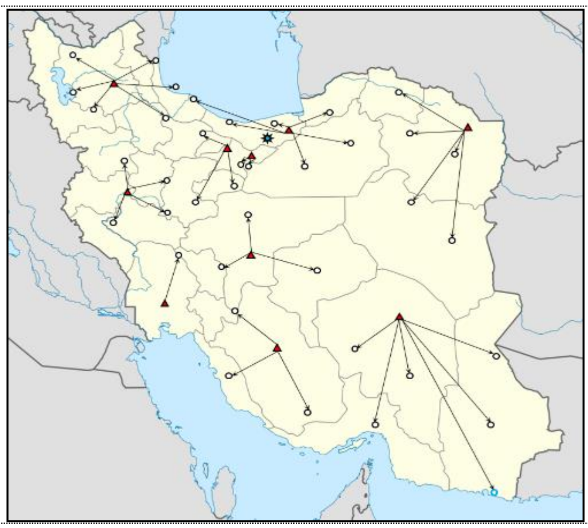

Figure 7. The network of Kalleh transportation for the solution of fuzzy model.

5.2. The Type II fuzzy approach

Our IT2FN in this model is the flow in the network. Define:

$$\tilde{w}_{ij} = ([w_{ij}^{11}, w_{ij}^{12}], w_{ij}^{m}, [w_{ij}^{21}, w_{ij}^{22}])\forall i, j \quad (57)$$

Now we utilize the IT2 approach for the case of paper. All other parameters in our case have the same amount as the previous section. Therefore, we obtain the following table:

Table 2. Feasibility, optimality, decision making indices and MF of objective value for *p*=10 (fuzzy type II approach)

k	α_k	$K_{\tilde{G}}^1(\tilde{z}^0(\alpha_k))$	$K_{\tilde{G}}^2(\tilde{z}^0(\alpha_k))$	$\mu_{\tilde{D}}^1(x^0(\alpha_k))$	$\mu_{\tilde{D}}^2(x^0(\alpha_k))$	$z^{11}(\alpha_k)$	$z^{12}(\alpha_k)$	$z^{m}(\alpha_k)$	$z^{21}(\alpha_k)$	$z^{22}(\alpha_k)$
٠	٠.١	1.0000	1.0000	0.10000	0.10000	82,064	84,576	85,786	86,622	89,072
١	٠.١٥	1.0000	1.0000	0.15000	0.15000	82,199	84,931	86,095	86,902	89,439
٢	٠.٢	1.0000	1.0000	0.20000	0.20000	82,332	85,285	86,404	87,181	89,807
٣	٠.٢٥	0.9873	0.9925	0.24683	0.24813	82,623	85,805	86,698	87,400	90,113
٤	٠.٣	0.9555	0.9885	0.28665	0.29655	82,859	86,269	87,112	87,684	90,486
٥	٠.٣٥	0.8653	0.9562	**0.30286**	**0.33467**	82,988	86,624	**87,421**	87,963	90,855
٦	٠.٤	0.7312	0.8355	0.29248	0.33420	83,116	86,980	87,730	88,241	91,224
٧	٠.٤٥	0.5726	0.6836	0.25767	0.30762	83,401	87,502	88,024	88,460	91,531
٨	٠.٥	0.3305	0.5227	0.16525	0.26135	83,526	87,858	88,333	88,738	91,901

It is clear from the table above that the best interval for $\mu_{\tilde{D}}(x^0(\alpha_k))$ occurs at *k*=5, which in the feasibility level is 0.35 and the optimality level interval is [0.8653, 0.9562]. The value of $z^m(\alpha_k)$ is 87,421 which shows 0.4% improvement in comparison with fuzzy type I approach and 2.1% improvement in comparison with crisp approach. The structure of the network for the solution and the position of hubs will remain same as fuzzy type I approach. (Figure 7)

6. Conclusion

This paper can be considered as the first attempt to solve an ordered p-hub median problem using a fuzzy programming approach. It's also the first time that the fuzzy type II approach is utilized to solve a hub location problem. We explained both crisp and fuzzy mathematical models and the solution method for fuzzy linear programming based on Yager's index. The method of the paper allows us to take a decision interactively with the DM. Through the idea of a feasible optimal solution in degree α, the DM has enough information to fix an aspiration level. The DM can also choose the degrees of feasibility that he/she is willing to admit depending on the context. It is important to highlight that the acceptable optimal solutions in degree α are not fuzzy quantities, which makes it easier to decide a simple way by solving a crisp parametric linear program. The DM also has additional information about the risk of violation of the constraints, and about the compatibility of the cost of the solution with his wishes for the values of the objective function. The DM can intervene in all the steps of the decision process which makes our approach very useful to be applied in a lot of real-world problems where the information is uncertain or incomplete. Finally, a computational test with the transportation network of Kalleh Dairy Co. was described. A possible extension for the work in this paper is to use the Total fuzzy type II concept instead of Interval fuzzy type II in the mathematical programming of the model.

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