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Efficient algorithms for uncapacitated facility location problem on uncertain environments

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In an uncapacitated facility location problem, the aim is to find the best locations for facilities on a specific network in order to service the existing clients at the maximum total profit or minimum cost. In this paper, we investigate the uncapacitated facility location problem where the profits of the demands and the opening costs of the facilities are uncertain values. We first present the belief degree-constrained, expected value and tail value at risk programming models of the problem under investigation. Then, we apply the concepts of the uncertainty theory to transform these uncertain programs into the corresponding deterministic optimization models. The efficient algorithms are provided for deriving the optimal solutions of the problem under investigation.

***Keywords**: uncapacitated facility location problem, combinatorial optimization, uncertain programming.*

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1. Introduction

One of the well-known models in optimization is the uncapacitated facility location problem (UFLP) in which the task is to find the best locations for establishing facilities in order to serve the existing clients in an optimal way. This problem is well-known to be NP-hard, but the specific solvable cases have been studied by the researchers up to now. For more details the interested reader is referred to [3, 9, 10, 18, 24].

In the real life, we are usually faced the situations where some input parameters of the UFLP problem are uncertain for example, the vertex weights, edge lengths, cost coefficients and profits of the problem may be uncertain. On the other hand, note that there exists various types of uncertainty in the literature. In particular, the uncertainty concept introduced by Liu [12] is a suitable tool to

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deal with these parameters which is actually based on the belief degree. On the issue of the facility location in uncertain environment, some papers have been appeared up to now. In 2012, Gao [6] considered the single facility location problems with uncertain demands and proposed a solution approach for it. Later, Wen et al. [28] investigated the capacitated facility location-allocation problem with uncertain demands. In 2016, Nguyen and Chi [19] studied the inverse 1-median location problem on a tree network with uncertain costs and showed that the inverse distribution function of the minimum cost can be found at $O(n^2 \log n)$ time. Soltanpour et al. [22] proposed linear time algorithms for finding the 1-center and 2-center of uncertain unweighted trees. The same authors [23] considered the inverse median location problem with uncertain vertex weights and modification costs. They presented a solution method with O(nlogn) time complexity for the problem with tail value at risk objective. Recently, Etemad et al. [5] developed a combinatorial algorithm for inverse median location problem in uncertain environment on block graphs. Further, for a survey on uncertain location problems, we refer the interested reader to [7, 8, 16, 21, 25, 30].

We know that the uncertainty leads to a risk. Hence, Liu [15] introduced the risk concept in the uncertain environment efficiently. On the other hand, risk measurement is one of the important steps in the decision making process and the risk metrics contain techniques and data sets used to calculate the risk value of the problem under investigation. Among them, Tail value at risk (TVaR) metric [20] is one of the measures of risk which is widely reliable for industry segments and market participants. For a survey on the risk management in the location problems with uncertain random and fuzzy variables, the reader is referred to [1, 2, 26, 27, 29].

In this paper, we investigate the UFLP model with uncertain profits of the demands and the uncertain setup costs. We propose efficient solution methods for solving the problem. The organization of this paper is as follows: In the next section, we first introduce the basic concepts from uncertainty theory and the TVaR metric in an uncertain environment. Moreover, we discuss about the optimization models in the uncertain environment. In Section 3, the mathematical formulation of the deterministic UFLP problem and an applicable solution algorithm for it are presented. Section 4, states the UFLP problem in the uncertain environment based on the belief degree of Liu and the efficient procedures are provided to find the α-optimal locations set (α-OLS), the expected-Optimal location set (E-OLS) and the TVaR-optimal location set (TVaR-OLS) for the problem under investigation in an uncertain network. The conclusion of the paper is resented in Section 5.

2. Preliminaries

In this section, we first present basic concepts from the uncertainty theory and the TVaR metric in an uncertain environment. Then, we discuss the uncertain optimization model and present a new model with TVaR objective and expected value constraints.

2.1. The uncertainty theory

Assume that Γ is a nonempty set and Θ is a σ-algebra over Γ. A set function $\mathcal{M}: \Theta \to [0,1]$ is said to be an uncertain measure if satisfies in normality, duality and subadditivity axioms. The triple $(\Gamma, \Theta, \mathcal{M})$ is named an uncertainty space.

Definition 2.1. (Liu [13]). Let $(\Gamma, \Theta, \mathcal{M})$ be an uncertainty space. A measurable function θ from $(\Gamma, \Theta, \mathcal{M})$ to the set of real numbers is called an uncertain variable.

Definition 2.2. (Liu [13]). Let Θ be an uncertain variable. For any real number, say x, the function $\Upsilon(x) = \mathcal{M}\{\theta \leq x\}$ is called an uncertainty distribution of θ.

Definition 2.3. (Liu [13]). Let $\theta_i, i = 1, \dots, n$, be the uncertain variables. We say $\theta_i, i = 1, \dots, n$, is independent if for any Borel sets $B_1, B_2, \dots, B_n$ of real numbers, the following equality is satisfied:

$$\mathcal{M}\{\bigcap_{i=1}^{n}\{\theta_i \in B_i\}\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\theta_i \in B_i\}.$$

Definition 2.4. (Liu [12]). The expected value of the uncertain variable θ is defined by

$$E[\theta] = \int_0^{+\infty} \mathcal{M}\{\theta \geq r\}\, dr - \int_{-\infty}^{0} \mathcal{M}\{\theta \leq r\} dr,$$

provided that at least one of the two integrals is finite.

Theorem 2.5. (Liu [13]). Let $\theta_i, i = 1, \dots, n$, be the independent uncertain variables and $\Upsilon_i^{-1}, i = 1, \dots, n$, be the inverse uncertainty distributions of $\theta_i, i = 1, \dots, n$, respectively. Further, let it be a strictly increasing function with respect to $x_i, i = 1, \dots, m$, and a strictly decreasing function with respect to $x_i,\ i = m+1, \dots, n$. Then the uncertain variable $\nu = f(\theta_1, \theta_2, \dots, \theta_n)$ has the following inverse uncertainty distribution

$$\Upsilon^{-1}(\alpha) = f\Big(\Upsilon_1^{-1}(\alpha), \dots, \Upsilon_m^{-1}(\alpha), \Upsilon_{m+1}^{-1}(1-\alpha), \dots, \Upsilon_n^{-1}(1-\alpha)\Big),$$

and has an expected value

$$E[\nu] = \int_0^1 f\Big(\Upsilon_1^{-1}(\alpha), \dots, \Upsilon_m^{-1}(\alpha), \Upsilon_{m+1}^{-1}(1-\alpha), \dots, \Upsilon_n^{-1}(1-\alpha)\Big) d\alpha.$$

2.2. The TVaR metric in an uncertain environment

The Risk demonstrates a situation, in which there is a chance of loss or danger. The quantification of risk is a key step towards the management and mitigation of risk. In this section, we present the definition of the TVaR metric to account the probability of loss and the severity of the loss in an uncertain environment [20]. In order to define the TVaR metric, we need to know the definition of loss function.

Definition 2.6. (Liu [15]). Consider $\theta_i, i = 1, \dots, n$, as the uncertain factors of a system. A function f is said to be a loss function if some specified loss occurs if and only if

$f(\theta_1, \theta_2, \dots, \theta_n) > 0.$

In the uncertain environment, the TVaR of loss function is defined as follows:

Definition 2.7. (Peng [20]). Let $\theta_i, i = 1, \dots, n$, be the uncertain factors and f be the loss function of a system. Then the TVaR of f is defined as

$$TVaR_\beta = \frac{1}{\beta}\int_0^\beta sup\,\{\lambda \mid \mathcal{M}\{f(\theta_1,\theta_2,\,\dots,\theta_n) \geq \lambda\} \geq \gamma\}d\gamma,$$

for any given risk confidence level $\beta \in (0,1]$.

Theorem 2.8. (Peng [20]). Let $\theta_i, i = 1, \dots, n$, be the uncertain factors of a system and $\Upsilon_i^{-1}, i = 1, 2, \dots, n$, be the inverse uncertainty distributions of $\theta_i, i = 1, \dots, n$ respectively. Also let the loss function $f(x_1, x_2, \dots, x_n)$ be a strictly increasing function with respect to $x_i, i = 1, \dots, m$, and a strictly decreasing function with respect to $x_i, i = \mathrm{m} + 1, \dots, n$ Then, for each risk confidence level $\beta \in (0,1]$. We have

$$TVaR_\beta = \frac{1}{\beta}\int_0^\beta f\Big(\Upsilon_1^{-1}(1-\gamma), \dots, \Upsilon_m^{-1}(1-\gamma), \Upsilon_{m+1}^{-1}(\gamma), \dots, \Upsilon_n^{-1}(\gamma)\Big)d\gamma.$$

2.3. Uncertain optimization

Let $x = (x_1, x_2, \dots, x_n)$ be a decision vector, and be an $\theta = (\theta_1, \theta_2, \dots, \theta_n)$ uncertain vector. Consider the following optimization model.

$$\begin{array}{ll} min \quad or \quad max & f(x,\theta) \\ \qquad s.t. & z_l(x) \leq 0, \;\; l = 1, \dots, m, \\ & x \geq 0, \end{array} \tag{1}$$

where f is an uncertain function and z_l, $l = 1, \dots, m$, are the crisp functions.

2.3.1. The belief degree-constrained programming model

Let a system contain independent uncertain variables $\theta_1, \theta_2, \dots, \theta_n$ with regular uncertainty distributions, say $\Upsilon_1, \Upsilon_2, \dots, \Upsilon_n$. Let $f(x_1, x_2, \dots, x_n)$ be a strictly increasing function with respect to $x_1, x_2, \dots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_2, \dots, x_n$. To find a solution with minimum $f(x,\theta)$ in the sense of an uncertain measure subject to a set of expected constraints, for each $\alpha \in (0,1]$ we can write the following optimization model

$$\begin{array}{ll} min \quad or \quad max & T \\ \qquad s.t. & \mathcal{M}\{f(x,\theta) \leq T\} \geq \alpha \\ & z_l(x) \leq 0, \quad l = 1, \dots, m, \\ & x \geq 0. \end{array} \tag{2}$$

Using Theorem 2.5 and the inverse uncertainty distributions, we can rewrite the model (2) as

$$\begin{array}{ll} \min \;\; or \;\; \max & f\left(\Upsilon_1^{-1}(\alpha), \dots, \Upsilon_m^{-1}(\alpha), \Upsilon_{m+1}^{-1}(1-\alpha), \dots, \Upsilon_n^{-1}(1-\alpha)\right) \\ \qquad s.t. & z_l(x) \leq 0 \qquad l = 1, \dots, m, \\ & x \geq 0. \end{array} \tag{3}$$

2.3.2. The expected value programming model

Consider the optimization model (1). To find a solution with minimum expected objective value subject to a set of expected constraints, we can equivalently write the following optimization model

$$\begin{array}{lll} \min \ or \ \max & E(\ f(x,\theta)) & \\ s.t. & z_l(x) \leq 0 & = 1, \dots, m, \\ & x \geq 0. & \end{array} \tag{4}$$

Using Theorem 2.5 and the inverse uncertainty distributions, we can reformulate the model (2) as

$$\begin{array}{ll} \min \ or \ \max & \int_0^1 f\left(\Upsilon_1^{-1}(\alpha), \dots, \Upsilon_m^{-1}(\alpha), \Upsilon_{m+1}^{-1}(1-\alpha), \dots, \Upsilon_n^{-1}(1-\alpha)\right) d\alpha \\ s.t. & z_l(x) \leq 0 \qquad l = 1, \dots, m, \\ & x \geq 0. \end{array} \tag{5}$$

2.3.3. The TVaR programming model

Since the objective function of the model (1) involves uncertainty, we apply the TvaR criterion as the objective metric of the problem under investigation. Thus, the model (1), for any risk confidence level $\beta \in (0,1]$, is transformed into the following form:

$$\begin{array}{ll} min & TVaR_\beta(x) = \frac{1}{\beta}\int_0^\beta \{sup\{\lambda | \mathcal{M}f(x,\theta) \geq \lambda\} \geq \gamma\} d\gamma \\ s.t. & z_l(x) \leq 0, \qquad l = 1, \dots, m, \\ & x \geq 0. \end{array} \tag{6}$$

According to Theorem 2.8, we can rewrite the model (6) as

$$\begin{array}{ll} \min & \frac{1}{\beta}\int_0^1 f\left(\mathrm{x}, \Upsilon_1^{-1}(1-\gamma), \dots, \Upsilon_m^{-1}(1-\gamma), \Upsilon_{m+1}^{-1}(1-\gamma), \dots, \Upsilon_n^{-1}(1-\gamma)\right) d\gamma \\ s.t. & z_l(x) \leq 0, \qquad l = 1, \dots, m, \\ & x \geq 0. \end{array} \tag{7}$$

3. The UFLP problem

In this section, we present the mathematical formulation of the UFLP problem and a well-known greedy heuristic algorithm for it. Define $I = \{1, \dots, m\}$ as the index set of location clients and $J = \{1, \dots, n\}$ as the index set of location facilities. The profit of the demand of client i from facility j is c_{ij}. Establishing a facility at location j involves a fixed cost f_j. Define y_{ij} to be the binary variable which is equal to 1 if demand point i is served by opened facility at location j, and define x_j to be one if facility at location j is closed, and zero; otherwise. The UFLP problem is mathematically defined as the following integer linear programming model [11]:

$$
\begin{aligned}
max \quad & -\sum_{j\in J} f_j x_j + \sum_{i\in I}\sum_{j\in J} c_{ij}\, y_{ij} & & \\
s.t. \quad & \sum_{j\in J} y_{ij} = 1, & i \in I, & \\
& y_{ij} \le x_j & i \in I, & \quad j \in J, \\
& x_j \in \{0,1\} & j \in J, & \\
& y_{ij} \ge 0 & i \in I, & \quad j \in J.
\end{aligned}
\tag{8}
$$

or

$$
\begin{aligned}
min \quad & \sum_{j\in J} f_j x_j - \sum_{i\in I}\sum_{j\in J} c_{ij}\, y_{ij} & & \\
& \sum_{j\in J} y_{ij} = 1, & i \in I, & \\
& y_{ij} \le x_j & i \in I, & \quad j \in J, \\
& x_j \in \{0,1\} & j \in J, & \\
& y_{ij} \ge 0 & i \in I, & \quad j \in J.
\end{aligned}
\tag{9}
$$

The objective function maximizes the total profit or minimizes the total set up cost. The first set of constraints state that all of the demand of client at location i must be assigned to a facility for all j. The second set of constraints state that each client at location i is serviced just by one established facility j. Finally, the two sets of the last constraints stand to the integrality and non-negativity of decision variables, respectively.

Note that the UFLP problem is NP-hard in general case as proved by Krarup and Pruzan [9]. But, some special cases of the problem are polynomially solvable.

The greedy heuristic algorithm

Here, we recall the greedy heuristic algorithm for the UFLP problem [3] as proceeds in the following:

I. The greedy heuristic algorithm starts with no facilities open.
II. Given a set S of open facilities, this algorithm adds to set S the facility location $j \notin S$ such that

$$\rho_j(S) = Z(S \cup \{j\}) - Z(S),$$

is as large as possible and positive that

$$Z(S) = \sum_{i\in I} \max_{j\in S} c_{ij} - \sum_{j\in S} f_j.$$

III. If there is no such $j \notin S$, (i.e. if $\rho_j(S) \le 0,\ \forall j \notin S$) then algorithm stops with S as location of facilities.

Based on the above statements, the greedy heuristic methods for the UFLP model, is summarized in Algorithm 1:

Algorithm 1. Solves UFLP
Begin
1: Set $S^0 = \emptyset$, $\rho_j(\emptyset) = \sum_{i \in I} c_{ij} - f_j$, $\forall j \in J$.

2: Set $t = 1$.

3: Derive $j_t = \text{argmax}\{\rho_j(S^{t-1})\ j \notin S^{t-1}\}$.

4: **If** $\rho_{j_t}(S^{t-1}) \leq 0$,, **then** stop; $S^{t-1}$is the location facilities with the corresponding objective value $Z^G = Z(S^{t-1})$, if $t > 1$. **If** $t = 1$, **then** set $S^1 = \{j_1\}$.

5: **If** $\rho_{j_t}(S^{t-1}) > 0$, **, then** set $S^t = S^{t-1} \cup \{j_t\}$ set $t = t + 1$ and return to 3.
End

Remark 3.1. The time complexity of Algorithm 1 is bounded by $O(n^2 m)$ (see [3]).

Observe that Algorithm 1 will be applied in proper form for solving the UFLP problem in uncertain environment.

4. The UFLP model in uncertain environment

In this section, we investigate the uncertain UFLP problem on networks and provide the efficient procedures to find the α-OLS, E-OLS and TVaR-OLS in an uncertain network. Now, consider the following assumptions:

1. The cost for establishing facility j is a positive uncertain variable η_j.
2. The profit of the demand of client i from facility j is a positive uncertain variable θ_{ij}.
3. All the uncertain variables η_j and θ_{ij} are independent.

Furthermore, we will assume that η_j and θ_{ij} have regular uncertainty distribution Φ_j and Υ_{ij}, respectively. Now, define $\eta = \{\eta_j | j \in J\}$ and $\theta = \{\theta_{ij} \mid i \in I, j \in J\}$. We denote the uncertain network as $N = (I, J, \eta, \theta)$. Obviously, the objective function of the problem is an uncertain variable.

Definition 4.1. Let $S \subseteq J$ and

$$y_{ij} = \begin{cases} 1, & \text{if client } i \text{ is served by facility } j \in S, \\ 0, & else. \end{cases}$$

and

$$x_j = \begin{cases} 1, & if\ \ j \in S \text{ is selected to open facility,} \\ 0, & else. \end{cases}$$

Then, S is called facilities location set (OLS) for UFLP if

$$\begin{cases} \sum_{j \in J} y_{ij} = 1, & \forall i \in I, \\ y_{ij} \leq x_j, & \forall i \in I, \forall j \in J. \end{cases}$$

Note that $x_j,\ y_{ij} \in \{0,1\}$ for all $i \in I, j \in J$.

Let S $\subseteq$ J be a facilities location set. We define

$$\mathrm{U(S)} = \sum_{\mathrm{i\in I}}\sum_{\mathrm{j\in J}} \theta_{\mathrm{ij}}\,\mathrm{y_{ij}} - \sum_{\mathrm{j\in J}} \eta_{\mathrm{j}}\mathrm{x_j}.$$

4.1. Chance-constrained UFL model

Definition 4.2. In the network N = (I, J, η, θ), let $S^\star$ be a facilities location. Then $S^\star$ is called the α-OLS if for any facilities location $S \subseteq J$ the following inequality holds:

$$max\,\{P \mid \mathcal{M}\,\{U(S^\star) \geq P\} \geq \alpha\} \geq max\,\{P \mid \mathcal{M}\{U(S) \geq P\} \geq \alpha\}.$$

Based on Definition 4.2, in order to find α-OLS, we propose the following optimization model:

$$\begin{array}{lll}
max & P & \\
s.t. & \mathcal{M}\left\{\displaystyle\sum_{i\in I}\sum_{j\in J}\theta_{ij}\,y_{ij} - \sum_{j\in J}\eta_j\,x_j \geq P \geq \alpha\right\}, & \\
 & \displaystyle\sum_{j\in J} y_{ij} = 1, & i \in I, \\
 & y_{ij} \leq x_j & i \in I, \quad j \in J, \\
 & x_j \in \{0,1\} & j \in J, \\
 & y_{ij} \geq 0 & i \in I, \quad j \in J,
\end{array} \tag{10}$$

where β is a predetermined confidence level provided by the decision-maker. According to Theorem 2.5, the model (10) can be equivalently reformulated as the following deterministic optimization model:

$$\begin{array}{lll}
max & \displaystyle\sum_{i\in I}\sum_{j\in J}\Upsilon_{ij}^{-1}(1-\alpha)y_{ij} - \sum_{j\in J}\Phi_j^{-1}(\alpha)x_j & \\
s.t. & \displaystyle\sum_{j\in J} y_{ij} = 1, & i \in I, \\
 & y_{ij} \leq x_j & i \in I, \quad j \in J, \\
 & x_j \in \{0,1\} & j \in J, \\
 & y_{ij} \geq 0 & i \in I, \quad j \in J.
\end{array} \tag{11}$$

In fact, the solution to the latest model is just UFLP of the deterministic network, and the profit of the demand of client at location i from facility at location j is $\Upsilon_{ij}^{-1}(1-\alpha)$ and the cost for establishing facility at location j is $\Phi_j^{-1}(\alpha)$. Based on this explanations, we conclude the following result:

Theorem 4.3. Let in the uncertain network N, the independent uncertain variables θ_{ij} and η_i have the regular uncertainty distributions Υ_{ij}, and Φ_i. Then, the α-OLS solution of uncertain network is the same optimal location set of UFLP on the corresponding deterministic network, where $\Upsilon_{ij}^{-1}(1-\alpha)$ is the profit of the demand of client i from facility j and , $\Phi_j^{-1}(\alpha)$ is the cost for establishing the facility j.

Altogether, for finding the α-OLS of the uncertain UFLP problem, our proposed solution approach is summarized as the following algorithm:

Algorithm 2. obtains α-OLS of UFLP on uncertain networks
Begin
1: Assign a predetermined confidence level α.
2: Calculate $\Upsilon_{ij}^{-1}(1-\alpha)$ and $\Phi_j^{-1}(\alpha)$.
3: Construct the corresponding deterministic network.
4: Using Algorithm 1 find the α-OLS.
End

Now, let us consider the following example to illustrate the efficiency of our solution approach.

Example 4.4. Let $m = 4$ and $n = 6$. Furthermore, let

$$(\eta_1, \dots, \eta_6) = \big(\mathcal{Z}(2,3,4), \mathcal{Z}(1,2,3), \mathcal{Z}(1,2,3), \mathcal{Z}(1,2,3), \mathcal{Z}(2,3,4), \mathcal{Z}(2,3,4)\big),$$

be the cost vector for establishing the facilities. Moreover, let the matrix of profit be given as follow:

$$[\theta_{ij}] = \begin{bmatrix} \mathcal{Z}(5,6,7) & \mathcal{Z}(5,6,7) & \mathcal{Z}(7,8,9) & \mathcal{Z}(5,6,7) & 0 & \mathcal{Z}(5,6,7) \\ \mathcal{Z}(5,6,7) & \mathcal{Z}(7,8,9) & \mathcal{Z}(5,6,7) & 0 & \mathcal{Z}(5,6,7) & \mathcal{Z}(5,6,7) \\ \mathcal{Z}(4,5,6) & 0 & \mathcal{Z}(2,3,4) & \mathcal{Z}(5,6,7) & \mathcal{Z}(2,3,4) & 0 \\ \mathcal{Z}(1,2,3) & \mathcal{Z}(2,3,4) & 0 & \mathcal{Z}(1,2,3) & \mathcal{Z}(3,4,5) & \mathcal{Z}(3,4,5) \end{bmatrix}.$$

For any $\alpha \in (0,1)$, the inverse uncertainty distribution of a zigzag uncertain variable $\mathcal{Z}(a,b,c)$ is as

$$\phi^{-1}(\alpha) = \begin{cases} (1-2\alpha)a + 2\alpha b, & \text{if } \alpha < 0.5, \\ (2-2\alpha)b + (2\alpha-1)c, & if\ \alpha \geq 0.5. \end{cases}$$

Applying the above input data, we are going to construct the deterministic network when $\alpha = 0.8$,

$$\big(\Phi_1^{-1}(\alpha), \dots, \Phi_6^{-1}(\alpha)\big) = (3.6, 2.6, 2.6, 2.6, 3.6, 3.6),$$

and

$$\left[\Upsilon_{ij}^{-1}(1-\alpha)^{-1}\right] = \begin{bmatrix} 5.4 & 5.4 & 7.4 & 5.4 & 0 & 5.4 \\ 5.4 & 7.4 & 5.4 & 0 & 5.4 & 5.4 \\ 4.4 & 0 & 2.4 & 5.4 & 2.4 & 0 \\ 1.4 & 2.4 & 0 & 1.4 & 3.4 & 3.4 \end{bmatrix}.$$

The execution of Algorithm 2 yields the following results in all iterations:

Iteration 1:

$$\left(\rho_1(\emptyset),\rho_2(\emptyset),\ \dots,\rho_6(\emptyset)\right) = (13, 12.6, 12.6, 8.6, 7.6, 10.6).$$

Since $j_1 = argmax_{j\notin\emptyset}\rho_j(\emptyset) = 1$ and $\rho_{j_1}(\emptyset) > 0$ then $S^1 = \{1\}$.

Iteration 2:

$$\left(\rho_2(S^1),\rho_3(S^1),\ \dots,\rho_6(S^1)\right) = (0.4, -0.6, -1.6, -1.6, -1.6),$$
$$j_2 = \arg\max_{j\notin S^1}\rho_j(S^1) = 2,$$
$$\rho_{j_2}(S^1) > 0,$$
$$S^2 = \{1, 2\}.$$

Iteration 3:

$$\left(\rho_3(S^2),\rho_4(S^2),\ \dots,\rho_6(S^2)\right) = (-0.6, -1.6, -2.6, -2.6),$$
$$j_3 = \arg\max_{j\notin S^2}\rho_j(S^2) = 3.$$

Since $\rho_{j_3}(S^2) = -0.6$, then the algorithm stops with $S^2 = \{1, 2\}$ as location of facilities.

4.2. The expected value of UFLP

Definition 4.5. In network $N = (I, J, \eta, \theta)$, let S^* be a subset of J. Then S^* is called E-OLS if for $S \subseteq J$ the following inequality

$$E\left(U(S^*)\right) \geq E\left(U(S)\right)$$

is satisfied.

Therefore, for obtaining the solution of the model (8) in the sense of expected value, we can consider the following optimization model:

$$\begin{array}{llll}
max & E\left(\sum_{i\in I}\sum_{j\in J}\theta_{ij}\, y_{ij} - \sum_{j\in J}\eta_j\, x_j\right) & & \\
s.t. & \sum_{j\in J} y_{ij} = 1, & i\in I, & \\
 & y_{ij} \leq x_j & i\in I, & j\in J, \\
 & x_j \in \{0,1\} & j\in J, & \\
 & y_{ij} \geq 0 & i\in I, & j\in J.
\end{array} \tag{12}$$

Finally, based on Theorem 2.5, we get.

$$\begin{array}{llll}
max & \sum_{i\in I}\sum_{j\in J}\left(\int_0^1 \Upsilon_{ij}^{-1}(\alpha)d\alpha\right) y_{ij} - \sum_{j\in J}\left(\int_0^1 \Phi_j^{-1}(1-\alpha)d\alpha\right) x_j & & \\
s.t. & \sum_{j\in J} y_{ij} = 1, & i\in I, & \\
 & y_{ij} \leq x_j, & i\in I, & j\in J, \\
 & x_j \in \{0,1\} & j\in J, &
\end{array}$$

$$y_{ij} \geq 0 \qquad\qquad i \in I, \qquad j \in J. \tag{13}$$

Now, according to the above discussions, we can get the following result:

Theorem 4.6. Let in the uncertain network N, the independent uncertain variables θ_{ij} and η_i have the regular uncertainty distributions Υ_{ij} and Φ_i, respectively. Then, the E-OLS solution of uncertain network is the same optimal location set of the corresponding deterministic network, where $\int_0^1 \Upsilon_{ij}^{-1}(\alpha)d\alpha$ is the profit of the demand of client i from the facility j, and $\int_0^1 \Phi_j^{-1}(1-\alpha)d\alpha$ is the cost for establishing facility j.

Now, our solution algorithm for finding the E-OLS of the uncertain UFLP model is summarized in Algorithm 3:

Algorithm 3. obtains E-OLS of UFLP on uncertain networks
Begin
1: Calculate $\int_0^1 \Upsilon_{ij}^{-1}(\alpha)d\alpha$ and $\int_0^1 \Phi_j^{-1}(1-\alpha)d\alpha$.

2: Construct the corresponding deterministic network.

3: Using Algorithm 1 find the E-OLS.
End

Now, we consider the following example to illustrate the effectiveness of our solution approach.

Example 4.7. Consider the input data of Example 4.4. According to definition of the expected value of a zigzag uncertain variable, we have:

$$\left(\int_0^1 \Phi_1^{-1}(1-\alpha)d\alpha, \dots, \int_0^1 \Phi_6^{-1}(1-\alpha)d\alpha\right) = (3,2,2,2,3,3),$$

and

$$\left[\int_0^1 \Upsilon_{ij}^{-1}(\alpha)d\alpha\right] = \begin{bmatrix} 6 & 6 & 8 & 6 & 0 & 6 \\ 6 & 8 & 6 & 0 & 6 & 6 \\ 5 & 0 & 3 & 6 & 3 & 0 \\ 2 & 3 & 0 & 2 & 4 & 4 \end{bmatrix}.$$

Note that if $\eta = \mathcal{Z}(a,b,c)$ and ϕ be its uncertainty distribution, then, we will get

$$E(\eta) = \int_0^1 \phi^{-1}(\alpha)d\alpha = (a+2b+c)/4.$$

Then, by applying Algorithm 1, we get:

Iteration 1:

$$\left(\rho_1(\emptyset), \rho_2(\emptyset), \dots, \rho_6(\emptyset)\right) = (16,15,15,12,10,13).$$

Since $j_1 = argmax_{j \notin \emptyset} \rho_j(\emptyset) = 1$ and $\rho_{j_1}(\emptyset) > 0$ then $S^1 = \{1\}$.

Iteration 2:

$$\big(\rho_2(S^1),\rho_3(S^1),\dots,\rho_6(S^1)\big) = (1,0,-1,-1,-1),$$
$$j_2 = \arg\max_{j\notin S^1}\rho_j(S^1) = 2,$$
$$\rho_{j_2}(S^1) > 0,$$
$$S^2 = \{1,2\}.$$

Iteration 3:

$$\big(\rho_3(S^2),\rho_4(S^2),\dots,\rho_6(S^2)\big) = (0,-1,-2,-2),$$
$$j_3 = \arg\max_{j\notin S^2}\rho_j(S^2) = 3.$$

Since $\rho_{j3}(S^2)=0$, then the algorithm stops with $S^2=\{1,2\}$ as the optimal location of facilities.

4.3. The TVaR of the UFLP

Consider the following uncertain UFLP model

$$\begin{aligned}
\min \quad & \sum_{j\in J}\eta_j x_j - \max\sum_{i\in I}\sum_{j\in J}\theta_{ij}\,y_{ij} \\
s.t. \quad & \sum_{j\in J} y_{ij} = 1, && i\in I,\\
& y_{ij}\le x_j && i\in I,\quad j\in J,\\
& x_j \in\{0,1\} && j\in J,\\
& y_{ij}\ge 0 && i\in I,\quad j\in J.
\end{aligned} \tag{14}$$

By using the TVaR metric, for any risk confidence level $\beta\in(0,1]$, the model (14) can be written as follows:

$$\begin{aligned}
\min \quad & \sum_{j\in J}\left[\frac{1}{\beta}\int_0^\beta \Phi_j^{-1}(\gamma)d\gamma\right]x_j - \sum_{i\in I}\sum_{j\in J}\left[\frac{1}{\beta}\int_0^\beta \Upsilon_{ij}^{-1}(1-\gamma)d\gamma\right]y_{ij} \\
s.t. \quad & \sum_{j\in J} y_{ij} = 1, && i\in I,\\
& y_{ij}\le x_j && i\in I,\quad j\in J,\\
& x_j \in\{0,1\} && j\in J,\\
& y_{ij}\ge 0 && i\in I,\quad j\in J.
\end{aligned} \tag{15}$$

To solve the model (15), we divide the decisions into two stages. The decision x is called the first-stage decision. After choosing the location x, the second stage is to solve the following optimization model:

$$\begin{aligned}
\mathbb{R}(x,\Upsilon^{-1}) = \max & \sum_{i\in I}\sum_{j\in J}\left[\frac{1}{\beta}\int_0^\beta \Upsilon_{ij}^{-1}(\gamma)d\gamma\right]y_{ij} \\
s.t. \quad & \sum_{j\in J} y_{ij} = 1, && i\in I,\\
& y_{ij}\le x_j && i\in I,\quad j\in J,
\end{aligned}$$

$$y_{ij} \geq 0 \qquad\qquad i \in I, \quad j \in J. \tag{16}$$

Here, the loss

$$L(x, \Phi^{-1}, \Upsilon^{-1}) = \left[\frac{1}{\beta}\int_0^\beta \Phi_j^{-1}(\gamma)d\gamma\right] x_j - \mathbb{R}(x, \Upsilon^{-1}), \tag{17}$$

in fact represents the risk measure of the UFLP problem under investigation.

Observe that we can find the optimal value of the model (15) by comparing $L(x, \Phi^{-1}, \Upsilon^{-1})$ for all values of the first-stage decision variable.

Example 4.8. Consider the input data of Example 4.4. Let us assume that the first-stage decision variable x takes the three values (1,0,0,0,0,0), (1,1,0,0,0,0), (1,1,1,0,0,0). Using the data of Example 4.4, we construct the corresponding deterministic network where $\beta = 0.8$. We get

$$\left(\frac{1}{\beta}\int_0^\beta \Phi_1^{-1}(\gamma)d\gamma, \ldots, \frac{1}{\beta}\int_0^\beta \Phi_6^{-1}(\gamma)d\gamma\right) = (3.2, 2.2, 2.2, 2.2, 3.2, 3.2),$$

and

$$\left[\frac{1}{\beta}\int_0^\beta \Upsilon_{ij}^{-1}(1-\gamma)d\gamma\right] = \begin{bmatrix} 6.2 & 6.2 & 8.2 & 6.2 & 0 & 6.2 \\ 6.2 & 8.2 & 6.2 & 0 & 6.2 & 6.2 \\ 5.2 & 0 & 3.2 & 6.2 & 3.2 & 0 \\ 2.2 & 3.2 & 0 & 2.2 & 4.2 & 4.2 \end{bmatrix}.$$

Note that if $\eta = \mathcal{Z}(a, b, c)$, then we will have

$$TVaR_\beta = \begin{cases} \beta b + (1-\beta)c, & \text{if} \quad \beta \leq 0.5, \\ \frac{a-2b+c}{4\beta} + (\beta - 1)a + (2-\beta)b, & \text{if} \quad \beta > 0.5. \end{cases}$$

Let $\Upsilon = (\Upsilon_{ij}^{-1})_{i\in I,\ j\in J}$ and $\Phi^{-1} = \left(\Phi_j^{-1}\right)_{j\in J}$. Recall that in our solution approach, we should take the following three steps:

1. Compute $\mathbb{R}(x, \Upsilon^{-1})$ by solving (16) for each x.
2. Calculate $L(x, \Phi^{-1}, \Upsilon^{-1})$ for each x by using (17).
3. Choose the optimal values by comparing $L(x, \Phi^{-1}, \Upsilon^{-1})$ for all x.

Now, If the first stage decision variable $x = (1, 0, 0, 0, 0, 0)$ is considered, then the optimal second-stage solution

$$y_{11} = y_{21} = y_{31} = y_{41} = 1, y_{ij} = 0, i = 1, 2, 3, 4, j = 2, 3, 4, 5, 6,$$

and the optimal value on the second stage $\mathbb{R}((1,0,0,0,0,0), \Upsilon^{-1}) = 19.8$ is resulted. Thus, from (17), we obtain $L((1,0,0,0,0,0), \Phi^{-1}, \Upsilon^{-1}) = -16.6$.

When we consider $x = (1,1,0,0,0,0)$, we get

$$y_{11} = y_{22} = y_{31} = y_{42} = 1, y_{ij} = 0,\ i = 1, 2, 3, 4,\ j = 3, 4, 5, 6,\ y_{21} = y_{41} = 0,$$

and $\mathbb{R}((1,1,0,0,0,0), \Upsilon^{-1}) = 22.8$, $L((1,1,0,0,0,0), \Phi^{-1}, \Upsilon^{-1}) = -17.4$.

Further, for $x = (1,1,1,0,0,0)$, we obtain

$$y_{13} = y_{22} = y_{31} = y_{42} = 1, y_{ij} = 0, i = 1,2,3,4, j = 4,5,6,$$

$$y_{11} = y_{12} = y_{23} = y_{21} = y_{32} = y_{33} = y_{41} = y_{43} = 0,$$

and $\mathbb{R}((1,1,1,0,0,0), \Upsilon^{-1}) = 24.8$, $L((1,1,1,0,0,0), \Phi^{-1}, \Upsilon^{-1}) = -17.2$.

Finally, comparing the objective values L, we get the optimal value $L(x, \Phi^{-1}, \Upsilon^{-1})$, with the optimal solution $x = (1,1,0,0,0,0)$.

5. Conclusion

In this paper, we investigated the uncapacitated facility location problem where the profits of demands and the opening costs of facilities are uncertain. We proposed efficient procedures to solve the problem under investigation on an uncertain network.

For future work, we can consider the calassical and inverse capacitated facility location models on uncertain environments and develop the efficient solution algorithms for them.

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