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Additive slacks- based measure with undesirable output and feedback for a two-stage structure

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This paper develops slacks-based measure (SBM) and additive SBM (ASBM) to evaluate efficiency of decision making units (DMUs) in a two-stage structure with undesirable outputs and feedback variables from the internal perspective. The SBM model is linearized for a specific weight and the ASBM model is reformulated as a second order cone program. The target values for all inputs, outputs (both desirable and undesirable) and intermediate products are provided. This study shows that unlike the SBM model, ASBM can be adapted to the preference of the decision maker by selecting the weights to aggregate stages in the network.

Keywords*: Network data envelopment analysis, feedback, undesirable output, SOCP.*

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1. Introduction

Data envelopment analysis (DEA) is a tool for efficiency evaluation of a set of decision making units (DMUs) proposed by Charnes et al. [4]. In the primary models, the intermediate products of the underlying network system are ignored, and the system is considered as a black box. However, this may produce misleading results [3, 13, 14]. Thus, various network DEA models have been developed to consider the internal structure of DMUs, where stages are connected by the intermediate products aside the main inputs and final outputs [6, 8, 15, 16, 19]. The network DEA models are classified as radial and non-radial models, where the latter one is typified by the network slacks-based measure (NSBM) of Tone et al. [27]. Unlike the radial measure, in the NSBM models different inputs need not to be reduced in the same proportion in the input models, and different outputs need not to be expanded in the same proportion in the output models [17]. Furthermore, the inputs can be reduced and the outputs can be expanded simultaneously.

On the other hand, the classical DEA models such as NSBM models rely on the assumption that inputs are minimized and outputs are maximized, while the production process may produce undesirable outputs such as waste or pollution alongside the desirable outputs. For treatment of undesirable outputs in the network structure, two assumptions are made. The first one is weak disposability assumption proposed by Fare et al. [9]. Under this assumption, undesirable outputs are treated in their original forms. The second assumption is strong disposability. For treating undesirable outputs under this assumption, there are three main approaches: undesirable outputs are considered as inputs, undesirable outputs are used as outputs and the SBM approach, which deals with the undesirable outputs through the slacks of undesirable outputs [26]. This approach can measure the inefficiencies in both inputs and outputs simultaneously. Lozano [21] addressed the efficiency assessment of general network structures that produce both desirable and undesirable outputs based

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on the weak disposability of the undesirable outputs proposed by Kuosmanen [18]. His approach identified all sources of inefficiency and computed values for all the variables. To identify the inefficiency of Chinese commercial banks, An et al. [2] developed an NSBM model, where the increase of desirable outputs and the decrease of undesirable outputs are simultaneously considered. In another study, to assess the performance of water resource system, Zhou et al. [28] considered an extended two-stage NSBM model. The results obtained by their proposed approach can help the government departments concerned with identifying inefficient stages. Recently, Shi et al. [24] proposed a new NSBM model with undesirable outputs to evaluate the performance of both serial and parallel production processes. Their model can provide more accurate information regarding the efficiency improvement for the decision makers.

While these studies extended the empirical literature of the NSBM models, they also identified the limitations of them. These limitations may prolong the achievement of accurate efficiency measurement and consequently limit the identification of effective improvement strategies. The NSBM models are required to specify the weights for combining efficiencies of the stages that are functions of slacks. Green et al. [12] modified the additive DEA model of Charnes et al. [5] into an additive slacks-based measure (ASBM) and produced a nonlinear model to guarantee the efficiency scores of DMUs to lie between zero and unity. Chen et al. [7] revisited the ASBM model developed by Green et al. [12] and applied it for internal and external evaluations of multi-stage network structures having internal inputs and outputs. Also, they showed that the network ASBM model could be solved by a second order cone program (SOCP). In the current study, we develop both NSBM and ASBM models for a two-stage structure including feedback and undesirable outputs from the internal point of view [7, 17]. The NSBM model is linearized for a specific weight, while the ASBM model is reformulated as an SOCP problem independent of weights. Both models are applied for the efficiency evaluation of DMUs and for determination of slacks and target values on a real dataset including feedback and undesirable outputs. Results show that the largest slacks are found in the desirable outputs of the first stage of the ASBM model. Also, for the efficient DUMs in the first stage, with increasing weight, the overall inefficiency is clearly decreased in the ASBM model.

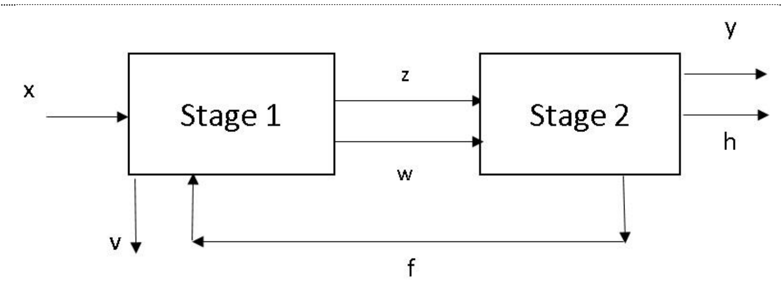


Figure 1. Two-stage system with feedback

The rest of our work is structured as follows. Section 2 focuses on the production possibility set (PPS) of the two-stage structure including feedback and undesirable outputs. Section 3 develops the NSBM model based on Kao's [17] approach. The ASBM model with its SOCP formulation is discussed in Section 4. Section 5 applies both models on the dataset of Li et al. [20] for the efficiency evaluations and comparison. Finally, Section 6 gives our concluding remarks.

2. The PPS in the two-stage structure

Suppose we have n DMUs, so that for $DMU_o, o \in \{1, \dots, n\}$, x is input to stage 1 and z, w and v are, respectively, desirable output, undesirable output jointly produced with desirable output, and desirable output not jointly produced with undesirable output in stage 1. Likewise, y is desirable output and h is undesirable output jointly produced with desirable output in stage 2 and f is the output of stage 2 used as input of stage 1 (see Fig. 1).

The PPS for the first and second stages are:

$$PPS^1 = \{(x, f, v, z, w) | (v, z, w) \text{ is produced from } (x, f)\},$$
$$PPS^2 = \{(z, w, h, y, f) | (h, y, f) \text{ is produced from } (z, w)\}.$$

We consider the undesirable outputs under weak disposability assumption of Kuosmanen [18]. Thus, PPSs of stages 1 and 2 turn to be:

$$PPS^1 = \Big\{(x, f, v, z, w) \Big| \sum_{k=1}^{K} \lambda_k^1 x_{nk} \le x_{no}, n = 1, \dots, N, \sum_{k=1}^{K} \theta_k^1 \lambda_k^1 v_{mk} \ge v_{mo}, m = 1, \dots, M, \sum_{k=1}^{K} \theta_k^1 \lambda_k^1 w_{jk} = w_{jo}, j = 1, \dots, J, \sum_{k=1}^{K} \theta_k^1 \lambda_k^1 z_{tk} \ge z_{to}, t = 1, \dots, T, \ \sum_{k=1}^{K} \lambda_k^1 f_{gk} \le f_{go}, g = 1, \dots, S, \sum_{k=1}^{K} \lambda_k^1 = 1, 0 \le \theta_k^1 \le 1, \lambda_k^1 \ge 0\Big\}, \quad (1)$$

$$PPS^2 = \Big\{(z, w, h, y, f) \Big| \sum_{k=1}^{K} \lambda_k^2 w_{jk} \le w_{jo}, j = 1, \dots, J, \sum_{k=1}^{K} \lambda_k^2 z_{tk} \le z_{to}, t = 1, \dots, T, \sum_{k=1}^{K} \theta_k^2 \lambda_k^2 f_{gk} \ge f_{go}, g = 1, \dots, S, \sum_{k=1}^{K} \theta_k^2 \lambda_k^2 y_{rk} \ge y_{ro}, r = 1, \dots, R, \sum_{k=1}^{K} \theta_k^2 \lambda_k^2 h_{ik} = h_{io}, i = 1, \dots, I, \sum_{k=1}^{K} \lambda_k^2 = 1, \ 0 \le \theta_k^2 \le 1, \lambda_k^2 \ge 0 \ \Big\}. \quad (2)$$

According to Fare et al. [10], the PPS of the network system is the aggregate PPS of all the stages. Thus, the PPS of the two-stage network given in Fig. 1 can be defined as follows:

$$PPS = \Big\{(x, f, v, z, w, h, y) \Big| \sum_{k=1}^{K} \lambda_k^1 x_{nk} \le x_{no}, n = 1, \dots, N, \sum_{k=1}^{K} \theta_k^1 \lambda_k^1 v_{mk} \ge v_{mo}, m = 1, \dots, M, \sum_{k=1}^{K} \theta_k^1 \lambda_k^1 w_{jk} = w_{jo}, j = 1, \dots, J, \sum_{k=1}^{K} \theta_k^1 \lambda_k^1 z_{tk} \ge z_{to}, t = 1, \dots, T, \ \sum_{k=1}^{K} \lambda_k^1 f_{gk} \le f_{go}, g = 1, \dots, S, \sum_{k=1}^{K} \lambda_k^2 w_{jk} \le w_{jo}, j = 1, \dots, J, \sum_{k=1}^{K} \lambda_k^2 z_{tk} \le z_{to}, t = 1, \dots, T, \sum_{k=1}^{K} \theta_k^2 \lambda_k^2 f_{gk} \ge f_{go}, g = 1, \dots, S, \sum_{k=1}^{K} \theta_k^2 \lambda_k^2 y_{rk} \ge y_{ro}, r = 1, \dots, R, \sum_{k=1}^{K} \theta_k^2 \lambda_k^2 h_{ik} = h_{io}, i = 1, \dots, I, \sum_{k=1}^{K} \lambda_k^1 = 1, \sum_{k=1}^{K} \lambda_k^2 = 1, 0 \le \theta_k^1 \le 1, 0 \le \theta_k^2 \le 1, \ \lambda_k^1 \ge 0, \lambda_k^2 \ge 0\Big\}. \quad (3)$$

There are two major views to impose constraints on the intermediate products. On the one hand, many studies followed the idea of continuity of Tone et al. [27] to develop NSBM models [7, 17, 22, 25]; that is,

$$\sum_{k=1}^{K} \theta_k^1 \lambda_k^1 z_{tk} = \sum_{k=1}^{K} \lambda_k^2 z_{tk}, t = 1, \dots, T. \quad (4)$$

This continuity condition has an economic interpretation, being that an equilibrium target amount for z_t that produces the highest efficiency for the system will finally be compromised between the stages [17]. On the other hand, the second view appropriates relational type of Kao [17], as developed and used in several studies [1, 11, 21, 23]

$$\sum_{k=1}^{K} \theta_k^1 \lambda_k^1 z_{tk} \ge \sum_{k=1}^{K} \lambda_k^2 z_{tk}, t = 1, \dots, T. \quad (5)$$

Note that (4) is a special case of (5) and the PPS with (4) is smaller than the PPS with (5) and imposes that the amount of each intermediate product produced in the network is sufficient to satisfy the amount of that intermediate product being consumed. Also, it is called free disposability of intermediate products so that if more intermediate consumed product is produced, then the required lower amount can also be produced [21].

3. The NSBM model

Organizations are continuously required to identify inefficiencies in their internal processes in order to make operational and strategic decisions for the achievement of management targets. The inability of primary SBM models in providing appropriate definition of the overall efficiency and the stages and the disregard of the intermediate products in the efficiency evaluation, led to the development of NSBM models. The NSBM model for the structure given in Fig. 1 is:

$$min\ E_{NSBM} = \frac{1-\frac{1}{2}(\frac{1}{N+S}(\sum_{n=1}^{N}\frac{s_n^-}{x_{no}}+\sum_{g=1}^{S}\frac{s_g^-}{f_{go}})+\frac{1}{J+T}(\sum_{t=1}^{T}\frac{s_t^{2-}}{z_{to}}+\sum_{j=1}^{J}\frac{s_j^{2-}}{w_{jo}}))}{1+\frac{1}{2}(\frac{1}{M+T}(\sum_{m=1}^{M}\frac{s_m^+}{v_{mo}}+\sum_{t=1}^{T}\frac{s_t^{1+}}{z_{to}})+\frac{1}{R+S}(\sum_{r=1}^{R}\frac{s_r^{2+}}{y_{ro}}+\sum_{g=1}^{S}\frac{s_g^+}{f_{go}}))} \quad (6)$$

$s.t.$

$$\sum_{k=1}^{K}\lambda_k^1 x_{nk}+s_n^- = x_{no}, n=1,\dots,N,$$

$$\sum_{k=1}^{K}\theta_k^1\lambda_k^1 v_{mk}-s_m^+ = v_{mo}, m=1,\dots,M,$$

$$\sum_{k=1}^{K}\theta_k^1\lambda_k^1 w_{jk} = w_{jo}, j=1,\dots,J,$$

$$\sum_{k=1}^{K}\lambda_k^2 w_{jk}+s_j^{2-} = w_{jo}, j=1,\dots,J,$$

$$\sum_{k=1}^{K}\theta_k^1\lambda_k^1 z_{tk}-s_t^{1+} = z_{to}, t=1,\dots,T,$$

$$\sum_{k=1}^{K}\lambda_k^2 z_{tk}+s_t^{2-} = z_{to}, t=1,\dots,T,$$

$$\sum_{k=1}^{K}\theta_k^2\lambda_k^2 f_{gk}-s_g^+ = f_{go}, g=1,\dots,S,$$

$$\sum_{k=1}^{K}\lambda_k^1 f_{gk}+s_g^- = f_{go}, g=1,\dots,S,$$

$$\sum_{k=1}^{K}\theta_k^2\lambda_k^2 y_{rk}-s_r^{2+} = y_{ro}, r=1,\dots,R,$$

$$\sum_{k=1}^{K}\theta_k^2\lambda_k^2 h_{ik} = h_{io}, i=1,\dots,I,$$

$$\sum_{k=1}^{K}\theta_k^1\lambda_k^1 z_{tk} \geq \sum_{k=1}^{K}\lambda_k^2 z_{tk}, t=1,\dots,T,$$

$$\sum_{k=1}^{K}\theta_k^1\lambda_k^1 w_{jk} \geq \sum_{k=1}^{K}\lambda_k^2 w_{jk}, j=1,\dots,J,$$

$$\sum_{k=1}^{K}\lambda_k^1 = 1,$$

$$\sum_{k=1}^{K}\lambda_k^2 = 1,$$

$$0 \leq \theta_k^i \leq 1, i=1,2,$$

$$s_n^-, s_j^{2-}, s_m^+, s_t^{1+}, s_t^{2-}, s_g^+, s_g^-, s_r^{2+}, \lambda_k^1, \lambda_k^2 \geq 0.$$

In model (6), $(s_n^-, s_j^{2-}, s_m^+, s_t^{1+}, s_t^{2-}, s_g^+, s_g^-, s_r^{2+})$ composes the slacks corresponding to inputs, intermediates between stages, output, feedback outputs, desirable outputs, and undesirable outputs, respectively. The stage efficiencies are computed as follow:

$$E^1 = \frac{1 - \frac{1}{N+S}\left(\sum_{n=1}^{N} \frac{s_n^-}{x_{no}} + \sum_{g=1}^{S} \frac{s_g^-}{f_{go}}\right)}{1 + \frac{1}{M+T}\left(\sum_{m=1}^{M} \frac{s_m^+}{v_{mo}} + \sum_{t=1}^{T} \frac{s_t^{1+}}{z_{to}}\right)}, \tag{7}$$

and

$$E^2 = \frac{1 - \frac{1}{J+T}\left(\sum_{t=1}^{T} \frac{s_t^{2-}}{z_{to}} + \sum_{j=1}^{J} \frac{s_j^{2-}}{w_{jo}}\right)}{1 + \frac{1}{R+S}\left(\sum_{r=1}^{R} \frac{s_r^{2+}}{y_{ro}} + \sum_{g=1}^{S} \frac{s_g^+}{f_{go}}\right)}. \tag{8}$$

On the other hand, since for the internal evaluation the overall efficiency is defined to be the weighted average of the efficiencies of these stages, we set the weights as $w_1 = \frac{\eta_1}{\eta_1+\eta_2}$ and $w_2 = \frac{\eta_2}{\eta_1+\eta_2}$, with η_1 and η_2 being denominator of (7) and (8), respectively, in order to linerize the objective function of model (6). Also, to transform the nonlinear constraints into linear ones, Kousmanen's [18] approach is used as follows:

$$\Lambda_k^i = \theta_k^i \lambda_k^i \ \ (\forall k, i = 1,2),$$

$$\Phi_k^i + \Lambda_k^i = \lambda_k^i \ \ (\forall k, i = 1,2).$$

Thus, we get the following linear model:

$$min\ E_{NSBM} = \beta - \frac{1}{2}\left(\frac{1}{N+S}\left(\sum_{n=1}^{N} \frac{s_n^-}{x_{no}} + \sum_{g=1}^{S} \frac{s_g^-}{f_{go}}\right) + \frac{1}{J+T}\left(\sum_{t=1}^{T} \frac{s_t^{2-}}{z_{to}} + \sum_{j=1}^{J} \frac{s_j^{2-}}{w_{jo}}\right)\right) \tag{9}$$

$s.t.$

$$\beta + \frac{1}{2}\left(\frac{1}{M+T}\left(\sum_{m=1}^{M} \frac{s_m^+}{v_{mo}} + \sum_{t=1}^{T} \frac{s_t^{1+}}{z_{to}} + \frac{1}{R+S}\left(\sum_{r=1}^{R} \frac{s_r^{2+}}{y_{ro}} + \sum_{g=1}^{S} \frac{s_g^+}{f_{go}}\right)\right)\right) = 1,$$

$$\sum_{k=1}^{K} (\Phi_k^1 + \Lambda_k^1) x_{nk} + s_n^- = \beta x_{no}, n = 1, \dots, N,$$

$$\sum_{k=1}^{K} \Lambda_k^1 v_{mk} - s_m^+ = \beta v_{mo}, m = 1, \dots, M,$$

$$\sum_{k=1}^{K} \Lambda_k^1 w_{jk} = \beta w_{jo}, j = 1, \dots, J,$$

$$\sum_{k=1}^{K} (\Phi_k^2 + \Lambda_k^2) w_{jk} + s_j^{2-} = \beta w_{jo}, j = 1, \dots, J,$$

$$\sum_{k=1}^{K} \Lambda_k^1 z_{tk} - s_t^{1+} = \beta z_{to}, t = 1, \dots, T,$$

$$\sum_{k=1}^{K} (\Phi_k^2 + \Lambda_k^2) z_{tk} + s_t^{2-} = \beta z_{to}, t = 1, \dots, T,$$

$$\sum_{k=1}^{K} \Lambda_k^2 f_{gk} - s_g^+ = \beta f_{go}, g = 1,\dots,S,$$

$$\sum_{k=1}^{K} (\Phi_k^1 + \Lambda_k^1) f_{gk} + s_g^- = \beta f_{go}, g = 1,\dots,S,$$

$$\sum_{k=1}^{K} \Lambda_k^2 y_{rk} - s_r^{2+} = \beta y_{ro}, r = 1,\dots,R,$$

$$\sum_{k=1}^{K} \Lambda_k^2 h_{ik} = \beta h_{io}, i = 1,\dots,I,$$

$$\sum_{k=1}^{K} \Lambda_k^1 z_{tk} \geq \sum_{k=1}^{K} (\Phi_k^2 + \Lambda_k^2) z_{tk}, t = 1,\dots,T,$$

$$\sum_{k=1}^{K} \Lambda_k^1 w_{jk} \geq \sum_{k=1}^{K} (\Phi_k^2 + \Lambda_k^2) w_{jk}, j = 1,\dots,J,$$

$$\sum_{k=1}^{K} (\Phi_k^1 + \Lambda_k^1) = \beta,$$

$$\sum_{k=1}^{K} (\Phi_k^2 + \Lambda_k^2) = \beta,$$

$$s_n^-, s_j^{2-}, s_m^+, s_t^{1+}, s_t^{2-}, s_g^+, s_g^-, s_r^{2+} \geq 0,$$

$$\Lambda_k^1, \Lambda_k^2, \Phi_k^1, \Phi_k^2 \geq 0, \beta > 0.$$

As we see, the weights w_1 and w_2 do not necessarily reflect the relative importance of the efficiencies of the stages. We do not know the weight values before the models are calculated. Indeed, the NSBM limitation in the case of internal evaluation is that the special set of weights with respect to stage efficiencies has to be assigned in order to ensure that the derived overall efficiency remains in a format of SBM that can be reformulated into a linear program. Furthermore, since the weights w_1 and w_2 are functions of the slacks, we may not consider cases where w_1 is much smaller or much greater than w_2.

4. The ASBM model

The ASBM network DEA model can relax the weighting requirement of the NSBM model and use weights to match the preferences of decision makers [7]. In the presence of undesirable outputs and feedback, the ASBM model for the internal evaluation of the structure given in Fig. 1 is as follows:

$$min E_{ASBM} = \frac{w_1}{A}\left(\sum \frac{x_{no} - s_n^-}{x_{no}} + \sum \frac{f_{go} - s_g^-}{f_{go}} + \sum \frac{z_{to}}{z_{to} + s_t^{1+}} + \sum \frac{v_{mo}}{v_{mo} + s_m^+}\right) + \quad (10)$$

$$\frac{w_2}{B}\left(\sum \frac{w_{jo} - s_j^{2-}}{w_{jo}} + \sum \frac{z_{to} - s_t^{2-}}{z_{to}} + \sum \frac{y_{ro}}{y_{ro} + s_r^{2+}} + \sum \frac{f_{go}}{f_{go} + s_g^+}\right)$$

$s.t.$

$$\sum_{k=1}^{K} \lambda_k^1 x_{nk} + s_n^- = x_{no}, n = 1,\dots,N,$$

$$\sum_{k=1}^{K} \theta_k^1 \lambda_k^1 v_{mk} - s_m^+ = v_{mo}, m = 1,\dots,M,$$

$$\sum_{k=1}^{K} \theta_k^1 \lambda_k^1 w_{jk} = w_{jo}, j = 1,\dots,J,$$

$$\sum_{k=1}^{K} \lambda_k^2 w_{jk} + s_j^{2-} = w_{jo}, j = 1,\dots,J,$$

$\sum_{k=1}^{K} \theta_k^1 \lambda_k^1 z_{tk} - s_t^{1+} = z_{to}, t = 1, \ldots, T,$

$\sum_{k=1}^{K} \lambda_k^2 z_{tk} + s_t^{2-} = z_{to}, t = 1, \ldots, T,$

$\sum_{k=1}^{K} \theta_k^2 \lambda_k^2 f_{gk} - s_g^{+} = f_{go}, g = 1, \ldots, S,$

$\sum_{k=1}^{K} \lambda_k^1 f_{gk} + s_g^{-} = f_{go}, g = 1, \ldots, S,$

$\sum_{k=1}^{K} \theta_k^2 \lambda_k^2 y_{rk} - s_r^{2+} = y_{ro}, r = 1, \ldots, R,$

$\sum_{k=1}^{K} \theta_k^2 \lambda_k^2 h_{ik} = h_{io}, i = 1, \ldots, I,$

$\sum_{k=1}^{K} \theta_k^1 \lambda_k^1 z_{tk} \geq \sum_{k=1}^{K} \lambda_k^2 z_{tk}, t = 1, \ldots, T,$

$\sum_{k=1}^{K} \theta_k^1 \lambda_k^1 w_{jk} \geq \sum_{k=1}^{K} \lambda_k^2 w_{jk}, j = 1, \ldots, J,$

$\sum_{k=1}^{K} \lambda_k^1 = 1,$

$\sum_{k=1}^{K} \lambda_k^2 = 1,$

$0 \leq \theta_k^i \leq 1, i = 1,2,$

$s_n^-, s_j^{2-}, s_m^+, s_t^{1+}, s_t^{2-}, s_g^+, s_g^-, s_r^{2+} \lambda_k^1, \lambda_k^2 \geq 0,$

where $A = N + S + T + M$, $B = J + T + R + S$ and w_1 and w_2 are user-specified weights such that $w_1 + w_2 = 1$. Since constraints of model (10) are nonlinear, we linearize them by Kousmanen's transformation perocedure. To do so, let $\Lambda_k^i = \theta_k^i \lambda_k^i$ $(\forall k, i = 1,2)$ and $\lambda_k^i = \Phi_k^i + \Lambda_k^i$ $(\forall k, i = 1,2)$ as before. Then, model (10) can be reformulated as follows:

$$minE_{ASBM} = \frac{w_1}{A}\left(\sum \frac{x_{no} - s_n^-}{x_{no}} + \sum \frac{f_{go} - s_g^-}{f_{go}} + \sum \frac{z_{to}}{z_{to} + s_t^{1+}} + \sum \frac{v_{mo}}{v_{mo} + s_m^+}\right) + \frac{w_2}{B}\left(\sum \frac{w_{jo} - s_j^{2-}}{w_{jo}} + \sum \frac{z_{to} - s_t^{2-}}{z_{to}} + \sum \frac{y_{ro}}{y_{ro} + s_r^{2+}} + \sum \frac{f_{go}}{f_{go} + s_g^+}\right) \quad (11)$$

$s.t.$

$\sum_{k=1}^{K} (\Phi_k^1 + \Lambda_k^1) x_{nk} + s_n^- = x_{no}, n = 1, \ldots, N,$

$\sum_{k=1}^{K} \Lambda_k^1 v_{mk} - s_m^+ = v_{mo}, m = 1, \ldots, M,$

$\sum_{k=1}^{K} \Lambda_k^1 w_{jk} = w_{jo}, j = 1, \ldots, J,$

$\sum_{k=1}^{K} (\Phi_k^2 + \Lambda_k^2) w_{jk} + s_j^{2-} = w_{jo}, j = 1, \ldots, J,$

$\sum_{k=1}^{K} \Lambda_k^1 z_{tk} - s_t^{1+} = z_{to}, t = 1, \ldots, T,$

$\sum_{k=1}^{K} (\Phi_k^2 + \Lambda_k^2) z_{tk} + s_t^{2-} = z_{to}, t = 1, \ldots, T,$

$\sum_{k=1}^{K} \Lambda_k^2 f_{gk} - s_g^+ = f_{go}, g = 1,\ldots,S,$

$\sum_{k=1}^{K} (\Phi_k^1 + \Lambda_k^1) f_{gk} + s_g^- = f_{go}, g = 1,\ldots,S,$

$\sum_{k=1}^{K} \Lambda_k^2 y_{rk} - s_r^{2+} = y_{ro}, r = 1,\ldots,R,$

$\sum_{k=1}^{K} \Lambda_k^2 h_{ik} = h_{io}, i = 1,\ldots,I,$

$\sum_{k=1}^{K} \Lambda_k^1 z_{tk} \geq \sum_{k=1}^{K} (\Phi_k^2 + \Lambda_k^2) z_{tk}, t = 1,\ldots,T,$

$\sum_{k=1}^{K} \Lambda_k^1 w_{jk} \geq \sum_{k=1}^{K} (\Phi_k^2 + \Lambda_k^2) w_{jk}, j = 1,\ldots,J,$

$\sum_{k=1}^{K} (\Phi_k^1 + \Lambda_k^1) = 1,$

$\sum_{k=1}^{K} (\Phi_k^2 + \Lambda_k^2) = 1,$

$s_n^-, s_j^{2-}, s_m^+, s_t^{1+}, s_t^{2-}, s_g^+, s_g^-, s_r^{2+} \geq 0,$

$\Lambda_k^1, \Lambda_k^2, \Phi_k^1, \Phi_k^2 \geq 0.$

The objective function of (11) is still nonlinear, and in the following theorem we see how to reformulate it as an SOCP.

Theorem 1 Model (11) is equivalent to the following SOCP*:*

$$min E_{ASBM} = \sum_{t=1}^{T} \xi_t^1 + \sum_{m=1}^{M} \xi_m^2 + \sum_{r=1}^{R} \xi_r^3 + \sum_{g=1}^{S} \xi_g^4 + \xi^5 \quad (12)$$

$s.t.$

$$\left\| \begin{bmatrix} 2\sqrt{w_1 z_{to}} \\ A\xi_t^1 - (z_{to} + s_t^{1+}) \end{bmatrix} \right\| \leq A\xi_t^1 + z_{to} + s_t^{1+}, t = 1,\ldots,T,$$

$$\left\| \begin{bmatrix} 2\sqrt{w_1 v_{mo}} \\ A\xi_m^2 - (v_{mo} + s_m^+) \end{bmatrix} \right\| \leq A\xi_m^2 + v_{mo} + s_m^+, m = 1,\ldots,M,$$

$$\left\| \begin{bmatrix} 2\sqrt{w_2 y_{ro}} \\ B\xi_r^3 - (y_{ro} + s_r^{2+}) \end{bmatrix} \right\| \leq B\xi_r^3 + y_{ro} + s_r^{2+}, r = 1,\ldots,R,$$

$$\left\| \begin{bmatrix} 2\sqrt{w_2 f_{go}} \\ B\xi_g^4 - (f_{go} + s_g^+) \end{bmatrix} \right\| \leq B\xi_g^4 + f_{go} + s_g^+, g = 1,\ldots,S,$$

$$\frac{w_1}{A}\left(\sum_{n=1}^{N} \frac{x_{no} - s_n^-}{x_{no}} + \sum_{g=1}^{S} \frac{f_{go} - s_g^-}{f_{go}}\right) + \frac{w_2}{B}\left(\sum_{j=1}^{J} \frac{w_{jo} - s_j^{2-}}{w_{jo}} + \sum_{t=1}^{T} \frac{z_{to} - s_t^{2-}}{z_{to}}\right) \leq \xi^5,$$

constraintsets of model (11)

Proof. Introducing ξ_t^1, ξ_m^2, ξ_r^3, ξ_g^4 and ξ^5, model (11) is equivalent to

$$min \quad \sum_{t=1}^{T} \xi_t^1 + \sum_{m=1}^{M} \xi_m^2 + \sum_{r=1}^{R} \xi_r^3 + \sum_{g=1}^{S} \xi_g^4 + \xi^5 \tag{13}$$

$$s.t.$$

$$\frac{w_1}{A}\left(\frac{z_{to}}{z_{to}+s_t^{1+}}\right) \le \xi_t^1, t = 1, \dots, T,$$

$$\frac{w_1}{A}\left(\frac{v_{mo}}{v_{mo}+s_m^{+}}\right) \le \xi_m^2, m = 1, \dots, M,$$

$$\frac{w_2}{B}\left(\frac{y_{ro}}{y_{ro}+s_r^{2+}}\right) \le \xi_r^3, r = 1, \dots, R,$$

$$\frac{w_2}{B}\left(\frac{f_{go}}{f_{go}+s_g^{+}}\right) \le \xi_g^4, g = 1, \dots, S,$$

$$\frac{w_1}{A}\left(\sum_{n=1}^{N} \frac{x_{no}-s_n^-}{x_{no}} + \sum_{g=1}^{S} \frac{f_{go}-s_g^-}{f_{go}}\right) + \frac{w_2}{B}\left(\sum_{j=1}^{J} \frac{w_{jo}-s_j^{2-}}{w_{jo}} + \sum_{t=1}^{T} \frac{z_{to}-s_t^{2-}}{z_{to}}\right) \le \xi^5,$$

constraintsets of model(11) .

The first four sets of constraints are equivalent to

$$w_1 z_{to} \le A\xi_t^1 (z_{to} + s_t^{1+}),\ t = 1, \dots, T,$$

$$w_1 v_{mo} \le A\xi_m^2 (v_{mo} + s_m^{+}),\ m = 1, \dots, M,$$

$$w_2 y_{ro} \le B\xi_r^3 (y_{ro} + s_r^{2+}),\ r = 1, \dots, R,$$

$$w_2 f_{go} \le B\xi_g^4 (f_{go} + s_g^{+}),\ g = 1, \dots, S.$$

These are further equivalent to the following conic constraints:

$$\left\|\begin{bmatrix} 2\sqrt{w_1 z_{to}} \\ A\xi_t^1 - (z_{to} + s_t^{1+}) \end{bmatrix}\right\| \le A\xi_t^1 + z_{to} + s_t^{1+},\ t = 1, \dots, T,$$

$$\left\|\begin{bmatrix} 2\sqrt{w_1 v_{mo}} \\ A\xi_m^2 - (v_{mo} + s_m^{+}) \end{bmatrix}\right\| \le A\xi_m^2 + v_{mo} + s_m^{+},\ m = 1, \dots, M,$$

$$\left\|\begin{bmatrix} 2\sqrt{w_2 y_{ro}} \\ B\xi_r^3 - (y_{ro} + s_r^{2+}) \end{bmatrix}\right\| \le B\xi_r^3 + y_{ro} + s_r^{2+},\ r = 1, \dots, R,$$

$$\left\|\begin{bmatrix} 2\sqrt{w_2 f_{go}} \\ B\xi_g^4 - (f_{go} + s_g^{+}) \end{bmatrix}\right\| \le B\xi_g^4 + f_{go} + s_g^{+},\ g = 1, \dots, S.$$

The last set of constraints in (13) is linear. Thus, we get the SOCP model (12).

After solving (12), efficiency of the stages are computed as follows:

$$E^1 = \frac{1}{N+S+T+M}\left(\sum \frac{x_{no}-s_n^-}{x_{no}} + \sum \frac{f_{go}-s_g^-}{f_{go}} + \sum \frac{z_{to}}{z_{to}+s_t^{1+}} + \sum \frac{v_{mo}}{v_{mo}+s_m^+}\right),$$

$$E^2 = \frac{1}{J+T+R+S}\left(\sum \frac{w_{jo}-s_j^{2-}}{w_{jo}} + \sum \frac{z_{to}-s_t^{2-}}{z_{to}} + \sum \frac{y_{ro}}{y_{ro}+s_r^{2+}} + \sum \frac{f_{go}}{f_{go}+s_g^+}\right).$$

As we see, the internal evaluation of the ASBM model is able to obtain the overall efficiency and the efficiency of stages simultaneously. Also, the harmonic mean is always less than or equal to the arithmetic mean. Thus effiicency derived by (9) is always less than the efficiency obtained by (12). On the other hand, the optimal solution $(s_n^{-*}, s_j^{2-*}, s_m^{+*}, s_t^{1+*}, s_t^{2-*}, s_g^{+*}, s_g^{-*}, s_r^{2+*})$ of models (9) and (12) determines the target values for each process:

$$\hat{x}_{no} = x_{no} - s_n^{-*}, n = 1,\ldots,N,$$

$$\hat{v}_{mo} = v_{mo} + s_m^{+*}, m = 1,\ldots,M, \quad (14)$$

$$\hat{w}_{jo} = w_{jo} - s_j^{2-*}, j = 1,\ldots,J,$$

$$\hat{z}_{to} = z_{to} + s_t^{1+*}, t = 1,\ldots,T,$$

$$\hat{z}_{to} = z_{to} - s_t^{2-*}, t = 1,\ldots,T,$$

$$\hat{f}_{go} = f_{go} + s_g^{+*}, g = 1,\ldots,S,$$

$$\hat{f}_{go} = f_{go} - s_g^{-*}, g = 1,\ldots,S,$$

$$\hat{y}_{ro} = y_{ro} + s_r^{2+*}, r = 1,\ldots,R,$$

where $(\hat{x}_{no}, \hat{v}_{mo}, \hat{w}_{jo}, \hat{z}_{to}, \hat{f}_{go}, \hat{y}_{ro})$ comprise the target inputs, desirable outputs and undesirable outputs and also the intermediate variables and feedbacks.

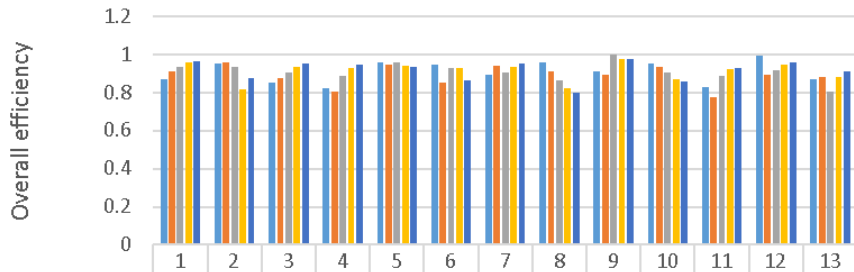
5. Numerical example

To evaluate the proposed models, here we evaluate the efficiencies of 31 regions in China, where the dataset is taken from Li et al. [20]. It has the ecosystem consisting of two distinct stages; stage 1 is the primary system and stage 2 is decontamination. In this two-stage structure, the three waste gasses are combined into a single factor (w), and are separated into treated gasses, as desirable outputs (y) and discard or untreated, as unsedirable outputs (h). At the same time, undesirable outputs w from the first stage are inputs to the decontamination stage. Also, recycled water is fed back into the first stage f. In addition, we set $w_1 = w_2 = \frac{1}{2}$ for the ASBM model. The efficiencies of stages 1 and 2 and the overall efficiency of the NSBM model are reported in second, third and fourth columns of Table 1, respectively. The last three columns of Table 1 also show the evaluation results based on the ASBM model. Eighteen DMUs are efficient in both models (9) and (12). Moreover, the overall efficiencies of DMUs which are provided by NSBM are all lower than the reported overall efficiencies using the ASBM. Also, the most inefficient unit in the NSBM model is DMU 28 and the most inefficient unit in the ASBM model is DMU 31.

Table 1. Efficiency results based on NSBM and ASBM models.

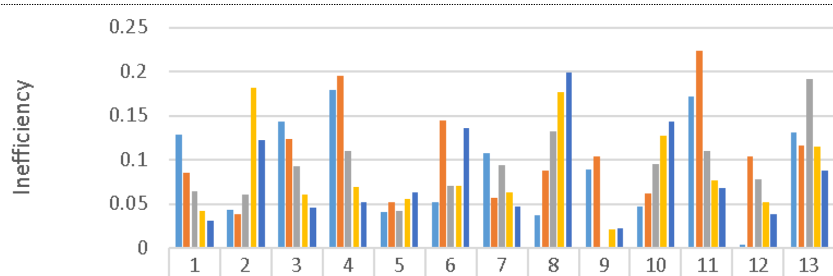
DMUs	Model (9)			Model (12)		
	E^1	E^2	E_{NSBM}	E^1	E^2	E_{ASBM}
1	1	1	1	1	1	1
2	1	1	1	1	1	1
3	1	1	1	1	1	1
4	1	1	1	1	1	1
5	1	1	1	1	1	1
6	1	0.7543	0.8748	1	0.8718	0.9359
7	0.8282	0.9102	0.8696	0.8828	0.9951	0.939
8	1	1	1	1	1	1
9	1	1	1	1	1	1
10	1	0.6551	0.8205	1	0.8139	0.907
11	1	1	1	1	1	1
12	1	0.6445	0.7896	1	0.7801	0.89
13	0.7124	1	0.8357	0.9157	1	0.9579
14	1	1	1	1	1	1
15	1	1	1	1	1	1
16	1	1	1	1	1	1
17	0.9149	0.9204	0.9177	0.9001	0.9579	0.929
18	1	1	1	1	1	1
19	1	1	1	1	1	1
20	1	0.7969	0.8883	1	0.8108	0.9054
21	1	1	1	1	1	1
22	0.5374	1	0.7119	0.7351	1	0.8675
23	1	0.8719	0.9316	1	0.9984	0.9992
24	1	1	1	1	1	1
25	1	1	1	1	1	1
26	1	1	1	1	1	1
27	0.6605	1	0.8055	0.8088	1	0.9044
28	1	0.4288	0.6098	1	0.7785	0.8893
29	1	1	1	1	1	1
30	1	0.6809	0.813	1	0.8433	0.9217
31	1	0.4373	0.6227	1	0.6159	0.808

The remaining thirteen DMUs exhibit varying degrees of inefficiencies. For example, DMUs 6, 10, 12, 20, 23, 28, 30, 31 are efficient in the first stage and inefficient in the second stage and thus overall they are inefficient. The ASBM model compared to the NSBM model is more flexible in terms of dealing with combining the weights of the network DEA model that can help to match the preference of a decision-maker. The trend of the overall efficiencies of thirteen inefficient DMUs with model (12) for different weights are depicted in Fig. 2: Efficiency 1 $(w_1 = \frac{1}{4}, w_2 = \frac{3}{4})$, Efficiency 2 $(w_1 = \frac{1}{3}, w_2 = \frac{2}{3})$, Efficiency 3 $(w_1 = \frac{1}{2}, w_2 = \frac{1}{2})$, Efficiency 4 $(w_1 = \frac{2}{3}, w_2 = \frac{1}{3})$, Efficiency 5 ($w_1 = \frac{3}{4}, w_2 = \frac{1}{4}$). 1=DMU 6, 2=DMU 7, 3=DMU 10, 4=DMU 12, 5=DMU 13, 6=DMU 17, 7=DMU 20, 8=DMU 22, 9=DMU 23, 10=DMU 27, 11=DMU 28, 12=DMU 30, 13=DMU 31.



	1	2	3	4	5	6	7	8	9	10	11	12	13
Efficiency 1	0.871	0.956	0.857	0.821	0.959	0.948	0.893	0.963	0.91	0.952	0.828	0.995	0.869
Efficiency 2	0.915	0.961	0.876	0.804	0.947	0.855	0.942	0.912	0.896	0.938	0.776	0.896	0.883
Efficiency 3	0.936	0.939	0.907	0.89	0.958	0.929	0.905	0.868	0.999	0.904	0.889	0.922	0.808
Efficiency 4	0.957	0.818	0.939	0.93	0.944	0.93	0.937	0.823	0.978	0.873	0.923	0.948	0.884
Efficiency 5	0.968	0.877	0.953	0.948	0.937	0.863	0.953	0.801	0.977	0.857	0.932	0.961	0.912

Figure 2. Trend of the overall efficiency of inefficient DMUs for model (12).



	1	2	3	4	5	6	7	8	9	10	11	12	13
Inefficiency 1	0.13	0.04	0.14	0.18	0.04	0.05	0.11	0.04	0.09	0.05	0.17	0	0.13
Inefficiency 2	0.09	0.04	0.12	0.2	0.05	0.14	0.06	0.09	0.1	0.06	0.22	0.1	0.12
Inefficiency 3	0.06	0.06	0.09	0.11	0.04	0.07	0.09	0.13	0	0.1	0.11	0.08	0.19
Inefficiency 4	0.04	0.18	0.06	0.07	0.06	0.07	0.06	0.18	0.02	0.13	0.08	0.05	0.12
Inefficiency 5	0.03	0.12	0.05	0.05	0.06	0.14	0.05	0.2	0.02	0.14	0.07	0.04	0.09

Figure 3. Trend of the overall inefficiency of inefficient DMUs for model (12).

Table 2. Slack variables of internal evaluation based on NSBM model (9).

DMUs	S_n^-			S_m^+		S_j^{2-}		S_t^{1+}	S_t^{2+}	S_g^-	S_g^+	S_r^{2+}	
1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	444	0	0	0	0
7	1778	1455	1.3	276.5	13.8	3.4	0	13.6	1.5	0.0001	0	0.045	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	113	0	0	308	0	0.01	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	65.6	0	0.01	19.45	0
13	0	53.1	5.9	7617	576	0	0	101	0	0.0045	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0
17	327	37.6	17.3	241.3	2.1	0	0	0.2	67.9	0.0052	0	0.001	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	10.5	0	0	0	0	0.01	0	0
21	0	0	0	0	0	0	0	0	0	0	0	0	0
22	0	1166	8.2	17648	42.4	0	0	68.5	0	0	0	0	0
23	0	0	0	0	0	0.1	0	0	0	0	0	9.608	0
24	0	0	0	0	0	0	0	0	0	0	0	0	0
25	0	0	0	0	0	0	0	0	0	0	0	0	0
26	0	0	0	0	0	0	0	0	0	0	0	0	0
27	964	1910	6.6	4311	0	0	0	130	0	0	0	0	0
28	0	0	0	0	0	18.3	0	0	0	0	0.01	5.627	0
29	0	0	0	0	0	0	0	0	0	0	0	0	0
30	0	0	0	0	0	0	0	0	6.1	0	0	1.023	0
31	0	0	0	0	0	131	0	0	0	0	0.01	6.553	0

According to Fig. 2, the overall efficiency of nine inefficient DMUs (6, 10, 12, 17 20, 23, 28, 30, 31) are improved with the increase of w_1 from $\frac{1}{4}$ to $\frac{3}{4}$. Indeed, the overall efficiency is enhanced if the preference of decision-maker is the primary stage. For instance, the efficiency of DMU 6 increases from 0.86 to 0.96. Also, the overall efficiencies of four DMUs (7, 13, 22 and 27) decrease with an increase in w_1. In summary, for DMUs that are efficient in the first stage, with increasing w_1, the inefficiency $(1 - E_{ASBM})$ clearly decreases, as shown in Fig. 3: Efficiency 1 ($w_1 = \frac{1}{4}, w_2 = \frac{3}{4}$), Efficiency 2 ($w_1 = \frac{1}{3}, w_2 = \frac{2}{3}$), Efficiency 3 ($w_1 = \frac{1}{2}, w_2 = \frac{1}{2}$), Efficiency 4 ($w_1 = \frac{2}{3}, w_2 = \frac{1}{3}$), Efficiency 5 ($w_1 = \frac{3}{4}, w_2 = \frac{1}{4}$). 1=DMU 6, 2=DMU 7, 3=DMU 10, 4=DMU 12, 5=DMU 13, 6=DMU 17, 7=DMU 20, 8=DMU 22, 9=DMU 23, 10=DMU 27, 11=DMU 28, 12=DMU 30, 13=DMU 31.

Table 3. Slack variables of internal evaluation based on ASBM model (12).

DMUs	S_n^-			S_m^+		S_j^{2-}		S_t^{1+}	S_t^{2-}	S_g^-	S_g^+	S_r^{2+}	
1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	1.3	0	0	0	0	0	0
3	0	0	0	0	0	0	4.6	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	452.5	0	0.0013	0	0
7	1897.1	1417.4	1.3	1649.7	15.3	0.2	0	24.5	0.6	0	0.0001	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	117.9	0	0	321.1	0	0.0075	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	19	0	0	0	0	0.033	29.3	0
13	2	2924.5	1.3	4239.8	22.1	0	2	26.19	0	0	0	0	11.9
14	0	0	0	0	0	4	45797	4.9	101.8	0	0.013	19.6	26512
15	0	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0
17	1545.5	164.9	8.9	10033	8.6	0	0	2.9	72.03	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	43.7	0	0	33.78	0	0.0197	0	0
21	0	0	0	0	0	0	0	0	0	0	0	0	0
22	609.2	426.3	17.1	12823	0	0	0	153	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0	0	0	0	0
24	0	0	0	0	0	0	0	0	0	0	0	0	0
25	0	0	0	0	0	0	0	0	0	0	0	0	0
26	0	0	0	0	0	0	0	0	0	0	0	0	0
27	1095.1	2861.8	6.7	4684.9	0	0	0	132.8	0	0	0	0	0
28	0	0	0	0	0	0.2	0.3	0	0	0	10.8	0	0
29	0	0	0	0	0	0	0	0	0	0	0	0	0
30	0	0	0	0	0	0	0	0	7.6	0	0.0014	1.29	0
31	0	0	0	0	0	301.9	0	0	113.5	0	0.005	0	0

The slack variables $(s_n^-, s_j^{2-}, s_m^+, s_t^{1+}, s_t^{2-}, s_g^+, s_g^-, s_r^{2+})$ of models (9) and (12) are summarized in Tables 2 and 3. We see that if a stage is efficienct, the slacks remain unchangeed. For instance, for DMUs 6, 10, 12, 20 28, 30 and 31, stage 1 is efficient, and all the slacks of stage 1 are zero. Also, the largest slacks are found in the desirable outputs and inputs of the first stage of the ASBM leading to increase of the outputs and decrease of the inputs. Moreover, for all the overall efficient DMUs, slacks in the inputs and outputs, even in the intermediate products are zero. The slacks in Tables 2 and 3 allow establishing the corresponding target values for each input and output (desirable and undesirable).

Table 4. Inputs and outputs targets based on ASBM model (12).

DMUs	Stage 1							Stage 2					
	$\hat{x}$			$\hat{f}$	$\hat{z}$	$\hat{v}$		$\hat{z}$	$\hat{w}$		$\hat{y}$		$\hat{f}$
1	231.7	8870.8	35.9	0.0099	342.6	17536.8	2069	342.6	19.6	338167	14.0175	337768	0.0099
2	441.1	8853.6	23.1	0.0046	157.6	12736.4	1413	157.6	15.5	642807	8.2764	642515	0.0046
3	6317.3	20106	195.3	0.0085	486.1	26008.9	7288	486.1	143	4338184	30.5686	4337976	0.0085
4	4055.8	8311.7	73.4	0.0029	328.2	11784.6	3611	328.2	56	3616588	13.4269	3616470	0.0029
5	7147.2	12174.6	184.4	0.002	445.1	15435.5	2490	445.1	123.1	3636835	10.2404	3636819	0.002
6	4085.3	24225.6	142.2	0.0106	683.4	24163	4389	230.9(-)	91.1	2821292	23.8663	2821166	0.0119(+)
7	3637.6(-)	8276(-)	128.5(-)	0.0061	78.9(-)	14234.6(+)	2765	102.8(-)	66.8(-)	1244080(-)	11.974	1244054	0.0062
8	11830	10400.5	358.9	0.0021	218.1	13473.5	3834	218.1	204.7	1994184	16.2568	1994170	0.0021
9	244	6961.2	116	0.0355	134.1	20047.6	2380	134.1	20	716984	21.8889	716636	0.0355
10	4763.8	36552.9	552.2	0.0282	657.1	53401	7920	336.1(-)	164(-)	2914784	59.7929	2913820	0.0357(+)
11	1413.4	13442.3	1468.9	0.0495	340.8	38600.9	5827	285.3	84.9	1450005	42.0466	1687970	0.0755
12	5730.2	16587.8	292.6	0.0174	330.2	16881.9	5988	330.2	129.2(-)	1907916	54.7188(+)	1906917	0.0505(+)
13	1328.1(-)	10926.2(-)	198.8(-)	0.0243(-)	196.3(-)	23719.1(+)	3770(+)	222.5	90.6	1091094	25.5998	1091048	0.0264
14	2464.1	11967.3	188.9	0.0085	311.1	25576	4559	214.3	103.7	1456358	21.6138	1528632	0.0385
15	7515.3	33538.2	221.8	0.0331	739.1	49274.1	9685	739.1	143.9	4183047	47.8769	4182611	0.0331
16	7926.4	21710.1	238.6	0.0142	209.5	29389.8	9406	209.5	134.5	3501630	40.3526	3501512	0.0142
17	3118.6(-)	16719.5(-)	290.3(-)	0.0252	282.6	32407(+)	5787(+)	213.5(-)	132.6	1612032	28.9946	1611859	0.0254
18	3789.4	15898.5	328.8	0.0402	190.3	21963.9	6639	190.3	136.5	1592897	30.3812	1592833	0.0402
19	2830.7	22005.9	451.5	0.0542	260.2	56805.7	10594	260.2	172.8	2430904	83.8009	2429144	0.0542
20	4217.5	10506.8	303	0.0121	190.5	12844.6	4862	156.7(-)	86.2(-)	1302101	24.5927	1301949	0.0318(+)
21	727.5	2755.8	45.3	0.0052	44.7	2810.8	887	44.7	20.7	154135	3.7051	154001	0.0052
22	1626.7(-)	9885.7(-)	65.8(-)	0.0092(-)	33.9(-)	24045.4(+)	2945	186.9	41.5	1129714	13.2288	1129686	0.0142
23	5947.4	18203	245.9	0.023	178.3	23694.5	8076	178.3	120.7	1819280	28.5777	1818588	0.023
24	4485.3	5949.1	100.8	0.0052	68.9	6783.3	3484	68.9	46.6	1899123	9.1403	1899117	0.0052
25	6072.1	8047.5	151.8	0.0111	132.4	101177.1	4959	132.4	84.9	1607208	15.3498	1607173	0.0111
26	361.6	696.7	29.8	0.0002	4	697	308	4	24.3	55088	0.4681	55084	0.0002
27	2955.2(-)	10360.5(-)	81.3(-)	0.0077(-)	47.7(-)	18958(+)	3753	180.6	51.4	2113935	12.8663	2113916	0.0086
28	4658.8	5365.8	123.1	0.0036	121.4	5528.8	2578	120.8(-)	80.3	1253438	17.0525(+)	1253208	0.0116(+)
29	542.7	1982.7	27.4	0.0007	24.1	1869.4	573	24.1	17.2	436305	2.1987	436302	0.0007
30	1107.1	1998.3	69.4	0.0011	55.7	2285.6	647	48.1(-)	31.3	1060375	5.1854	1060369	0.0025(+)
31	4124.6	6572.9	590.1	0.0041	255.1	7250	2233	141.6(-)	94.2(-)	2311731	19.7459(+)	2311684	0.0096(+)

Table 4 shows the corresponding targets of model (12) that are calculated using (14). Note that in Table 4, the $(+)$ and $(-)$ signs show increase and decrease in dataset in the target values, respectively. Results show a high potential for improvement in outputs of both stages using the ASBM model. For example, for DMU 28, output y increases from 6.27 to 17.05 and for DMU 31 increases from 9.37 to 19.74. Also, a significant increase is seen in output v for DMUs 7, 13, 17, 22, and 27. Likewise, there is potential for reducing input x for DMUs 7, 13, 17, 22, and 27. Numerical results show that the ASBM model has detected a large inefficiency in the inefficient DMUs and estimated a potential improvement of both inputs and outputs.

6. Conclusions

We developed NSBM and ASBM models for a two-stage structure in the presence of undesirable outputs and feedback variable under the weak disposability assumption. Unlike the NSBM model using the weights as functions of slack variables for each DMU, the ASBM model has flexibility in choosing the stage weights. This property makes the ASBM usable when the combining weights are required to be the same across all units and adaptable to the preferences of the decision maker. The NSBM model is linearized by the Charnes-Cooper transformation and the nonlinear ASBM model is reformulated as an SOCP, being a convex optimization problem. The application of both models on a real dataset shows that the ASBM model has more flexibility in the performance evaluation, the efficiency estimation and the target value determination.

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