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Computing maximum proportion and most violated sets

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In Fisher's and Arrow-Debreu's market equilibrium models with linear utilities, a set B of buyers and a set G of divisible goods, suppose that there are some buyers with surplus money w.r.t current prices of goods. If there does not exists an equilibrium, then, there are some buyers with surplus money w.r.t the given prices. A set of buyers with surplus money called a violated set. Computing this set helps to find the set of buyers with maximum surplus money w.r.t the given prices. In this paper, two new kinds of violated sets are defined, which called maximum proportion and most violated sets. We present an algorithm to compute a maximum proportion set, which runs in at most $|B|$ maximum flow computations. Also, we show that the set of all buyers B is a most violated set.

Keywords: *The market equilibrium problem, Fisher's and Arrow-Debreu's models, Violated sets.*

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1. Introduction

Fisher's and Arrow-Debreu's market equilibrium models are the two fundamental models within mathematical economics. In the both model, the purpose is to compute an equilibrium. In 1954, Arrow and Debreu [1] proved that the market equilibrium always exists if the utility functions are concave. The result is prominently mentioned in their Nobel prize laudation and the market is usually referred to as the Arrow-Debreu market, which considers a more general model in which each buyer i starts with an initial endowment $\{e_{i1}, e_{i2}, .., e_{i|G|}\}$ of goods, where e_{ij} is the initial proportion of good j possessed by buyer i. If P is a vector of prices for the goods, then the value of the goods *for* buyer i is $e_i(P) = \sum_{j \in G} e_{ij} p_j$. The first polynomial time algorithm for the linear Arrow-Debreu mode is given by Jain [15], it is based on solving a convex program using the ellipsoid algorithm. Another polynomial-time algorithm was given by Ye [19], it is based on solving a convex program using the interior-point method. The algorithm in [19] runs in $O(n^4 L)$ time, where $n = |B| + |G|$ and L is the bit-length of the input data u_{ij} (which u_{ij} is the utility of buyer i purchasing all of good j).

Jain, Mahdian and Saberi [16] considered approximate utility maximization and gave a combinatorial method to compute an ε-approximate solution, which runs in $O(1/\varepsilon)$ calls of the algorithm in [4]. Devanur and Vazirani[6] improved the running time to $O((n^7/\varepsilon)\log n/\varepsilon)$. This running time avoids dependence on the size of the integers in the problem instance. Garg and Kapoor [9] relaxed the definition of approximation by permitting purchases to violate their optimality conditions by ε. Under this revised notion of approximation, they developed an $O((n^3/\varepsilon)\log n/\varepsilon)$ time algorithm. Ghiyasvand and Orlin [13] developed an approximation algorithm that runs in $O(n^3/\varepsilon)$ time using a new definition of approximation. Duan and Mulhern[7]

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presented the first combinatorial polynomial time algorithm for computing the equilibrium of the Arrow-Debreu market model with linear utilities. Devanur et al. [5] presented a rational convex program for linear Arrow-Debreu markets. Finally, Garg and Vazirani [11] obtained a linear complementarity problem formulation that captures exactly the set of equilibria for Arrow-Debreu markets with SPLC utilities and SPLC production, and gave a complementary pivot algorithm for finding an equilibrium. Some new results of the market problems presented by [2,4,8,10,14].

In Fisher'model [3], all initial endowments are in dollars: each buyer i has a fixed amount of money e_i and it does not change by increasing or decreasing the prices. Devanur et al. [5] gave the first polynomial time algorithm for computing an equilibrium, using $O(n^4(\log n + n\log U_{\max} + \log M))$ max-flow computations, where M depends on the endowments and $U_{\max}$ is the maximum utility. Finally, Orlin [17] developed the first strongly polynomial time algorithm for finding the market equilibrium, which runs in $O(n^4 \log n)$ time.

Consider a market consisting a set B of buyers and a set G of divisible goods. We are given for each buyer i the amount e_i of money she possesses and for each good j one unit of good. Let u_{ij} denote the utility derived by i on obtaining a unit amount of good j. Let $P = (p_1, p_2, ..., p_{|G|})$ denote a vector of prices. If at these prices' buyer i is given good j, she derives u_{ij} / p_i amount of utility per unit amount of money spent. Define

$$\alpha_i = \max_{j\in G} \frac{u_{ij}}{p_j}.$$

Clearly buyer i will be happiest with goods that maximize u_{ij} / p_j. This motivates defining a bipartite graph $D = (G, B)$, which for each $i \in B$ and $j \in G$, edge (i, j) is in D iff $\alpha_i = u_{ij} / p_j$. Direct edge of D from G to B and assign a capacity of infinity to all these edges. Introduce source vertex s, sink vertex t, a directed edge from s to each vertex $j \in G$, with a capacity of p_j, and a directed edge from each vertex $i \in B$ to t with a capacity of e_i. This network is clearly a function of the current prices P and defined by $N(P)$. An *equilibrium* is obtained w.r.t. the prices P iff $(\{s\})$ and $(\{s\} \cup G \cup B)$ are two min-cuts in $N(P)$. On the other hand, an equilibrium is obtained w.r.t prices P iff the following conditions are satisfied.

Condition-1: There exists a maximum flow x from node s to node t such that $x_{si} = p_i$, for each $i \in G$.
Condition-2: There exists a maximum flow x from node s to node t such that $x_{jt} = e_j$, for each $j \in B$.

Supposing that Condition-1 is satisfied, but Condition-2 is not. Thus, there are some buyers with surplus money w.r.t the current prices P. For satisfying Condition-2, we should increase the prices. Ghiyasvand [12] called a set of buyers with surplus money as a violated set and defined a kind of violated sets called maximum mean, then computed a maximum mean violated set in $O(mn\log(n^2/m))$, where m is the number of pairs (i, j) such that buyer i has some utility for purchasing good j.

This paper defines two new kinds of violated sets, which are maximum proportion and most violated sets. Then, an algorithm to compute a maximum proportion set is presented, which runs in at most $|B|$ maximum flow computations. Finally, we show that the set of all buyers B is a most violated set. Computing a maximum mean, most violated, or maximum proportion set help to know the set of buyers with maximum surplus money w.r.t the current prices P.

This paper consists of four sections in addition to Introduction section. Section 2 defines the most violated and maximum proportion sets. In Section 3, a maximum proportion set is computed in $|B|$ maximum flow computations. Section 4 shows that the set of all buyers B is a most violated set.

2. Violated sets

A directed graph D is a pair $D = (N, A)$ where N is a set of nodes and A is a set of ordered pairs of nodes, called arcs. We denote an arc from node i to node j by (i, j) and also associate with each arc a capacity c_{ij} that denotes the maximum amount that can flow on the arc. If two sets S and $\bar{S}$ form a nontrivial partition of N then, we define $cut(S) = \{(i, j) \in A \mid i \in S, j \notin S\}$, where $\bar{S} = N - S$. We refer to a cut as $s-t$ cut if $s \in S$ and $t \notin S$. *The capacity of* $cut(S)$ is defined as:

$$K(S) = \sum_{(i,j)\in(S,\bar{S})} c_{ij}. \tag{1}$$

An $s-t$ cut whose capacity is minimum among all $s-t$ cuts is called a *minimum cut*.

Theorem 2.1 (Max-flow min-cut theorem). The maximum value of the flow from a source node s to a sink node t in a capacitated network equals the minimum capacity among all $s-t$ cuts. ■

For each $T \subseteq B$, define its money $m(T) = \sum_{j\in T} e_j$. Also, w.r.t prices P, define $m(S) = \sum_{i\in S} p_i$, for each $S \subseteq G$. For $T \subseteq B$ and $S \subseteq G$, define its neighborhood in $N(P)$ by

$$\Omega(T) = \{i \in G \mid \exists j \in T, (i, j) \in N(P)\},$$

and

$$\Gamma(S) = \{j \in B \mid \exists i \in S, (i, j) \in N(P)\}.$$

Lemma 2.1 (Ghiyasvand[12]). For given prices P in $N(P)$, there exists a maximum flow x from node s to node t such that $x_{jt} = e_j$, for each $j \in B$ if and only if

$$\text{for each } T \subseteq B: \quad m(\Omega(T)) \geq m(T). \quad ■$$

For given prices P and each set $T \subseteq B$, we define *the value of set* T by

$$V^P(T) = m(T) - m(\Omega(T)).$$

If Condition-1 is satisfied, then, by Lemma 2.1, an equilibrium is obtained w.r.t. prices P if and only if for every set $T \subseteq B$:

$$V^P(T) \leq 0.$$

A set $T \subseteq B$ is called a *violated set* if $V^P(T) > 0$. If Condition-1 is satisfied but an equilibrium is not obtained w.r.t prices P, then Lemma 2.1 says that there are some violated sets in $N(P)$, w.r.t. the current prices P. *The mean value of set* T is defined by

$$\overline{V}^P(T) = \frac{V^P(T)}{|T|},$$

and a *maximum mean set* is computed by

$$\overline{T^*} = \underset{T \in B}{Max} \overline{V}^P(T).$$

This paper defines two new kinds of violated sets. We call *the proportion of a set* T by

$$\Upsilon(T) = \frac{m(T)}{m(\Omega(T))},$$

and *a maximum proportion set* Z is defined by

$$\Upsilon(Z) = \underset{T \in B}{Max} \Upsilon(T).$$

Also, $\widetilde{T}^* \subseteq B$ is *a most violated set* w.r.t prices P if

$$V^P(\widetilde{T}^*) = \underset{T \in B}{Max} V^P(T).$$

By Lemma 2.1, if the Condition-1 is satisfied, an equilibrium is obtained w.r.t. prices P if and only if

(1) For every set $T \subseteq B$: $\overline{V}^P(T) \leq 0$, or

(2) For every set $T \subseteq B$: $V^P(T) \leq 0$, or

(3) For every set $T \subseteq B$: $\Upsilon(T) \leq 1$.

Example 2.1. In Figure 1, consider two sets $T_1 = \{1,2,3\}$ and $T_2 = \{3,4\}$. We have $\Omega(T_1) = \{a,b\}$, $m(T_1) = 100 + 60 + 20 = 180$, and $m(\Omega(T_1)) = 60$, so

$$\overline{V}^P(T_1) = \frac{m(T_1) - m(\Omega(T_1))}{|T_1|} = 40.$$

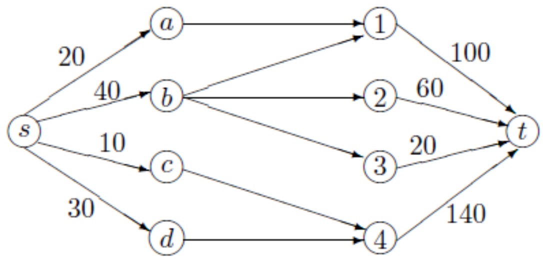

Figure 1. A network $N(P)$ with $G=\{a,b,c,d\}$, $p_a=20$, $p_b=40$, $p_c=10$, $p_d=30$, $B=\{1,2,3,4\}$, $e_1=100$, $e_2=60$, $e_3=20$, and $e_4=140$.

Also, by $\Omega(T_2)=\{b,c,d\}$, $m(T_2)=20+140=160$, and $m(\Omega(T_2))=40+10+30=80$, we get

$$\overline{V}^P(T_2)=\frac{m(T_2)-m(\Omega(T_2))}{|T_2|}=40.$$

Hence,

$$\overline{V}^P(T_1)=\overline{V}^P(T_2),$$

which means sets T_1 and T_2 have no difference with respect to the definition of the mean value for violated sets. The proportion of sets T_1 and T_2 are

$$\Upsilon(T_1)=\frac{m(T_1)}{m(\Omega(T_1))}=\frac{180}{60}=3,$$

And

$$\Upsilon(T_2)=\frac{m(T_2)}{m(\Omega(T_2))}=\frac{160}{80}=2.$$

Thus, the sets T_1 and T_2 are different with respect to the definition of the proportion for violated sets. By definitions, $m(\Omega(T\ ))$ is the maximum amount of money spent by the buyers of T with respect to the current prices P. Hence, $\Upsilon(T_1)=3$ means that the maximum amount of money spent by the buyers of T_1 is $1/3$ of their money, i.e.

$$m(\Omega(T_1))=\frac{1}{3}m(T_1).$$

Also, by $\Upsilon(T_2)=3$, the maximum amount of money spent by the buyers of T_2 is $1/3$ of their money, which means

$$m(\Omega(T_2))=\frac{1}{2}m(T_2).$$

3. Computing a maximum proportion set

In this section, an algorithm to compute a maximum proportion set is presented. If $\Upsilon(Z) \le 1$ then, by Lemma 2.1, $\{s\} \cup G \cup B$ is a minimum cut in $N(P)$. Supposing that we multiply prices of all goods in G by $\phi > 0$, then the network $N(P)$ changes to $N(\phi P)$.

Lemma 3.1. If $\phi \ge \Upsilon(Z)$, then, for every maximum flow x from node s to node t in network $N(\phi P)$, we have $x_{jt} = e_j$, for each $j \in B$. Also, for $\phi < \Upsilon(Z)$, such a maximum flow does not exist.

Proof. By the definition of a maximum proportion set Z, we get

$$\phi < \Upsilon(Z) \text{ if and only if } \phi < \frac{m(T)}{m(\Omega(T))},$$

for each set T. Hence, by Lemma 2.1, we conclude the claims. ■

Definition 3.1. Supposing that, for each maximum flow x from node s to node t in $N(\phi P)$, there exists at least one node $j \in B$ such that $x_{jt} \ne e_j$. Let $\hat{Z} = Z \cap B_1$, $H = \Omega(\hat{Z}) \cap G_1$ and $\tilde{Z} = Z - \hat{Z}$, where $\{s\} \cup G_1 \cup B_1$ is a min-cut in $N(\phi P)$. Figure 2 shows the sets $\hat{Z}$, H, $\tilde{Z}$, B_1, B_2, G_1 and G_2, where $G_2 = G - G_1$ and $B_2 = B - B_1$.

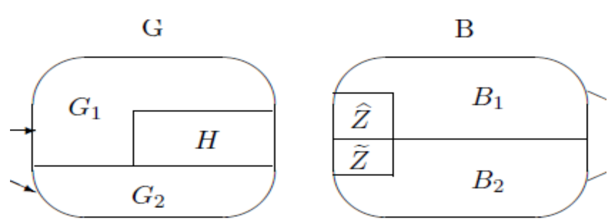

Figure 2. The sets $\hat{Z}$, H, $\tilde{Z}$, B_1, B_2, G_1 and G_2.

The following lemma presents two properties of these sets.

Lemma 3.2. Supposing that, for each maximum flow x from node s to node t in $N(\phi P)$, there exists at least one node $j \in B$ such that $x_{jt} \ne e_j$. Let $\{s\} \cup G_1 \cup B_1$ be a minimum cut in $N(\phi P)$. If $\hat{Z}$ is not empty, then

(a) $m(\hat{Z}) \le \phi \times m(H)$.

(b) $Z \ne \hat{Z}$.

Proof.

(a) By Figure 2,

$$K(\{s\}\cup(G_1-H)\cup(B_1-\hat{Z}))=\phi\times m(G_2)+\phi\times m(H)+m(B_1)-m(\hat{Z}), \tag{2}$$

and

$$K(\{s\}\cup G_1\cup B_1)=\phi\times m(G_2)+m(B_1). \tag{3}$$

If $m(\hat{Z})>\phi\times m(H)$, then, by (2) and (3),

$$K(\{s\}\cup(G_1-H)\cup(B_1-\hat{Z}))\le K(\{s\}\cup G_1\cup B_1),$$

which is a contradiction with the minimality of cut $\{s\}\cup G_1\cup B_1$.

(b) By Lemma 3.1 and the assumption of this lemma, we get $\phi<\Upsilon(Z)$. On the other hand, by $H\subseteq\Omega(Z)$, we have

$$\Upsilon(Z)\times m(H)\le\Upsilon(Z)\times m(\Omega(Z))=m(Z).$$

Hence $\phi\times m(H)<m(Z)$, which means by (a), $Z\neq\hat{Z}$. ■

Lemma 3.3. If, for each maximum flow x from node s to node t in $N(\phi P)$, there exists at least one node $j\in B$ such that $x_{jt}\neq e_j$, then $Z\subseteq B_2$.

Proof. Supposing that for the sake of contradiction, the set Z does not belong to the set B_2, which means, by Figure 2, the set $\hat{Z}$ is not empty. Thus, by Lemma 3.2(b), we have $Z\neq\hat{Z}$, so, the set $\hat{Z}$ is not empty. By Lemma 3.1 and the assumption, we have $\phi<\Upsilon(Z)$, so, by Lemma 3.2(a),

$$m(\hat{Z})<\Upsilon(Z)\times m(H). \tag{4}$$

By Definition 3.1, we get $\hat{Z}=Z\cap B_1$ and $\tilde{Z}=Z-\hat{Z}$. Also, $\{s\}\cup G_1\cup B_1$ is a min-cut in $N(\phi P)$, which means sets $\hat{Z}$ and $\tilde{Z}$ are in different sides of the minimum cut $\{s\}\cup G_1\cup B_1$. Hence, we get $\Omega(\tilde{Z})\cap H=\phi$. Consequently, by the definitions, we have $\Omega(\tilde{Z})\cup H\subseteq\Omega(Z)$, which means

$$m(\Omega(\tilde{Z}))+m(H)\le m(\Omega(Z)). \tag{5}$$

On the other hand, by Figure 2 and the definition of $\Upsilon(Z)$, we have

$$m(\Omega(Z))=\frac{m(\tilde{Z})+m(\hat{Z})}{\Upsilon(Z)}.$$

Thus, by (5),

$$\Upsilon(Z)\times m(\Omega(\tilde{Z}))+\Upsilon(Z)\times m(H)\le m(\hat{Z})+m(\tilde{Z}),$$

which means, by (4),

$$\Upsilon(Z) \times m(\Omega(\tilde{Z})) < m(\tilde{Z}),$$

contradicting the definition of $\Upsilon(Z)$. ■

Lemma 3.4. Supposing that, for each maximum flow x from node s to node t in $N(\phi P)$, there exists at least one node $j \in B$ such that $x_{jt} \neq e_j$, where $\phi = \dfrac{m(B)}{m(G)}$. Then, for each minimum cut $\{s\} \cup G_1 \cup B_1$ in $N(\phi P)$, we have

(a) $K(\{s\}) = K(\{s\} \cup G_1 \cup B_1)$.

(b) $B \neq B_2$.

Proof.

(a) By (1) and Figure 2, we get

$$K(\{s\}) = \phi \times m(G),$$

and

$$K(\{s\} \cup G \;\; \cup B \;\;) = m(B).$$

Thus, by $\phi = \dfrac{m(B)}{m(G)}$, Claim (a) is true.

(b) If $B = B_2$, then B_1 is empty, which means, by $B_1 = \Gamma(G_1)$, the set G_1 is empty. Thus, $\{s\} \cup G_1 \cup B_1 = \{s\}$ is a minimum cut in $N(\phi P)$. Hence, by Part *(a)*, $\{s\} \cup G \cup B$ is a minimum cut, so, there exists a maximum flow x from node s to node t in $N(\phi P)$ such that $x_{jt} = e_j$, for each $j \in B$, which is a contradiction. ■

Algorithm 3.1 computes a maximum proportion set. The next theorem proves this claim and computes its running time.

Theorem 3.1.

(a) At the end of Algorithm 3.1, a maximum proportion set is computed.

(b) The complexity of Algorithm 3.1 is at most $|B|$ maximum flow computations.

Proof. By Lemma 3.3 and Lemma 3.4, after at most $|B|$ iterations, we have a maximum flow x from node s to node t in $N(\phi P)$ such that $x_{jt} = e_j$, for each $j \in B$. On the other hand, in each iteration, we have

$$\phi = \frac{m(B_2)}{m(G_2)} \leq \Upsilon(Z).$$

Thus, by Lemma 3.1 after at most $|B|$ iterations, we get $\phi = \Upsilon(Z)$. In each iteration, the algorithm computes a maximum flow. ■

Algorithm 3.1.

Input: A bipartite graph $D = (G, B)$.

Output: A maximum proportion set Z.

Begin

Form network $N(\phi P)$, where $\phi = m(B)/m(G)$;

Compute a maximum flow x from node s to node t in $N(\phi P)$;

While there exists a $j \in B$ such that $x_{jt} \neq e_j$ **do**

Begin

Compute a minimum cut $\{s\} \cup G_1 \cup B_1$ in $N(\phi P)$;

Let $B_2 = B - B_1$ and $G_2 = G - G_1$;

Let $B = B_2, G = G_2$ and $\phi = m(B_2)/m(G_2)$;

Compute a maximum flow x from node s to node t in $N(\phi P)$;

End;

End.

Algorithm 3.1. Computing a maximum proportion set.

Orlin [18] presented an algorithm to solve the maximum flow problem, which runs in $O(mn)$ time. Consequently, by Theorem 3.1, a maximum proportion set is computed in $O(|B|mn)$ time using Orlin's algorithm in each iteration of Algorithm 3.1.

4. Computing a most violated set

In this section, we show if Condition-1 is satisfied, then, the set B is a most violated set. For it, we define the network $H(P)$ in a similar way of the definition of $N(P)$. Direct edges from B to G and assign a capacity of infinity to all these edges. Introduce source vertex s and a directed edge from s

to each $i \in B$ with a capacity of e_i. Introduce sink node t and a directed edge from each vertex $j \in G$ to t with a capacity of p_j (Figure 3 shows the network $H(P)$).

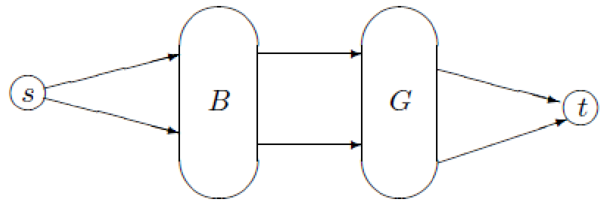

Figure 3. The network $H(P)$.

Lemma 4.1. If $\{s\} \cup T \cup \Omega(T)$ is an $s-t$ minimum cut in $H(P)$, then, the set T is a most violated set in $N(P)$.

Proof. All nodes of $\Omega(T)$ are in the s side of the $s-t$ min cut, because each edge from B to G has a capacity of infinity. Be the definitions, we have

$$K(\{s\} \cup T \cup \Omega(T)) = m(B-T) + m(\Omega(T)),$$

or

$$K(\{s\} \cup T \cup \Omega(T)) = m(B) - (m(T) - m(\Omega(T))).$$

Thus, minimizing the capacity of the cut $\{s\} \cup T \cup \Omega(T)$ is equivalent to maximizing $m(T) - m(\Omega(T))$. ■

Theorem 4.1. If Condition-1 is satisfied, then, the set B is a most violated set.

Proof. Assume set T is a most violated set such that set T is a strict subset of B. Condition-1 is satisfied, so

$$m(B-T) \geq m(G - m(T)). \tag{6}$$

In the network $H(P)$, the capacity of cut $\{s\} \cup B \cup \Omega(G)$ is:

$$K(\{s\} \cup B \cup G) = m(G) = m(B) - (m(B) - m(G))$$

$$= m(B) - (m(T) - m(\Omega(T)) + m(B-T) - m(G - \Omega(G))).$$

Hence, by (6), we get

$$K(\{s\} \cup B \cup G) \leq m(B) - (m(T) - m(\Omega(T)) = K(\{s\} \cup T \cup \Omega(T)).$$

which means, by Lemma 4.1, the set B is a most violated set. ■

5. Conclusion

Given Fisher's and Arrow-Debreu's market equilibrium models with linear utilities, a set of buyers and a set of divisible goods, suppose that there are some buyers with surplus money w.r.t current prices of goods. Ghiyasvand (2012) called a set of buyers with surplus money as a violated set and computed a kind of violated set called maximum mean set. This paper presented two new kinds of violated sets, which called maximum proportion and most violated sets. An algorithm to compute a maximum proportion set was presented, which runs in at most $|B|$ maximum flow computations. Also, we showed that the set of all buyers B is a most violated set computation.

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