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An approximate method for solving optimal control problems with Chebyshev cardinal wavelets

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In this paper, an efficient method is proposed for solving nonlinear quadratic optimal control problems with inequality constraints. The method is based upon Chebyshev cardinal wavelets. The operational matrix of integration is given for related procedures. This matrix is used to reduce the solution of the nonlinear constrained optimal control to a nonlinear programming one to which existing well-developed algorithms may be applied. Finally, the applicability and validity of method are shown by numerical results of some examples. Moreover, the comparison with the existing results show the preference of this method.

***Keywords**: Optimal control problem, Chebyshev cardinal wavelets, Operational matrix, Nonlinear programming.*

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1. Introduction

Solving an optimal control problem is complicated. Due to the complexity of the majority of the applications, optimal control problems are often solved numerically. Numerical methods for solving optimal control problems date back to nearly seven decades, the 1950s with the work of Bellman et al. [2, 3, 4]. Numerical methods for solving optimal control problems are divided into two major classifications, direct methods and indirect methods. In an indirect method, the calculus of variations [14, 19] is applied to determine the first-order optimality conditions of the original optimal control problem. The indirect approach leads to a multiple-point boundary-value problem that is solved to determine the candidate optimal trajectories called extremals. Each of the computed extremals is then examined to see if it is a local minimum, maximum, or a saddle point. Among the locally optimizing solutions, the specific extremal with the lowest price is selected. One of the widely used methods to solve optimal control problems is the direct method. There is a large number of studies that apply this method to solve optimal control problems (see for example [5, 6, 9, 16, 21, 22, 24] and the references therein). This method transforms the optimal control problem into a mathematical programming problem by using either the discretization technique [5, 6] or the parameterization technique [9, 21, 22, 24]. The discretization technique converts the optimal control problem into a nonlinear programming problem with a large number of unknown parameters and a large number of constraints [6]. On the other hand, parameterizing the control variables [9, 24] needs the integration of the state equations. While the simultaneous parameterization of both the state variables and the control variables [24], results in a nonlinear programming problem with a large number of parameters and a large number of equality constraints. In the last several years, various methods have been proposed to solve these problems. Yen and Nagurka [33] proposed a method based on the state parameterization, using Fourier series, to solve the linear-quadratic optimal control problem (with equal number of state variables and control variables) subject to state

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and control inequality constraints. Also, Razzaghi and Elnagar proposed a method to solve the unconstrained linear-quadratic optimal control problem with equal number of state and control variables [31]. Their approach is based on using the shifted Legendre polynomials in order to parameterize the derivation of each of the state variables. In [16] Jaddu and Shimemura proposed a method to solve the linear-quadratic and the nonlinear optimal control problems by using Chebyshev polynomials to parameterize some of the state variables, then the remaining state variables and the control variables are determined from the state equations. The approach proposed in [24] is based on approximating the state variables and control variables with hybrid functions.
The aim of this paper is developing a computational approach for solving nonlinear constrained quadratic optimal control problems by using Chebyshev cardinal wavelets. The method is based on approximating the state variables and the control variables with Chebyshev cardinal wavelets [13]. We remind that the Chebyshev cardinal wavelets have advantages such as cardinality, orthogonality and spectral accuracy. The key advantage of these new wavelets, in comparison with other popular wavelets, such as the Chebyshev wavelets [12] and the Legendre wavelets [10], is their cardinality property. The cardinality property saves us from computing some integrals that are often appeared in evaluating the coefficients of the Chebyshev cardinal wavelets expansion of a function. In fact, the intended coefficients are achieved by computing the values of the considered function at some grid points which are also utilized in generating these wavelets. It must be noted that Chebyshev cardinal wavelets contain both features of the wavelets and Chebyshev cardinal functions.

This paper considers the following sections: In Section 2 we describe the basic formulation of the Chebyshev cardinal wavelets required for our subsequent development. Section 3 is devoted to the formulation of optimal control problems. Section 4 summarizes the application of these methods to the optimal control problems and we report our numerical findings and demonstrate the accuracy of the proposed methods. Sections 5 completes this paper with a brief conclusion.

2. Properties of Chebyshev cardinal wavelets

The Chebyshev cardinal wavelets are reviewed in summary, and interested properties are presented in this section.

2.1. The Chebyshev cardinal wavelets

By using the process of building polynomial wavelets which has been defined in [11, 13], we can define the Chebyshev cardinal wavelets over $[0,1]$ as follows:

$$\hat{\psi}_{nm}(t)=\begin{cases}\sqrt{\dfrac{2M}{\pi}}\,2^{\frac{k}{2}}C_m(2^{k+1}t-2n+1) & \dfrac{n-1}{2^k}\le t<\dfrac{n}{2^k},\\ 0 & \text{otherwise,}\end{cases} \tag{1}$$

where t is an independent variable defined on $[0,1]$, k is an arbitrary non-negative integer, $n=1,2,\dots,2^k$ and C_m is the Chebyshev cardinal function of order m. Note that the coefficient $\sqrt{\dfrac{2M}{\pi}}$ is used for normality. The set $\{\hat{\psi}_{nm}(t)\,|\,n=1,2,\dots,2^k, m=1,2,\dots,M,\ M\in\mathbb{N}\}$ generates an orthonormal basis for $L^2_{w_n}[0,1]$ (the subscript indicates weighted orthogonality), i.e.

$$\langle\hat{\psi}_{nm}(t),\hat{\psi}_{n'm'}(t)\rangle_{w_n}=\int_0^1\hat{\psi}_{nm}(t)\hat{\psi}_{n'm'}(t)w_n(t)\mathrm{d}t=\begin{cases}1 & (n,m)=(n',m'),\\ 0 & (n,m)\ne(n',m'),\end{cases} \tag{2}$$

where

$$w_n(t)=\begin{cases} w(2^{k+1}t-2n+1) & \frac{n-1}{2^k}\le t<\frac{n}{2^k}, \\ 0 & otherwise. \end{cases} \quad (3)$$

By the usage of some simplifications, Eq. (1) can be rephrased in the following form

$$\hat{\psi}_{nm}(t)=\begin{cases} \sqrt{\frac{2M}{\pi}}2^{\frac{k}{2}}\prod_{l=1,l\neq m}^{M}(\frac{t-\xi_{nl}}{\xi_{nm}-\xi_{nl}}) & \frac{n-1}{2^k}\le t<\frac{n}{2^k}, \\ 0 & otherwise, \end{cases} \quad (4)$$

where $\xi_{nm}=\frac{1}{2^{k+1}}(\eta_m+2n-1)$ for $n=1,2,\dots,2^k$ and $m=1,2,\dots,M$, and the values $\eta_m=-cos(\frac{(2m-1)\pi}{2M})$ are the zeros of the Chebyshev polynomial [7] of order M defined over the interval $[-1,1]$ for $m=1,2,\dots,M$.

For constructing a wavelet basis with the interpolation property, we assume a revised form of Eq. (4) as follows:

$$\psi_{nm}(t)=\begin{cases} \prod_{l=1,l\neq m}^{M}(\frac{t-\xi_{nl}}{\xi_{nm}-\xi_{nl}}) & \frac{n-1}{2^k}\le t<\frac{n}{2^k}, \\ 0 & otherwise. \end{cases} \quad (5)$$

We remind that the set $\{\psi_{nm}(t)\,|\,n=1,2,\dots,2^k,\, m=1,2,\dots,M,\, M\in\mathbb{N}\}$ forms an orthogonal basis with respect to the weight function $w_n(t)$ for $L^2_{w_n}[0,1]$ and

$$\langle\psi_{nm}(t),\psi_{n'm'}(t)\rangle_{w_n}=\int_0^1\psi_{nm}(t)\psi_{n'm'}(t)w_n(t)\mathrm{d}t=\begin{cases} \frac{\pi}{M\,2^{k+1}} & (n,m)=(n',m'), \\ 0 & (n,m)\neq(n',m'). \end{cases} \quad (6)$$

2.2. Function of a variable approximation

Any function $u(t)\in L^2_{w_n}[0,1]$ can be approximated by the Chebyshev cardinal wavelets as follows:

$$u(t)\simeq\sum_{n=1}^{2^k}\sum_{m=1}^{M}c_{nm}\psi_{nm}(t)=C^T\Psi(t), \quad (7)$$

where

$$C=[c_{11},c_{12},\dots,c_{1M}\,|\,c_{21},c_{22},\dots,c_{2M}\,|\dots|\,c_{2^k1},c_{2^k2},\dots,c_{2^kM}]^T, \quad (8)$$

$$\Psi(t)=[\psi_{11}(t),\psi_{12}(t),\dots,\psi_{1M}(t)\,|\,\psi_{21}(t),\dots,\psi_{2M}(t)\,|\dots|\,\psi_{2^k1}(t),\dots,\psi_{2^kM}(t)]^T, \quad (9)$$

and

$$c_{nm}=u(\xi_{nm}),\quad n=1,2,\dots,2^k,\quad m=1,2,\dots,M. \quad (10)$$

It has to be considered that c_{nm} are the items of the vector C.

2.3. Function of two variables approximation

Let $u(t,s)$ be a function of two variables defined for $t\in[0,1]$ and $s\in[0,1]$. Then $u(t,s)$ can be expanded as following,

$$u(t,s)=\Psi^T(t)U\Psi(s). \tag{11}$$

The following explanation clarifies the above statement:

Remark 2.3.1. Eq. (7) can be expressed in a more simple form as follows

$$v(t);\ \sum_{i=1}^{\hat{m}} v_i\psi_i(t)=V^T\Psi(t), \tag{12}$$

where $\hat{m}=2^kM$, $v_i=v_{nm}$ and $\psi_i(t)=\psi_{nm}$ for the index $i=(n-1)M+m.$

Remark 2.3.2. The Chebyshev cardinal wavelets can be used to expand any function $u\in L^2_{w_{n.n'}}([0,1]\times[0,1])$,

$$u(x,t);\ \sum_{p=1}^{\hat{m}}\sum_{q=1}^{\hat{m}} u(x_p,t_q)\psi_p(t)\psi_q(t)=\Psi^T(t)U\Psi(s), \tag{13}$$

where $U=[u_{pq}]$ and its elements are computed as $u_{pq}=u(x_p,t_q)$.

For example, we have compared the graph of the function $u(x,t)=sin(10xt)-t$ (Fig. 1) with Fig. 2 and Fig. 3, for $M=3,k=1$ and $M=3,k=2$, respectively.

2.4. The operational matrix of integration

The operational matrix of integration of the Chebyshev cardinal wavelets have been derived in [13]. The integration of the vector $\Psi(t)$ defined in Eq. (9) can be approximated as

$$\int_0^t\Psi(\tau)\mathrm{d}\tau;\ \ P\Psi(t), \tag{14}$$

where P is called the operational matrix of integration for Chebyshev cardinal wavelets which is an $\hat{m}$ order square matrix and has the following form

$$P=\begin{pmatrix} A & B & B & B & \dots & B\\ 0 & A & B & B & \dots & B\\ 0 & 0 & \ddots & \ddots & \ddots & B\\ \vdots & \vdots & \ddots & A & B & B\\ 0 & 0 & \dots & 0 & A & B\\ 0 & 0 & 0 & \dots & 0 & A\end{pmatrix}_{\hat{m}\times\hat{m}}. \tag{15}$$

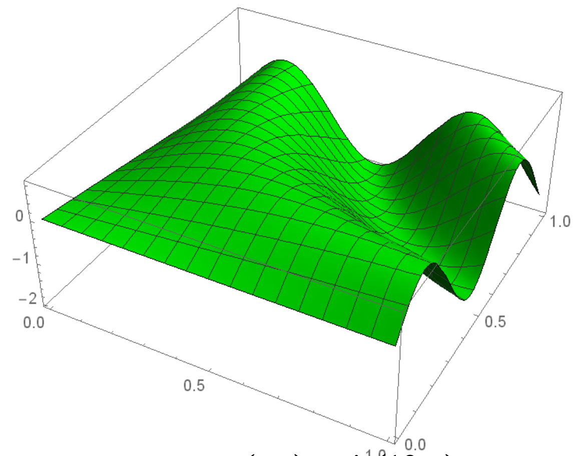

Figure 1. $u(x,t) = sin(10xt) - t$

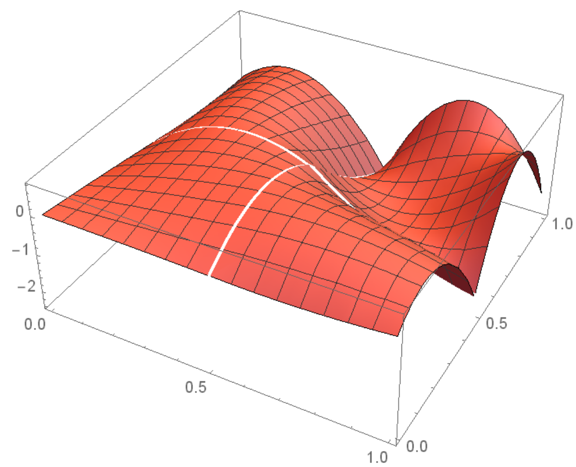

Figure 2. Approximation of $u(x,t)$ with $M = 3,\ k = 1$

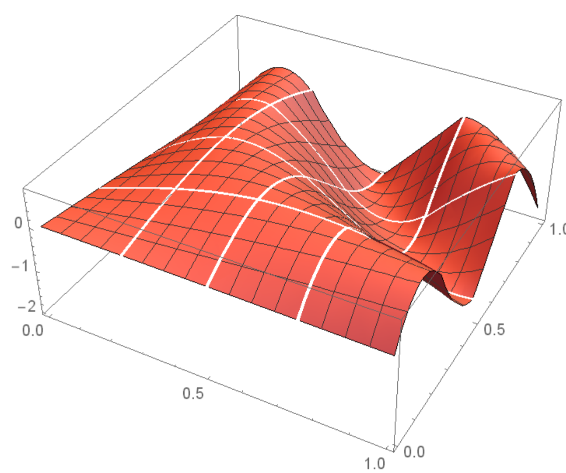

Figure 3. Approximation of $u(x,t)$ with $M = 3,\ k = 2$

In Eq. (15), $A=[a_{ij}]$ and $B=[b_{ij}]$ are $M\times M$ matrices which their elements are obtained by the following relations:

$$a_{ij}=\frac{1}{2^{k+1}}\int_{-1}^{\eta_j}\prod_{l=1,l\neq i}^{M}(\frac{\tau-\eta_l}{\eta_i-\eta_l})\mathrm{d}\tau,\qquad b_{ij}=\frac{1}{2^{k+1}}\int_{-1}^{1}\prod_{l=1,l\neq i}^{M}(\frac{\tau-\eta_l}{\eta_i-\eta_l})\mathrm{d}\tau. \tag{16}$$

As an illustrative example for $M=2,\ k=1$ and $M=3,\ k=1$, we have

$$P=\begin{pmatrix} \frac{1}{8}-\frac{1}{16\sqrt{2}} & \frac{1}{8}+\frac{3}{16\sqrt{2}} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{8}-\frac{3}{16\sqrt{2}} & \frac{1}{8}+\frac{1}{16\sqrt{2}} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{8}-\frac{1}{16\sqrt{2}} & \frac{1}{8}+\frac{3}{16\sqrt{2}} \\ 0 & 0 & \frac{1}{8}-\frac{3}{16\sqrt{2}} & \frac{1}{8}+\frac{1}{16\sqrt{2}} \end{pmatrix}_{4\times 4},$$

and

$$P=\begin{pmatrix} \frac{1}{18}-\frac{1}{32\sqrt{3}} & \frac{1}{18}+\frac{1}{8\sqrt{3}} & \frac{1}{18}+\frac{\sqrt{3}}{32} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{5}{36}-\frac{1}{4\sqrt{3}} & \frac{5}{36} & \frac{5}{36}+\frac{1}{4\sqrt{3}} & \frac{5}{18} & \frac{5}{18} & \frac{5}{18} \\ \frac{1}{18}-\frac{\sqrt{3}}{32} & \frac{1}{18}-\frac{1}{8\sqrt{3}} & \frac{1}{18}+\frac{1}{32\sqrt{3}} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ 0 & 0 & 0 & \frac{1}{18}-\frac{1}{32\sqrt{3}} & \frac{1}{18}+\frac{1}{8\sqrt{3}} & \frac{1}{18}+\frac{\sqrt{3}}{32} \\ 0 & 0 & 0 & \frac{5}{36}-\frac{1}{4\sqrt{3}} & \frac{5}{36} & \frac{5}{36}+\frac{1}{4\sqrt{3}} \\ 0 & 0 & 0 & \frac{1}{18}-\frac{\sqrt{3}}{32} & \frac{1}{18}-\frac{1}{8\sqrt{3}} & \frac{1}{18}+\frac{1}{32\sqrt{3}} \end{pmatrix}_{6\times 6}.$$

2.5. The integration of the cross product

The integration of the cross product of two Chebyshev cardinal wavelets vectors $\Psi(t)$ is

$$D=\int_0^1\Psi(t)\Psi(t)^T\,\mathrm{d}t=\begin{pmatrix} L & 0 & \dots & 0 \\ 0 & L & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & L \end{pmatrix}_{2^k M\times 2^k M}, \tag{17}$$

where L is an $M\times M$ matrix.

3. Problem statement

Consider the following class of nonlinear systems with inequality constraints,

$$\dot{x}(t) = F(t, x(t), u(t)) \tag{18}$$

$$S_j(t, x(t), u(t)) \le 0, \qquad j = 1, 2, \ldots, w, \qquad t \in [0,1] \tag{19}$$

$$x(0) = x_0 \tag{20}$$

where $x(t)$ and $u(t)$ are $j_1 \times 1$ and $j_2 \times 1$ state and control vectors, respectively. The aim of this paper is finding the numerical approximation of optimal control u(t) and the corresponding state trajectory $x(t)$, $0 \le t \le 1$ satisfying Eqs. (18)–(20) while minimizing (or maximizing) the quadratic performance index

$$J = \frac{1}{2} x^T(1) \mathbf{Z} x(1) + \frac{1}{2} \int_0^1 (x^T(t) \mathbf{Q}(t) x(t) + u^T(t) \mathbf{R}(t) u(t)) \mathrm{d}t, \tag{21}$$

where $\mathbf{Z}$ and $\mathbf{Q}(t)$ are positive semi-definite matrices, and $\mathbf{R}(t)$ is a positive definite matrix.

4. Description of the proposed method via examples

4.1. Example 1

This example is adapted from [17]. Find the control vector $u(t)$ which minimizes

$$J = \frac{1}{2} \int_0^1 (x_1^2(t) + u^2(t)) \mathrm{d}t, \tag{22}$$

subject to

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \tag{23}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}, \tag{24}$$

and the following inequality control constraint

$$|u(t)| \le 1. \tag{25}$$

We solve this problem by choosing $M = 5$ and $k = 2$. Let

$$\dot{x}_1(t) = W_1^T \Psi(t), \tag{26}$$

$$\dot{x}_2(t) = W_2^T \Psi(t), \tag{27}$$

$$u(t)=U_2^T\Psi(t), \tag{28}$$

where W_1, W_2 and U_2 can obtained similarly to Eq. (8) and $\Psi(t)$ is given in Eq. (9). By expanding $x_2(0)$ in terms of Chebyshev cardinal wavelets, we get

$$x_2(0)=[10,10,\ldots,10]^T\,\Psi(t)=E_2^T\Psi(t). \tag{29}$$

Integrating of Eqs. (26) and (27) from 0 to t, we obtain

$$x_1(t)=W_1^T P\Psi(t), \tag{30}$$

$$x_2(t)=W_2^T P\Psi(t)+E_2^T\Psi(t), \tag{31}$$

where P is the operational matrix of integration given in Eq. (14). By substituting Eqs. (26)–(31) in Eqs. (23) and (25), we have

$$(W_1^T-W_2^T P-E_2^T)\Psi(t)=0, \tag{32}$$

$$(W_2^T+W_2^T P+E_2^T-U_2^T)\Psi(t)=0, \tag{33}$$

$$|U_2^T\Psi(t)|-1\le 0. \tag{34}$$

We collocate Eqs. (32)–(34) at $\xi_i, i=1,2,\ldots,2^k M$ given in Eq. (4), so

$$(W_1^T-W_2^T P-E_2^T)\Psi(\xi_i)=0, \tag{35}$$

$$(W_2^T+W_2^T P+E_2^T-U_2^T)\Psi(\xi_i)=0, \tag{36}$$

$$|U_2^T\Psi(\xi_i)|-1\le 0. \tag{37}$$

By substituting Eqs. (28) and (30) in Eq. (22), we have

$$J=\frac{1}{2}(W_1^T PD_1P^T W_1+U_2^T D_1 U_2), \tag{38}$$

where D_1 can be calculated similarly to Eq. (17). The problem has now been reduced to a parameter optimization problem as follows. Find vectors W_1, W_2 and U satisfying Eqs. (35)–(37) while minimizing Eq. (38). This problem can be solved by using package of Mathematica 7. In Table 1, the minimum of J using the rationalized Haar functions [29], hybrid of block-pulse and Legendre polynomials [21], hybrid of block-pulse and Bernoulli polynomials [24], linear B-spline functions [8], presented method together with the exact solution are listed. Also, Fig. 4 shows the behavior of state variables and control function obtained with $M=5$ and $k=2$.

4.2. Example 2

This example is adapted from [18] and also studied by using generalized gradient method [25], classical Chebyshev [32], Fourier-based state parametrization [34], rationalized Haar approach [29], hybrid of block-pulse and Legendre polynomials [21], hybrid of block-pulse and Bernoulli polynomials [24], interpolating scaling functions [9] and Linear B-spline functions [8]. Find the control vector $u(t)$ which minimizes

$$J = \frac{1}{2}\int_0^1 (x_1^2(t) + x_2^2(t) + 0.005u^2(t))\mathrm{d}t, \tag{39}$$

subject to

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \tag{40}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \tag{41}$$

Table 1. Estimated values of J for Example 1

Method	J
Rationalized Haar functions [29]	
$K = 4$	8.07473
$K = 5$	8.07065
Hybrid of block-pulse and Legendre [21]	
$N = 4, M_1 = 3$	8.07059
$N = 4, M_1 = 4$	8.07056
Hybrid of block-pulse and Bernoulli [24]	
$N = 4, M = 2$	8.07058
$N = 4, M = 3$	8.07055
Linear B-spline functions [8]	
$M = 8$	8.07055438812380
Presented Method	
$M = 5, k = 2$	8.070554085321948
Exact	8.07054

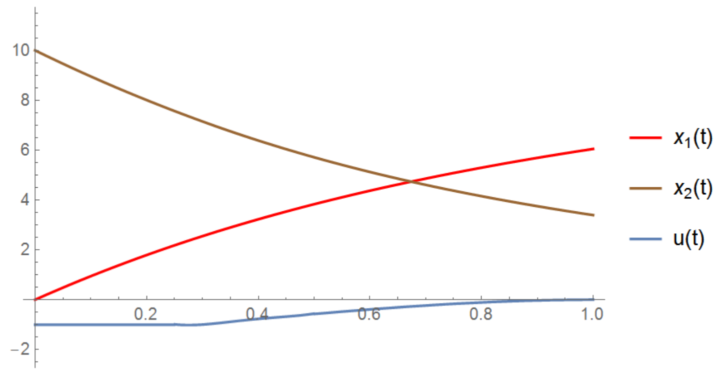

Figure 4. State variable $x_1(t)$, $x_2(t)$ and $u(t)$ obtained for $M=5$ and $k=2$ in Example 1

and the following state variable inequality constraint

$$x_2(t) \le r(t), \tag{42}$$

where

$$r(t) = 8(t-0.5)^2 - 0.5, \qquad 0 \le t \le 1. \tag{43}$$

We solve this problem by choosing $M=5$ and $k=2$. Let

$$\dot{x}_1(t) = W_3^T \Psi(t), \tag{44}$$

$$\dot{x}_2(t) = W_4^T \Psi(t), \tag{45}$$

$$u(t) = U_3^T \Psi(t), \tag{46}$$

where W_3, W_4 and U_3 can obtained similarly to Eq. (8). By expanding $r(t)$ and $x_2(0)$ in terms of Chebyshev cardinal wavelets, we get

$$r(t) = R^T \Psi(t), \tag{47}$$

$$x_2(0) = [-1,-1,\ldots,-1,-1]\Psi(t) = E_3^T \Psi(t) \tag{48}$$

Integrating, Eqs. (44) and (45) from 0 to t, we have

$$x_1(t) = W_3^T P \Psi(t), \tag{49}$$

$$x_2(t) = W_4^T P \Psi(t) + E_3^T \Psi(t), \tag{50}$$

where P is the operational matrix of integration given in Eq. (14). By substituting Eqs. (44)–(50) in Eqs. (40) and (42), and collocate the resulting equations at ξ_i given in Eq. (4), we obtain

$$(W_3^T - W_4^T P - E_3^T)\Psi(\xi_i) = 0, \quad (51)$$

$$(W_4^T + W_2^T P + E_3^T - U_3^T)\Psi(\xi_i) = 0, \quad (52)$$

$$(W_4 P + E_3^T - R^T)\Psi(\xi_i) \le 0. \quad (53)$$

By substituting Eqs. (46), (49) and (50) in Eq. (39), we have

$$J = (W_3^T P D_2 P^T W_3 + (W_4^T P + E_3^T) D_2 (P^T W_4 + E_3) + 0.005 U_3^T D_2 U_3), \quad (54)$$

where D_2 can be calculated similarly to Eq. (17). The problem has now been reduced to a parameter optimization problem as follows. Find vectors W_3, W_4 and U_3 satisfying Eqs. (51)–(53) while minimizing Eq. (54). This problem can be solved by using package of Mathematica 7 . In Table 2, we compare the minimum of J using the proposed method with other solutions in the literature. The computational result for $x_2(t)$ for $M = 5$ and $k = 2$ together with $r(t)$ are given in Fig. 5.

Table 2. Estimated values of J for Example 2

Method	J
Generalized gradient [25]	0.17800000
Classical Chebyshev [32]	
$M_2 = 6, K_1 = 12$	0.19600000
$M_2 = 11, K_1 = 22$	0.17880000
$M_2 = 13, K_1 = 26$	0.17358000
Fourier-based [34]	
$M_3 = 5$	0.17115
$M_3 = 7$	0.17096
$M_3 = 9$	0.17013
Rationalized Haar functions [29]	
$K = 64, w = 100$	0.170115
$K = 128, w = 100$	0:170103
Hybrid of block-pulse and Legendre [21]	
$N = 4, M_1 = 3$	0.17013645
$N = 4, M_1 = 4$	0.17013640
Hybrid of block-pulse and Bernoulli [24]	
$N = 4, M = 3$	0.1700305
$N = 4, M = 4$	0.1700301

Interpolating scaling functions [9]	
$n=4, r=5$	0.16982646
$n=5, r=5$	0.16982636
Linear B-spline functions [8]	
$M=8$	0.169811048165412
Present Method	
$M=5, k=2$	0.169677247546684

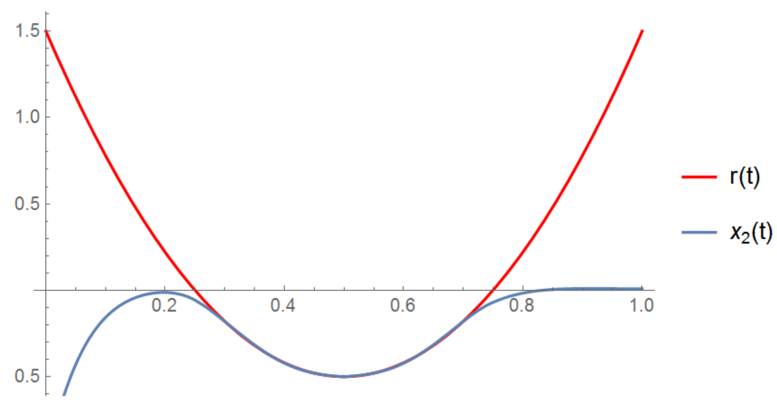

Figure 5: State variable $x_2(t)$ and $r(t)$ obtained for $M=5$ and $k=2$ in Example 2

5. Conclusion

In the presented study, the Chebyshev cardinal wavelets are used to solve nonlinear constrained optimal control problems. The problem has been reduced to a problem of solving a nonlinear programming one to which existing well-developed algorithms may be applied. The matrices P, and D in Eqs. (14) and (17) have large numbers of zero elements and they are sparse, hence the proposed method is very attractive and reduces the computer memory. Illustrative examples are given to demonstrate the validity and applicability of the proposed techniques.

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