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A Wide Neighborhood Primal-dual Predictor-corrector Interior-point Algorithm with New Corrector Directions for Linear Optimization

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In this paper, we present a new primal-dual predictor-corrector interior-point algorithm for linear optimization problems. In each iteration of this algorithm, we use the new wide neighborhood proposed by Darvay and Takács. Our algorithm computes the predictor direction, then the predictor direction is used to obtain the corrector direction. We show that the duality gap reduces in both predictor and corrector steps. Moreover, we conclude that the complexity bound of this algorithm coincides with the best-known complexity bound obtained for small neighborhood algorithms. Eventually, numerical results show the capability and efficiency of the proposed algorithm.

***Keywords**: Linear optimization, Interior-point methods, Predictor-corrector methods, Wide neighborhood.*

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1. Introduction

Interior point methods (IPMs) have been very successful in solving linear optimization (LO) problems. After the seminal paper of Karmarkar [7], various IPMs for solving LO problems have been introduced by several researchers. To see the main results in this field, we refer the reader to Roos et al. [11] and Wright [14]. The IPMs can be categorized in different ways such as primal-dual path-following methods, affine-scaling, feasible and infeasible IPMs. Among the different types of IPMs, the primal-dual predictor-corrector methods are the most effective methods for solving wide classes of optimization problems. The predictor-corrector IPMs was proposed by Mizuno et al. [10]. The IPMs are also distinguished in terms of the step length. There is short- and large-update methods, that work in small and wide neighborhoods of the central path, respectively. The large-update methods are better in practice while the short-update methods give better theoretical results.

In 2005, Ai and Zhang [1] presented a new class of primal-dual path-following interior-point algorithm for solving monotone linear complementarity problems (LCPs) based on new wide neighborhood. Their algorithm decomposes the classical Newton direction into two orthogonal directions, corresponding to the negative and positive parts of the right-hand side of the centering equation. They proved that the algorithm has the same theoretical complexity as a small neighborhood algorithm. The Ai-Zhang’s technique was later extended to semidefinite optimization (SDO) and

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second-order cone optimization (SOCO) by Li and Terlaky [9] and Feng [6], respectively. Liu et al. [8] introduced the first primal-dual second-order corrector interior-point algorithm for LO problems based on Ai-Zhang's wide neighborhood. Recently, Darvay and Takács [5] proposed a new wide neighborhood of the central path for LO based on an algebraic equivalent transformation of the centering equation of the central path and presented a large-update algorithm. Moreover, they derived $O(\sqrt{n}L)$ iteration complexity for the proposed algorithm, the same as the best theoretical complexity for small neighborhood algorithms, where n is the number of variables and L is the input data length.

Motivated by the mentioned results, we propose a new predictor-corrector interior-point algorithm for LO problems based on the wide neighborhood introduced by Darvay and Takács [5]. At each iteration, our algorithm applies the Darvay-Takács directions and computes predictor direction and new corrector directions. Moreover, we prove that the proposed algorithm has the same iteration complexity as a small neighborhood algorithm. In this way, we overcome the problem of theoretical inefficiency of the large-update interior-point algorithms. Numerical results show that the proposed algorithm is efficient.

The rest of the paper is organized as follows. In Sect. 2, we introduce the LO problem and Ai-Zhang's method for Darvay's direction, and then we describe Darvay-Takács's wide neighborhood. In Sect. 3, we propose a new primal-dual predictor-corrector interior-point algorithm based on Darvay-Takács's wide neighborhood. In Sect. 4, we prove the global convergence of the proposed algorithm and derive the polynomial complexity bound of the algorithm. Numerical results are presented in Sect. 5. At the end, some concluding remarks are given in Sect. 6.

The following notations are used throughout the paper. The Euclidean norm and the one-norm of a vector are denoted by $\|.\|$ and $\|.\|_1$, respectively. For vectors $u, v \in \mathbb{R}^n$, we denote the Hadamard product of u and v by $uv = (u_1v_1, \ldots, u_nv_n)^T$. The positive and negative parts of $u_i \in \mathbb{R}$ are denoted by $u_i^+ := max\{u_i, 0\}$ and $u_i^- := min\{u_i, 0\}$, for i = 1, ..., n. Thus, for any $u = (u_1, \ldots, u_n)^T \in \mathbb{R}^n$, we denote $u^+ := (u_1^+, \ldots, u_n^+)^T$ and $u^- := (u_1^-, \ldots, u_n^-)^T$, such that $u = u^+ + u^-$. Finally, we denote the minimal component of any vector $u \in \mathbb{R}^n$ by u_{min}.

2. Preliminaries

In this section, we outline some basic facts about IPMs, and then, we describe Darvay-Takács's wide neighborhood. We consider the primal-dual pair of LO problems in standard form

$$(P) \quad min \ \{c^Tx\colon \ Ax = b, \quad x \geq 0\},$$
$$(D) \quad max \ \{b^Ty\colon \ A^Ty + s = c, \quad s \geq 0\},$$

where $A \in \mathbb{R}^{m\times n}$ has full row rank, $c, x, s \in \mathbb{R}^n$ and $b, y \in \mathbb{R}^m$. The feasibility set of (P) and (D) is defined as follows:

$$\mathcal{F} := \{(x, y, s)\colon \quad Ax = b, \quad A^Ty + s = c, \quad x \geq 0, \quad s \geq 0\},$$

and the strictly feasibility set of (P) and (D) is defined by

$$\mathcal{F}^0 := \{(x, y, s)\colon \quad Ax = b, \quad A^Ty + s = c, \quad x > 0, \quad s > 0\}.$$

By applying the self-dual embedding model proposed by Ye et al. [15] and Terlaky [13], we can assume that both (P) and (D) satisfy the interior-point condition (IPC), i.e., $\mathcal{F}^0$ is nonempty. The optimality conditions for (P) and (D) can be written as follows:

$$\begin{aligned} Ax &= b, \qquad x \geq 0, \\ A^T y + s &= c, \qquad s \geq 0, \\ xs &= 0. \end{aligned} \tag{1}$$

The basic approach of primal-dual IPMs is to replace the third equation in system (1) by the parameterized equation $xs = \mu e$, where $0 < \mu = \frac{x^T s}{n}$ and e is the all-one vector. Then, we consider the following perturbed system:

$$\begin{aligned} Ax &= b, \qquad x \geq 0, \\ A^T y + s &= c, \qquad s \geq 0, \\ xs &= \mu e. \end{aligned} \tag{2}$$

For each $\mu > 0$, system (2) has a unique solution, which is denoted by $(x(\mu), y(\mu), s(\mu))$. This solution is called a μ-center of the primal-dual pair (P) and (D). The set of μ-centers with all $\mu > 0$ gives the central path of (P) and (D) which is denoted as follows:

$$\mathcal{C} := \{(x, y, s) \in \mathcal{F}^0 \; : \; xs = \mu e, \mu > 0\}.$$

Therefore, as μ goes to zero, $(x(\mu), y(\mu), s(\mu))$ converges to a pair of optimal solutions of (P) and (D). Now, we apply the algebraic equivalent transformation introduced by Darvay [2]. Assume that $\phi: [0, \infty) \to [0, \infty)$ be a continuously differentiable function such that $\phi'(t) > 0$ for all $t > 0$. Hence, we rewrite the system (2) in the following form:

$$\begin{aligned} Ax &= b, \qquad x \geq 0, \\ A^T y + s &= c, \qquad s \geq 0, \\ \phi\left(\frac{xs}{\tau\mu}\right) &= \phi(e), \end{aligned} \tag{3}$$

where $\tau \in (0,1)$ is the centering parameter and $\phi(.)$ is the vector-valued function induced by the real-valued function $\phi(t)$ such that $\phi(\frac{xs}{\tau\mu}) = (\phi(\frac{x_i s_i}{\tau\mu}))_{1 \leq i \leq n}$. Applying Newton's method to (3) and using $\phi(t) = \sqrt{t}$ leads to the system (4):

$$\begin{aligned} A\Delta x &= 0, \\ A^T \Delta y + \Delta s &= 0, \\ s\Delta x + x\Delta s &= 2(\sqrt{\tau\mu xs} - xs). \end{aligned} \tag{4}$$

The main idea of Ai-Zhang's method [1] is to decompose the Newton direction into negative and positive parts corresponding to the negative and positive parts of the right-hand side of the third equation of Newton search directions system. Based on this idea, Darvay and Takács [5] obtained the following two systems:

$$\begin{aligned} A\Delta x_- &= 0, \\ A^T \Delta y_- + \Delta s_- &= 0, \\ s\Delta x_- + x\Delta s_- &= 2(\sqrt{\tau\mu xs} - xs)^-, \end{aligned} \tag{5}$$

and

$$\begin{aligned} A\Delta x_+ &= 0, \\ A^T\Delta y_+ + \Delta s_+ &= 0, \\ s\Delta x_+ + x\Delta s_+ &= 2(\sqrt{\tau\mu xs} - xs)^+. \end{aligned} \tag{6}$$

In the classical primal-dual IPM, all the iterates must remain in a certain neighborhood of the central path. One of the popular neighborhoods is the so-called small neighborhood, defined as

$$\mathcal{N}_2(\theta) \coloneqq \{(x,y,s) \in \mathcal{F}^0 \ : \ \| \ xs - \mu e \ \| \leq \theta\mu\},$$

where $\theta \in (0,1)$. Another popular neighborhood is the large neighborhood (negative infinity neighborhood), defined as

$$\mathcal{N}_\infty^-(1-\rho) \coloneqq \{(x,y,s) \in \mathcal{F}^0 \ : \ \ xs \geq \rho\mu e\},$$

where $\rho \in (0,1)$. The wide neighborhood introduced by Ai and Zhang [1], defined as

$$\mathcal{N}(\tau,\beta) \coloneqq \{(x,y,s) \in \mathcal{F}^0 \ : \ \| \ (\tau\mu e - xs)^+ \ \| \leq \beta\tau\mu\},$$

where $\beta, \tau \in (0,1)$. In this paper, we use the new wide neighborhood $\mathcal{W}(\tau,\beta)$ presented by Darvay and Takács [5], which is defined as follows

$$\mathcal{W}(\tau,\beta) \coloneqq \{(x,y,s) \in \mathcal{F}^0 \ : \ \| \ (\sqrt{\tau\mu}e - \sqrt{xs})^+ \ \| \leq \sqrt{\beta\tau\mu}\},$$

where $\beta, \tau \in (0,1)$. $\mathcal{W}(\tau,\beta)$ is a wide neighborhood, since $\mathcal{N}(\tau,\beta) \subseteq \mathcal{W}(\tau,\beta)$.

3. Primal-dual predictor-corrector algorithm

In this section, we propose a new primal-dual predictor-corrector algorithm for LO. We assume that the algorithm starts with an iterate $(x,y,s) \in \mathcal{W}(\tau,\frac{\beta}{2})$. We obtain the predictor directions by substituting $\tau = 0$ in (5) and (6). Therefore, the predictor directions are computed by the following system

$$\begin{aligned} A\Delta x_-^a &= 0, \\ A^T\Delta y_-^a + \Delta s_-^a &= 0, \\ s\Delta x_-^a + x\Delta s_-^a &= (-2xs)^-. \end{aligned} \tag{7}$$

Since $(-2xs)^+ = 0$, the Newton direction corresponding to the positive part is zero, i.e., $\Delta x_+^a = \Delta y_+^a = \Delta s_+^a = 0$. Then, we compute the largest step size $\bar{\alpha}_a \in [0,1]$, such that

$$\big(x(\alpha_a), y(\alpha_a), s(\alpha_a)\big) \in \mathcal{W}(\tau,\beta), \qquad \forall\alpha_a \in [0,\bar{\alpha}_a]. \tag{8}$$

The new iterate is defined as follows

$$\big(x(\alpha_a), y(\alpha_a), s(\alpha_a)\big) \coloneqq (x,y,s) + \alpha_a(\Delta x_-^a, \Delta y_-^a, \Delta s_-^a). \tag{9}$$

We use the directions Δx_-^a and Δs_-^a obtained from solving (7) to compute the negative part of the corrector directions in the following system:

$$\begin{aligned} A\Delta x_-^c &= 0, \\ A^T\Delta y_-^c + \Delta s_-^c &= 0, \\ s(\alpha_a)\Delta x_-^c + x(\alpha_a)\Delta s_-^c &= 2(\sqrt{\tau\mu(\alpha_a)x(\alpha_a)s(\alpha_a)} - x(\alpha_a)s(\alpha_a))^- - \alpha_a\Delta x_-^a\Delta s_-^a, \end{aligned} \tag{10}$$

and we obtain the positive part of the corrector directions by solving the following system

$$\begin{aligned} A\Delta x_+^c &= 0, \\ A^T\Delta y_+^c + \Delta s_+^c &= 0, \\ s(\alpha_a)\Delta x_+^c + x(\alpha_a)\Delta s_+^c &= 2(\sqrt{\tau\mu(\alpha_a)x(\alpha_a)s(\alpha_a)} - x(\alpha_a)s(\alpha_a))^+. \end{aligned} \tag{11}$$

Finally, the step size $\alpha = (\alpha_1, \alpha_2) \in [0,1]^2$ is computed such that $(x(\alpha), y(\alpha), s(\alpha)) \in \mathcal{W}(\tau, \frac{\beta}{2})$, and the new iterate is defined as follows

$$(x(\alpha), y(\alpha), s(\alpha)) := (x(\alpha_a), y(\alpha_a), s(\alpha_a)) + \alpha_1(\Delta x_-^c, \Delta y_-^c, \Delta s_-^c) + \alpha_2(\Delta x_+^c, \Delta y_+^c, \Delta s_+^c). \tag{12}$$

Now we present our new primal-dual predictor-corrector algorithm as follows:

Primal-dual predictor-corrector algorithm

Input:
Accuracy parameter $\epsilon > 0$;
neighborhood parameters, $0 < \tau \leq \frac{1}{16}$ and $0 < \beta \leq \frac{1}{18}$;
a strictly feasible point $(x^0, y^0, s^0) \in \mathcal{N}(\tau, \frac{\beta}{2}) \subseteq \mathcal{W}(\tau, \frac{\beta}{2})$.
Set $k := 0$;
If $(x^0)^T s^0 \leq \epsilon$, then stop; otherwise, go to the predictor step.

Predictor step
Compute the search direction $(\Delta x_-^{a,k}, \Delta y_-^{a,k}, \Delta s_-^{a,k})$ by (7);
Set $\alpha_a^k = \frac{1}{4}\sqrt{\frac{\beta\tau}{2n}}$;
Compute $(x(\alpha_a^k), y(\alpha_a^k), s(\alpha_a^k))$ by (9);
If $x(\alpha_a^k)^T s(\alpha_a^k) \leq \epsilon$, then stop; otherwise, go to the corrector step.

Corrector step
Compute the corrector directions $(\Delta x_-^{c,k}, \Delta y_-^{c,k}, \Delta s_-^{c,k})$ by (10) and $(\Delta x_+^{c,k}, \Delta y_+^{c,k}, \Delta s_+^{c,k})$ by (11);
Set $\alpha_2^k = 1$ and compute the largest step size $\alpha_1^k \in [\sqrt{\frac{\beta\tau}{2n}}, 1]$, such that
$(x(\alpha^k), y(\alpha^k), s(\alpha^k)) \in \mathcal{W}(\tau, \frac{\beta}{2})$;
Compute $(x(\alpha^k), y(\alpha^k), s(\alpha^k))$ by (12);
Set $(x^{k+1}, y^{k+1}, s^{k+1}) := (x(\alpha^k), y(\alpha^k), s(\alpha^k))$ and $k := k + 1$;
If $x(\alpha^k)^T s(\alpha^k) \leq \epsilon$, then stop; otherwise, go to the predictor step.

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4. Analysis of the algorithm

Before starting the algorithm analysis, we define the following notations:

$$v = \sqrt{xs}, \quad v(\alpha_a) = \sqrt{x(\alpha_a)s(\alpha_a)}, \quad dx_-^a = \frac{v\Delta x_-^a}{x}, \quad ds_-^a = \frac{v\Delta s_-^a}{s}$$

$$dx_-^c = \frac{v(\alpha_a)\Delta x_-^c}{x(\alpha_a)},\ ds_-^c = \frac{v(\alpha_a)\Delta s_-^c}{s(\alpha_a)},\ dx_+^c = \frac{v(\alpha_a)\Delta x_+^c}{x(\alpha_a)},\ ds_+^c = \frac{v(\alpha_a)\Delta s_+^c}{s(\alpha_a)}, \tag{13}$$

$$\mathcal{J} \coloneqq \{1,2,\dots,n\},\ \mathcal{J}^+ \coloneqq \{i \in \mathcal{J} : \ \sqrt{\tau\mu(\alpha_a)} - v(\alpha_a)_i > 0\},$$

$$\mathcal{J}^- \coloneqq \{i \in \mathcal{J} \ : \ \sqrt{\tau\mu(\alpha_a)} - v(\alpha_a)_i \le 0\}. \tag{14}$$

The following technical results are used to analyze the algorithm.

Proposition 1. (Proposition 3.1 in Ai and Zhang [1]) For any $u, v \in \mathbb{R}^n$ and $p \ge 1$, we have

$$\| (u+v)^+ \|_p \le \| u^+ \|_p + \| v^+ \|_p, \quad \| (u+v)^- \|_p \le \| u^- \|_p + \| v^- \|_p$$

Lemma 1. (Lemma 3.5 in Ai and Zhang [1]) Let $u, v \in \mathbb{R}^n$ be such that $u^T v \ge 0$, and let $r = u + v$. Then, we have $\| (uv)^- \|_1 \le \| (uv)^+ \|_1 \le \frac{1}{4} \| r \|^2$.

Lemma 2. (Lemma 5.3 in Wright [14]) Let $u, v \in \mathbb{R}^n$ be such that $u^T v \ge 0$, then

$$\| uv \| \le 2^{-\frac{3}{2}} \| u + v \|^2.$$

Lemma 3. (Lemma 3.4 in Liu et al. [8]) suppose $(x, y, s) \in \mathcal{F}^0$ and $z + 2xs \le 0$. Let $(\Delta x, \Delta y, \Delta s)$ be the solution of

$$A\Delta x = 0,$$
$$A^T\Delta y + \Delta s = 0,$$
$$s\Delta x + x\Delta s = z.$$

If $(x + t_0\Delta x)(s + t_0\Delta s) > 0$ for some $0 < t_0 \le 1$, then $(x + t\Delta x, s + t\Delta s) > 0$ for all $0 \le t \le t_0$.

4.1. Analysis of the predictor step

Using (9) and the third equation of the system (7), we have

$$x(\alpha_a)s(\alpha_a) = (x + \alpha_a\Delta x_-^a)(s + \alpha_a\Delta s_-^a) = (1 - 2\alpha_a)xs + \alpha_a^2(\Delta x_-^a\Delta s_-^a). \tag{15}$$

Since $(\Delta x_-^a)^T\Delta s_-^a = 0$, we obtain

$$\mu(\alpha_a) \coloneqq \frac{x(\alpha_a)^T s(\alpha_a)}{n} = (1 - 2\alpha_a)\mu. \tag{16}$$

Lemma 4. Suppose that $(x,y,s) \in \mathcal{N}(\tau,\frac{\beta}{2}) \subseteq \mathcal{W}(\tau,\frac{\beta}{2})$, then the largest possible value of step size α_a satisfying (8) is given by $\bar{\alpha}_a \geq \frac{1}{1+\sqrt{1+\frac{2n}{\beta\tau}}}$.

Proof. We have

$$\begin{aligned}\|(\sqrt{\tau\mu(\alpha_a)}e-\sqrt{x(\alpha_a)s(\alpha_a)})^+\| &= \left\|\frac{(\tau\mu(\alpha_a)e-x(\alpha_a)s(\alpha_a))^+}{\sqrt{\tau\mu(\alpha_a)}e+\sqrt{x(\alpha_a)s(\alpha_a)}}\right\| \\ &\leq \frac{1}{\sqrt{\tau\mu(\alpha_a)}}\|(\tau\mu(\alpha_a)e-x(\alpha_a)s(\alpha_a))^+\| \\ &= \frac{1}{\sqrt{\tau\mu(\alpha_a)}}\|((1-2\alpha_a)(\tau\mu e-xs)-\alpha_a^2(\Delta x_-^a\Delta s_-^a))^+\| \\ &\leq \frac{1}{\sqrt{\beta\tau\mu(\alpha_a)}}((1-2\alpha_a)\|(\tau\mu e-xs)^+\|+\alpha_a^2\|(-\Delta x_-^a\Delta s_-^a)^+\|),\end{aligned}$$

where the second equality is obtained from (15) and (16) and the second inequality is derived from proposition 1 and the fact that $\beta \leq 1$. Now, from system (7) and Lemma 1, for $u := x^{-1/2}s^{1/2}\Delta x_-^a$, $v := x^{1/2}s^{-1/2}\Delta s_-^a,\ r := -2(xs)^{1/2}$, we have

$$\|(\Delta x_-^a\Delta s_-^a)^-\|_1 \leq \|(\Delta x_-^a\Delta s_-^a)^+\|_1 \leq \frac{1}{4}\left\|2(xs)^{1/2}\right\|^2 = n\mu.$$

According to what was mentioned and the fact that $(x,y,s) \in \mathcal{N}(\tau,\frac{\beta}{2}) \subseteq \mathcal{W}(\tau,\frac{\beta}{2})$, we conclude that

$$\|(\sqrt{\tau\mu(\alpha_a)}e-\sqrt{x(\alpha_a)s(\alpha_a)})^+\| \leq \frac{1}{\sqrt{\beta\tau\mu(\alpha_a)}}\left((1-2\alpha_a)\frac{\beta}{2}\tau\mu+\alpha_a^2 n\mu\right).$$

To obtain the iterate in the neighborhood $\mathcal{W}(\tau,\beta)$, the following relation must be established,

$$\frac{1}{\sqrt{(1-2\alpha_a)\beta\tau\mu}}\left((1-2\alpha_a)\frac{\beta}{2}\tau\mu+\alpha_a^2 n\mu\right) \leq \sqrt{\beta\tau\mu(\alpha_a)} = \sqrt{(1-2\alpha_a)\beta\tau\mu}. \quad (17)$$

It can be easily proved that the largest α_a satisfying (17) is the positive root of the quadratic function $g(\alpha_a) := n\alpha_a^2+\beta\tau\alpha_a-\frac{\beta\tau}{2}$, which is $\frac{1}{1+\sqrt{1+\frac{2n}{\beta\tau}}}$. Thus, for all $0 \leq \alpha_a \leq \frac{1}{1+\sqrt{1+\frac{2n}{\beta\tau}}}$, we have $g(\alpha_a) \leq 0$. Therefore, the proof of the lemma is completed. □

4.2. Analysis of the corrector step

We define

$$dx(\alpha) = \alpha_1 dx_-^c + \alpha_2 dx_+^c, \quad ds(\alpha) = \alpha_1 ds_-^c + \alpha_2 ds_+^c, \quad (18)$$

and

$$h(\alpha) = v(\alpha_a)^2 + 2v(\alpha_a)\big(\alpha_1(\sqrt{\tau\mu(\alpha_a)}e - v(\alpha_a))^- + \alpha_2(\sqrt{\tau\mu(\alpha_a)}e - v(\alpha_a))^+\big) - \alpha_1\alpha_a dx_-^a ds_-^a. \quad (19)$$

Therefore, we have

$$x(\alpha)s(\alpha) = h(\alpha) + dx(\alpha)ds(\alpha). \quad (20)$$

Remark 1. If $\alpha_a \leq \frac{\alpha_1}{4} \leq \frac{\alpha_2}{4}\sqrt{\frac{\beta\tau}{2n}}$, then $\alpha_a \leq \frac{1}{1+\sqrt{1+\frac{2n}{\beta\tau}}}$. Therefore, from Lemma 4 we conclude that $(x(\alpha_a), y(\alpha_a), s(\alpha_a)) \in \mathcal{W}(\tau,\beta)$.

Corollary 1. If $(x(\alpha_a), y(\alpha_a), s(\alpha_a)) \in \mathcal{W}(\tau,\beta)$, then

$$v_{min}(\alpha_a) \geq (1-\sqrt{\beta})\sqrt{\tau\mu(\alpha_a)}.$$

Proof. Since $(x(\alpha_a), y(\alpha_a), s(\alpha_a)) \in \mathcal{W}(\tau,\beta)$, we have

$$\left\|(\sqrt{\tau\mu(\alpha_a)}e - \sqrt{x(\alpha_a)s(\alpha_a)})^+\right\| \leq \sqrt{\beta\tau\mu(\alpha_a)}.$$

For $i \in \mathcal{I}^+$, we have $\sum_{i\in\mathcal{I}^+}\left(\sqrt{\tau\mu(\alpha_a)} - \sqrt{x(\alpha_a)_i s(\alpha_a)_i}\right)^2 \leq \beta\tau\mu(\alpha_a)$ and since $\sqrt{\tau\mu(\alpha_a)} - \sqrt{x(\alpha_a)_i s(\alpha_a)_i} > 0$, we conclude that

$$\sqrt{\tau\mu(\alpha_a)} - \sqrt{x(\alpha_a)_i s(\alpha_a)_i} \leq \sqrt{\beta\tau\mu(\alpha_a)}.$$

Therefore, $\sqrt{x(\alpha_a)_i s(\alpha_a)_i} \geq (1-\sqrt{\beta})\sqrt{\tau\mu(\alpha_a)}$. Also, for $i \in \mathcal{I}^-$, we have $\sqrt{x(\alpha_a)_i s(\alpha_a)_i} \geq \sqrt{\tau\mu(\alpha_a)} \geq (1-\sqrt{\beta})\sqrt{\tau\mu(\alpha_a)}$. Thus, for all $i \in \mathcal{I}$, we have $\sqrt{x(\alpha_a)_i s(\alpha_a)_i} \geq (1-\sqrt{\beta})\sqrt{\tau\mu(\alpha_a)}$. Therefore, we obtain

$$v_{min}(\alpha_a) \geq (1-\sqrt{\beta})\sqrt{\tau\mu(\alpha_a)}.$$

This completes the proof. □

Proposition 2. We have $dx_-^{a^T} ds_-^a = dx_-^{c^T} ds_-^c = dx_+^{c^T} ds_-^c = 0$, and $dx(\alpha)^T ds(\alpha) = 0$.

Proof. From the first two equations of systems (7), (10) and (11) and the notations given in (13), we conclude the proof directly. □

Lemma 5. Suppose that $(x,y,s) \in \mathcal{F}^0$ and $(\Delta x_-^a, \Delta y_-^a, \Delta s_-^a)$ be the solutions of (7). Then, for each $1 \leq i \leq n$,

$$(dx_-^a)_i (ds_-^a)_i \leq v_i^2.$$

Proof. From the third equation of system (7), we have

$$(dx_{-}^{a})_i + (ds_{-}^{a})_i = (-2v_i)^- \leq 0. \tag{21}$$

Then, $(dx_{-}^{a})_i(ds_{-}^{a})_i \leq \frac{((dx_{-}^{a})_i+(ds_{-}^{a})_i)^2}{4} = v_i^2$. Thus, the proof of the lemma is completed. □

Remark 2. From Lemma 5 and (15), we have $(dx_{-}^{a})_i + (ds_{-}^{a})_i = -2v_i \leq -2v(\alpha_a)_i$. Therefore, we obtain $(dx_{-}^{a})_i(ds_{-}^{a})_i \leq v(\alpha_a)_i^2$.

Lemma 6. Suppose $(x(\alpha_a), y(\alpha_a), s(\alpha_a)) \in \mathcal{W}(\tau,\beta)$ and $0 \leq \alpha_a \leq \frac{\alpha_1}{4} \leq \frac{\alpha_2}{4}\sqrt{\frac{\beta\tau}{2n}}$. Then

$$\|(dx(\alpha)ds(\alpha))^-\|_1 \leq \|(dx(\alpha)ds(\alpha))^+\|_1 \leq \frac{1}{4}\left(\sqrt{6} + \frac{12\sqrt{\beta}}{(47\sqrt{2})(1-\sqrt{\beta})}\right)^2 \alpha_2^2\beta\tau\mu(\alpha_a)$$

Proof. From systems (7), (10), (11) and proposition 2, we have

$$dx(\alpha) + ds(\alpha) = 2\alpha_1(\sqrt{\tau\mu(\alpha_a)}e - v(\alpha_a))^- + 2\alpha_2(\sqrt{\tau\mu(\alpha_a)}e - v(\alpha_a))^+ \\ - \alpha_1\alpha_a v(\alpha_a)^{-1}dx_{-}^{a}ds_{-}^{a}.$$

Then, using Lemma 1, we have

$$\|(dx(\alpha)ds(\alpha))^-\|_1 \leq \|(dx(\alpha)ds(\alpha))^+\|_1 \leq \frac{1}{4}\|dx(\alpha) + ds(\alpha)\|^2$$
$$= \frac{1}{4}\left\|2\alpha_1(\sqrt{\tau\mu(\alpha_a)}e - v(\alpha_a))^- + 2\alpha_2(\sqrt{\tau\mu(\alpha_a)}e - v(\alpha_a))^+ - \alpha_1\alpha_a v(\alpha_a)^{-1}dx_{-}^{a}ds_{-}^{a}\right\|^2$$
$$\leq \frac{1}{4}\left(\left\|2\alpha_1(\sqrt{\tau\mu(\alpha_a)}e - v(\alpha_a))^- + 2\alpha_2(\sqrt{\tau\mu(\alpha_a)}e - v(\alpha_a))^+\right\| + \frac{\alpha_1\alpha_a}{v_{min}(\alpha_a)}\|dx_{-}^{a}ds_{-}^{a}\|\right)^2.$$

Using lemma assumptions, we conclude that

$$\left\|2\alpha_1(\sqrt{\tau\mu(\alpha_a)}e - v(\alpha_a))^- + 2\alpha_2(\sqrt{\tau\mu(\alpha_a)}e - v(\alpha_a))^+\right\|^2$$
$$= 4\alpha_1^2\left\|(\sqrt{\tau\mu(\alpha_a)}e - v(\alpha_a))^-\right\|^2 + 4\alpha_2^2\left\|(\sqrt{\tau\mu(\alpha_a)}e - v(\alpha_a))^+\right\|^2$$
$$\leq 4\alpha_1^2\sum_{i\in I^-}(\sqrt{\tau\mu(\alpha_a)} - v(\alpha_a)_i)^2 + 4\alpha_2^2\beta\tau\mu(\alpha_a)$$
$$\leq 4\alpha_1^2\sum_{i\in I^-}v(\alpha_a)_i^2 + 4\alpha_2^2\beta\tau\mu(\alpha_a)$$
$$= 4\alpha_1^2 n\mu(\alpha_a) + 4\alpha_2^2\beta\tau\mu(\alpha_a)$$
$$\leq 6\alpha_2^2\beta\tau\mu(\alpha_a). \tag{22}$$

We obtain the following result from Proposition 2, Lemma 2 and system (7),

$$\| dx_-^a ds_-^a \| \leq 2^{-\frac{3}{2}} \| dx_-^a + ds_-^a \|^2 = 2^{-\frac{3}{2}} \| (-2v)^- \|^2 = \sqrt{2} n\mu. \quad (23)$$

Using (22), (23) and Corollary 1, we have

$$\|(dx(\alpha)ds(\alpha))^+\|_1 \leq \frac{1}{4}\left(\sqrt{6}\sqrt{\alpha_2^2\beta\tau\mu(\alpha_a)} + \frac{\sqrt{2}\alpha_1\alpha_a n\mu}{1-\sqrt{\beta}\sqrt{\tau\mu(\alpha_a)}}\right)^2$$

$$= \frac{1}{4}\left(\sqrt{6}\sqrt{\alpha_2^2\beta\tau\mu(\alpha_a)} + \frac{\sqrt{2}\alpha_1\alpha_a n\sqrt{\mu(\alpha_a)}}{(1-\sqrt{\beta})\sqrt{\tau}(1-2\alpha_a)}\right)^2$$

$$\leq \frac{1}{4}\left(\sqrt{6}\sqrt{\alpha_2^2\beta\tau\mu(\alpha_a)} + \frac{\beta\sqrt{\tau\mu(\alpha_a)}\alpha_2^2}{4\sqrt{2}(1-\sqrt{\beta})(1-2\alpha_a)}\right)^2$$

$$\leq \frac{1}{4}\left(\sqrt{6} + \frac{12\sqrt{\beta}}{(47\sqrt{2})(1-\sqrt{\beta})}\right)^2 \alpha_2^2\beta\tau\mu(\alpha_a).$$

Where the first equality is obtained from (16) and the second and third inequalities are derived from the fact that $\alpha_a \leq \frac{\alpha_1}{4} \leq \frac{\alpha_2}{4}\sqrt{\frac{\beta\tau}{2n}}$. □

Lemma 7. Suppose $(x(\alpha_a), y(\alpha_a), s(\alpha_a)) \in \mathcal{W}(\tau,\beta)$, $\alpha = (\alpha_1, \alpha_2)$ and $\mu(\alpha) = \frac{x(\alpha)^T s(\alpha)}{n}$. Then

1) $\mu(\alpha) \geq (1-2\alpha_1)\mu(\alpha_a)$.
2) $\mu(\alpha) \leq \left(1 - 2\alpha_1(1-\tau)\frac{1-\sqrt{\beta}}{2-\sqrt{\beta}} + \frac{2\alpha_2}{\sqrt{n}}\tau\sqrt{\beta}\right)\mu(\alpha_a)$.

Proof. Using (20) and proposition 2, similar to the proof of lemma 5 in Darvay et al. [4], the proof is completed. □

Using (19) and Remark 2, if $i \in \mathcal{I}^+$, we have

$$\begin{aligned} h(\alpha)_i &= v(\alpha_a)_i^2 + 2v(\alpha)_i\left(\alpha_1(\sqrt{\tau\mu(\alpha_a)} - v(\alpha_a)_i)^- + \alpha_2(\sqrt{\tau\mu(\alpha_a)} - v(\alpha_a)_i)^+\right) \\ &\quad -\alpha_1\alpha_a(dx_-^a)_i(ds_-^a)_i \\ &\geq v(\alpha_a)_i^2 + 2\alpha_2 v(\alpha_a)_i(\sqrt{\tau\mu(\alpha_a)} - v(\alpha_a)_i) - \alpha_1\alpha_a v(\alpha_a)_i^2 \\ &= (1-\alpha_1\alpha_a)v(\alpha_a)_i^2 + 2\alpha_2 v(\alpha_a)_i(\sqrt{\tau\mu(\alpha_a)} - v(\alpha_a)_i) > 0. \end{aligned} \quad (24)$$

On the other hand, if $i \in \mathcal{I}^-$, we have

$$\begin{aligned} & h(\alpha)_i = v(\alpha)_i^2 + 2v(\alpha)_i\left(\alpha_1(\sqrt{\tau\mu(\alpha_a)} - v(\alpha_a)_i)^- + \alpha_2(\sqrt{\tau\mu(\alpha_a)} - v(\alpha_a)_i)^+\right) \\ & -\alpha_1\alpha_a(dx_-^a)_i(ds_-^a)_i \\ & \geq v(\alpha_a)_i^2 + 2\alpha_1 v(\alpha_a)_i(\sqrt{\tau\mu(\alpha_a)} - v(\alpha_a)_i) - \alpha_1\alpha_a v(\alpha_a)_i^2 \\ & = (1-2\alpha_1-\alpha_1\alpha_a)v(\alpha_a)_i^2 + 2\alpha_1\sqrt{\tau\mu}v(\alpha_a)_i. \end{aligned} \quad (25)$$

Using Lemma 3 for $t_0 = 1$ and (20), we deduce

$$(x + \Delta x(\alpha))(s + \Delta s(\alpha)) = x(\alpha)s(\alpha) = h(\alpha) + dx(\alpha)ds(\alpha). \tag{26}$$

We prove the strict feasibility of the new iterates in the following lemma.

Lemma 8. Suppose $(x(\alpha_a), y(\alpha_a), s(\alpha_a)) \in \mathcal{W}(\tau, \beta), \tau \le \frac{1}{16}$ and $\beta \le \frac{1}{18}$. *If* $\alpha_a = \frac{\alpha_1}{4} = \frac{\alpha_2}{4}\sqrt{\frac{\beta\tau}{2n}}$ and $\alpha_2 = 1$, then $x(\alpha) > 0$ and $s(\alpha) > 0$.

Proof. By applying the definition of $(\Delta x(\alpha), \Delta y(\alpha), \Delta s(\alpha))$ and systems (10) and (11), we can write the following system

$$\begin{gathered} A\Delta x(\alpha) = 0 \\ A^T\Delta y(\alpha) + \Delta s(\alpha) = 0 \\ s(\alpha_a)\Delta x(\alpha) + x(\alpha_a)\Delta s(\alpha) = 2\alpha_1(\sqrt{\tau\mu(\alpha_a)x(\alpha_a)s(\alpha_a)} - x(\alpha_a)s(\alpha_a))^- \\ +2\alpha_2(\sqrt{\tau\mu(\alpha_a)x(\alpha_a)s(\alpha_a)} - x(\alpha_a)s(\alpha_a))^+ - \alpha_1\alpha_a\Delta x_-^a\Delta s_-^a. \end{gathered}$$

According to the system of Lemma 3, we get

$$\begin{aligned} z &= 2\alpha_1(\sqrt{\tau\mu(\alpha_a)x(\alpha_a)s(\alpha_a)} - x(\alpha_a)s(\alpha_a))^- + 2\alpha_2(\sqrt{\tau\mu(\alpha_a)x(\alpha_a)s(\alpha_a)} - x(\alpha_a)s(\alpha_a))^+ \\ &\quad -\alpha_1\alpha_a\Delta x_-^a\Delta s_-^a \\ &= 2\alpha_1 v(\alpha_a)(\sqrt{\tau\mu(\alpha_a)}e - v(\alpha_a))^- + 2\alpha_2 v(\alpha_a)(\sqrt{\tau\mu(\alpha_a)}e - v(\alpha_a))^+ - \alpha_1\alpha_a dx_-^a ds_-^a. \end{aligned}$$

Since $\alpha_2 = 1, \tau \le \frac{1}{16}, \beta \le \frac{1}{18}, \alpha_a = \frac{\alpha_1}{4} = \frac{\alpha_2}{4}\sqrt{\frac{\beta\tau}{2n}}$, and $n \ge 1$, we have $\alpha_1 \le \frac{1}{24},\ \alpha_a \le \frac{1}{96}$. These imply that $1 - 2\alpha_1 - \alpha_1\alpha_a > 0$. Therefore, using (24) and (25), we conclude that $h(\alpha)_i > 0$ for all $i \in \mathcal{I}$. Now, due to the definition of $h(\alpha)$, we have

$$z + 2x(\alpha_a)s(\alpha_a) = v(\alpha_a)^2 + h(\alpha) > 0. \tag{27}$$

By applying (20), Lemma 6, (24) and Corollary 1, we obtain for $i \in \mathcal{I}^+$:

$$\begin{aligned} x(\alpha)_i s(\alpha)_i &= h(\alpha)_i + dx(\alpha)_i ds(\alpha)_i \ge h(\alpha)_i - \|(dx(\alpha)ds(\alpha))^-\|_1 \\ &\ge (1 - \alpha_1\alpha_a)v(\alpha_a)_i^2 - \frac{1}{4}\left(\sqrt{6} + \frac{12\sqrt{\beta}}{47\sqrt{2}(1-\sqrt{\beta})}\right)^2 \alpha_2^2\beta\tau\mu(\alpha_a) \\ &\ge (1 - \frac{\beta\tau}{8n})(1-\sqrt{\beta})^2\tau\mu(\alpha_a) - \frac{1}{4}\left(\sqrt{6} + \frac{12\sqrt{\beta}}{47\sqrt{2}(1-\sqrt{\beta})}\right)^2 \beta\tau\mu(\alpha_a) \\ &\ge \left(\frac{2303}{2304}(\frac{3\sqrt{2}-1}{3\sqrt{2}})^2 - \frac{1}{72}\left(\sqrt{6} + \frac{12}{47\sqrt{2}(3\sqrt{2}-1)}\right)^2\right)\tau\mu(\alpha_a) > 0. \end{aligned}$$

Moreover, for $i \in \mathcal{I}^-$, the inequality $x(\alpha)_i s(\alpha)_i > 0$ can be proved similarly by using (20), Lemma 6, (25) and Corollary 1. Thus, we conclude that $x(\alpha)s(\alpha) > 0$. Due to relation (27), we use Lemma 3 and deduce that $x(\alpha) > 0,\ s(\alpha) > 0$, which completes the proof. □

Lemma 9. Assuming that the assumptions of Lemma 8 hold, then we have

$$\|(\tau\mu(\alpha)e - h(\alpha))^+\| \le \left(1 - \frac{7(14-9\sqrt{2})}{8(12-\sqrt{2})}\alpha_1\right)\beta\tau\mu(\alpha_a).$$

Proof. For $i \in \mathcal{J}^+$, we use the second part of Lemma 7 and (24) and conclude that

$$\begin{aligned}
\tau\mu(\alpha) - h(\alpha)_i &\le \tau\left(1 - 2(1-\tau)\frac{1-\sqrt{\beta}}{2-\sqrt{\beta}}\alpha_1 + \frac{2\tau\sqrt{\beta}}{\sqrt{n}}\alpha_2\right)\mu(\alpha_a) \\
&\quad -(1-\alpha_1\alpha_a)v(\alpha_a)_i^2 - 2\alpha_2 v(\alpha_a)_i(\sqrt{\tau\mu(\alpha_a)} - v(\alpha_a)_i) \\
&\le \tau\left(1 - \alpha_1(2(1-\tau)\frac{1-\sqrt{\beta}}{2-\sqrt{\beta}} - 2\sqrt{2}\sqrt{\tau})\right)\mu(\alpha_a) \\
&\quad -(1-\alpha_1\alpha_a)v(\alpha_a)_i^2 - 2\alpha_2 v(\alpha_a)_i(\sqrt{\tau\mu(\alpha_a)} - v(\alpha_a)_i) \\
&\le \tau(1 - \frac{7(14-9\sqrt{2})}{8(12-\sqrt{2})}\alpha_1)\mu(\alpha_a) - (1-\alpha_1\alpha_a)v(\alpha_a)_i^2 - 2v(\alpha_a)_i(\sqrt{\tau\mu(\alpha_a)} - v(\alpha_a)_i) \\
&= (\sqrt{\tau\mu(\alpha_a)} - v(\alpha_a)_i)^2 + \alpha_1\left(\alpha_a v(\alpha_a)_i^2 - \frac{7(14-9\sqrt{2})}{8(12-\sqrt{2})}\tau\mu(\alpha_a)\right) \\
&\le (\sqrt{\tau\mu(\alpha_a)} - v(\alpha_a)_i)^2 + \alpha_1\left(\frac{1}{96}v(\alpha_a)_i^2 - \frac{7(14-9\sqrt{2})}{8(12-\sqrt{2})}\tau\mu(\alpha_a)\right) \\
&\le (\sqrt{\tau\mu(\alpha_a)} - v(\alpha_a)_i)^2 - \frac{7(14-9\sqrt{2})}{8(12-\sqrt{2})}\alpha_1(\tau\mu(\alpha_a) - v(\alpha_a)_i^2) \\
&\le (1 - \frac{7(14-9\sqrt{2})}{8(12-\sqrt{2})}\alpha_1)(\sqrt{\tau\mu(\alpha_a)} - v(\alpha_a)_i)^2.
\end{aligned}$$

The last inequality is derived from the fact that $\tau\mu(\alpha_a) - v(\alpha_a)_i^2 \ge (\sqrt{\tau\mu(\alpha_a)} - v(\alpha_a)_i)^2$ for any $i \in \mathcal{J}^+$. On the other hand, we have $1 - 2\alpha_1 - \alpha_1\alpha_a > 0$. Applying the second part of Lemma 7 and (25), we obtain for $i \in \mathcal{J}^-$

$$\begin{aligned}
\tau\mu(\alpha) - h(\alpha)_i &\le \tau\left(1 - 2(1-\tau)\frac{1-\sqrt{\beta}}{2-\sqrt{\beta}}\alpha_1 + \frac{2\tau\sqrt{\beta}}{\sqrt{n}}\alpha_2\right)\mu(\alpha_a) \\
&\quad -(1-2\alpha_1-\alpha_1\alpha_a)v(\alpha_a)_i^2 - 2\alpha_1\sqrt{\tau\mu(\alpha_a)}v(\alpha_a)_i \\
&\le \tau\left(1 - \frac{6\sqrt{2}-2}{6\sqrt{2}-1}(1-\tau)\alpha_1 + \frac{2\tau\sqrt{\beta}}{\sqrt{n}}\alpha_2\right)\mu(\alpha_a) \\
&\quad -(1-2\alpha_1-\alpha_1\alpha_a)\tau\mu(\alpha_a) - 2\alpha_1\tau\mu(\alpha_a) \\
&= \left(-\frac{6\sqrt{2}-2}{6\sqrt{2}-1}(1-\tau) + 2\sqrt{2}\sqrt{\tau} + \alpha_a\right)\alpha_1\tau\mu(\alpha_a) \\
&\le \left(-\frac{15(6\sqrt{2}-2)}{16(6\sqrt{2}-1)} + \frac{1}{\sqrt{2}} + \frac{1}{96}\right)\alpha_1\tau\mu(\alpha_a) \le 0.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\|(\tau\mu(\alpha)e-h(\alpha))^+\| &\le \left(1-\frac{7(14-9\sqrt{2})}{8(12-\sqrt{2})}\alpha_1\right)\left\|((\sqrt{\tau\mu(\alpha_a)}-v(\alpha_a))^+)^2\right\|\\ &\le \left(1-\frac{7(14-9\sqrt{2})}{8(12-\sqrt{2})}\alpha_1\right)\left\|(\sqrt{\tau\mu(\alpha_a)}-v(\alpha_a))^+\right\|^2\\ &\le \left(1-\frac{7(14-9\sqrt{2})}{8(12-\sqrt{2})}\alpha_1\right)\beta\tau\mu(\alpha_a),\end{aligned}$$

where the last inequality is obtained from $(x(\alpha_a), y(\alpha_a), s(\alpha_a)) \in \mathcal{W}(\tau,\beta)$. This completes the proof of the lemma. □

The following lemma shows that the new iterates of the algorithm lie in the wide neighborhood $\mathcal{W}(\tau,\frac{\beta}{2})$.

Lemma 10. Assuming that the assumptions of Lemma 8 is true, then $(x(\alpha), y(\alpha), s(\alpha)) \in \mathcal{W}(\tau,\frac{\beta}{2})$.

Proof. Applying (20), Lemma 6, Lemma 9 and Proposition 1, we conclude that

$$\begin{aligned}\left\|(\sqrt{\tau\mu(\alpha)}e-\sqrt{x(\alpha)s(\alpha)})^+\right\| &= \left\|\frac{(\tau\mu(\alpha)e-x(\alpha)s(\alpha))^+}{\sqrt{\tau\mu(\alpha)}e+\sqrt{x(\alpha)s(\alpha)}}\right\|\\ &\le \frac{1}{\sqrt{\tau\mu(\alpha)}}\|(\tau\mu(\alpha)e-x(\alpha)s(\alpha))^+\|\\ &= \frac{1}{\sqrt{\tau\mu(\alpha)}}\|(\tau\mu(\alpha)e-h(\alpha)-dx(\alpha)ds(\alpha))^+\|\\ &\le \frac{1}{\sqrt{\tau\mu(\alpha)}}(\|(\tau\mu(\alpha)e-h(\alpha))^+\|+\|(-dx(\alpha)ds(\alpha))^+\|)\\ &= \frac{1}{\sqrt{\tau\mu(\alpha)}}(\|(\tau\mu(\alpha)e-h(\alpha))^+\|+\|(dx(\alpha)ds(\alpha))^-\|)\\ &\le \frac{1}{\sqrt{\tau\mu(\alpha)}}\left(\left(1-\frac{7(14-9\sqrt{2})}{8(12-\sqrt{2})}\alpha_1\right)\beta\tau\mu(\alpha_a)+\frac{1}{4}\left(\sqrt{6}+\frac{12\sqrt{\beta}}{47\sqrt{2}(1-\sqrt{\beta})}\right)^2\alpha_2^2\beta\tau\mu(\alpha_a)\right)\\ &\le \frac{1}{\sqrt{\tau\mu(\alpha)}}\left(1+\frac{1}{4}\left(\sqrt{6}+\frac{12\sqrt{\beta}}{47\sqrt{2}(1-\sqrt{\beta})}\right)^2\right)\beta\tau\mu(\alpha_a). \qquad (28)\end{aligned}$$

Now, we use the first part of Lemma 7, $\alpha_1 = \sqrt{\frac{\beta\tau}{2n}}$ and $n \ge 1$, therefore, we obtain

$$\mu(\alpha) \ge (1-2\alpha_1)\mu(\alpha_a) \ge (1-\sqrt{2\beta\tau})\mu(\alpha_a), \qquad (29)$$

which yields

$$\mu(\alpha_a) \le \frac{\mu(\alpha)}{(1-\sqrt{2\beta\tau})}. \qquad (30)$$

We substitute (30) into (28) and deduce

$$
\begin{aligned}
\|(\sqrt{\tau\mu(\alpha)}e-\sqrt{x(\alpha)s(\alpha)})^{+}\| &\leq \frac{1}{\sqrt{\tau\mu(\alpha)}}\left(1+\frac{1}{4}\left(\sqrt{6}+\frac{12\sqrt{\beta}}{47\sqrt{2}(1-\sqrt{\beta})}\right)^{2}\right)\frac{\beta\tau\mu(\alpha)}{(1-\sqrt{2\beta\tau})} \\
&= \left(1+\frac{1}{4}\left(\sqrt{6}+\frac{12\sqrt{\beta}}{47\sqrt{2}(1-\sqrt{\beta})}\right)^{2}\right)\frac{\sqrt{2\beta}}{(1-\sqrt{2\beta\tau})}\sqrt{\frac{\beta}{2}\tau\mu(\alpha)} \\
&\leq \left(1+\frac{1}{4}(\sqrt{6}+\frac{12}{47\sqrt{2}(3\sqrt{2}-1)})^{2}\right)\frac{4}{11}\sqrt{\frac{\beta}{2}\tau\mu(\alpha)} \\
&\leq 0.934\sqrt{\frac{\beta}{2}\tau\mu(\alpha)} \leq \sqrt{\frac{\beta}{2}\tau\mu(\alpha)}.
\end{aligned}
$$

Hence $(x(\alpha), y(\alpha), s(\alpha)) \in \mathcal{W}(\tau, \frac{\beta}{2})$, which completes the proof. □

Theorem 1. Suppose $\tau \leq \frac{1}{16},\ \beta \leq \frac{1}{18},\ \alpha_2 = 1,\ \alpha_a = \frac{1}{4}\sqrt{\frac{\beta\tau}{2n}}$ and $\alpha_1 = \sqrt{\frac{\beta\tau}{2n}}$. Then, the algorithm will terminate in $O(\sqrt{n}log\frac{(x^0)^T s^0}{\epsilon})$ iterations with the solution such that $x^T s \leq \epsilon$.

Proof. From Remark 1 and Lemma 10, we have

$$
(x(\alpha_a), y(\alpha_a), s(\alpha_a)) \in \mathcal{W}(\tau, \beta), \quad (x(\alpha), y(\alpha), s(\alpha)) \in \mathcal{W}(\tau, \frac{\beta}{2}).
$$

Using the second part of Lemma 7, we obtain

$$
\begin{aligned}
\mu(\alpha) &\leq \left(1-2(1-\tau)\frac{1-\sqrt{\beta}}{2-\sqrt{\beta}}\alpha_1+\frac{2\tau\sqrt{\beta}}{\sqrt{n}}\alpha_2\right)\mu(\alpha_a) \\
&\leq \left(1-\frac{7(14-9\sqrt{2})}{8(12-\sqrt{2})}\alpha_1\right)\mu(\alpha_a) \\
&= \left(1-\frac{7(14-9\sqrt{2})\sqrt{\beta\tau}}{8\sqrt{2}(12-\sqrt{2})\sqrt{n}}\right)\mu(\alpha_a).
\end{aligned}
$$

By Theorem 3.2 in Wright [14], the desired result is obtained. □

5. Numerical results

In this section, we compare the proposed primal-dual predictor-corrector algorithm in this paper (algorithm a) with the second-order corrector algorithm presented in Liu et al. [8] (algorithm b) and the primal-dual predictor-corrector algorithm proposed in Sayadi Shahraki et al. [12] (algorithm c). The test problems are taken from Netlib test collection and implemented in MATLAB R2016a on an Intel Core i5 (2.5GHz) under Windows 10. We use the self-dual embedding technique presented by Terlaky [13] to obtain the strictly feasible vectors $x^0 = ones(n,1), y^0 = ones(m,1), s^0 =$

$ones(n,1)$, as starting points of the algorithm. We set $\beta = \frac{1}{20}$ and $\tau = \frac{1}{16}$ for all three algorithms. We stop the iteration of algorithms if the relative duality gap satisfies $\frac{x^T s}{(x^0)^T s^0 + 1} \leq 10^{-8}$. For "algorithm a", we use bisection in closed interval $[\frac{1}{1+\sqrt{1+\frac{2n}{\beta\tau}}}, 1]$ and repeat this procedure at most ten times to determine the greatest α_a such that $(x(\alpha_a), y(\alpha_a), s(\alpha_a)) \in \mathcal{W}(\tau, \beta)$. Moreover, we repeat bisection procedure ten times in closed interval $[\sqrt{\frac{\beta\tau}{2n}}, 1]$ to obtain the greatest α_1 such that $(x(\alpha), y(\alpha), s(\alpha)) \in \mathcal{W}(\tau, \frac{\beta}{2})$. For "algorithm b" and "algorithm c", we also use bisection procedure in closed interval $[\sqrt{\frac{\beta\tau}{2n}}, 1]$ to determine the greatest α_1 such that $(x(\alpha), y(\alpha), s(\alpha)) \in \mathcal{N}(\tau, \frac{\beta}{2})$. The number of iterations (It) and CPU times (Time) are presented in Table 1. The numerical results show that the presented algorithm in this paper is efficient and reliable.

Table 1. Numerical results

	Algorithm (a)		Algorithm (b)		Algorithm (c)	
Problem	It.	Time	It.	Time	It.	Time
adlittle	13	0.2496	20	0.3028	20	0.5426
afiro	8	0.0758	17	0.1049	17	0.2018
bandm	20	3.4097	31	3.8159	31	4.4606
beaconfd	10	0.9487	18	1.3390	18	1.6518
blend	9	0.2019	17	0.3076	17	0.3666
capri	19	2.8912	35	3.7838	35	4.7748
e226	20	2.5650	31	3.0646	31	3.7748
kb2	9	0.1330	13	0.1640	13	0.1955
lotfi	15	1.4553	29	3.1054	29	2.5571
scagr7	12	0.4422	20	0.6754	20	0.7678
scagr25	15	5.4726	25	6.7255	25	8.1264
scsd1	11	1.7705	17	2.0271	17	2.2994
scsd6	14	7.2206	21	8.9914	21	11.6456
sc50a	10	0.1045	15	0.1280	15	0.1749
sc50b	8	0.0918	13	0.1229	13	0.1509
sc105	10	0.2858	14	0.3139	14	0.3897
sc205	11	0.7369	16	0.7653	16	0.9337
vtp-base	18	1.4524	34	2.2141	34	2.4858

6. Conclusions

In this paper, we presented a new primal-dual predictor-corrector interior-point algorithm for LO problems based on Darvay-Takács's wide neighborhood. We proved that the complexity bound of this algorithm is $O(\sqrt{n} log \frac{(x^0)^T s^0}{\epsilon})$, which coincides with the best-known complexity bound obtained for short-update algorithms. Moreover, we provided some numerical results, which show the efficiency and accuracy of our algorithm for solving LO problems. Finally, for future research, the proposed algorithm can be extended to SDO and SOCO.

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