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Hybrid model for evaluation of DMUs with principles of strong and weak disposability in the presence of grey undesirable factors

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In the evaluation of decision-making units with classical models of data envelopment analysis, it is assumed that the factors are deterministic. In some decision-making problems, the amount of inputs or outputs of the units is not exactly known and it is a three-parameter interval in grey form. In this case, it is recommended to choose the factors from their center of gravity. In the classic models of data envelopment analysis, all factors are also considered desirable, but in real problems there are undesirable factors too which cannot be used to evaluate problems with undesirable inputs and undesirable outputs. In this paper, a model is presented for calculating the efficiency of decision making units in the presence of the center of gravity of undesirable three-parameter interval grey undesirable factors based on the combination of strong and weak disposability principles. To this end, the proposed method is discussed with a practical example. According to the obtained results, it was found that using this method is more reliable for managers to make decisions. Also, by reducing the undesirable outputs of the units, their desirable outputs increased.

Keywords*: Data Envelopment Analysis, Center of gravity, Efficiency, Strong and weak disposability.*

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1. Introduction

The non-parametric method of data envelopment analysis (DEA) measures the efficiency of decision making units (DMUs) with multiple inputs and outputs. [4, 6]. This method is used to evaluate banks, companies, industries, etc. [8, 11, 26, 27 and 48]. DEA is based on existing data and some of the main criteria for representing the set of measurable productive activities, the limits of the efficiency

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boundary or the production possibility set (PPS) [3,12]. Nasseri et al by using some virtual units into the PPS, ranked all efficient DMUs [28]. In DEA, different methods are provided for ranking DMUs. One of these methods is the Cross-efficiency method. Nasseri and Kiaei have used simultaneous optimization of inputs and outputs for ranking DMUs more accurately. In this case, the selection of zero weights in inputs and outputs is avoided as much as possible [29]. Pourmahmoud and Norouzi Bene used DEA to evaluate units with ordinal data [36]. Pourmahmoud and Kaheh obtained cost efficiency of DMUs with approximate method [35]. If the data of DMUs are not certain, it is not possible to measure the efficiency using traditional models of DEA. So, non-deterministic DEA models should be used to calculate the efficiency of such units. Ebrahimnejad et al proposed a DEA model by a fuzzy stochastic variables [13]. Teimourzadeh et al used fuzzy DEA method to calculate and evaluate the road safety index [43]. To calculate the efficiency of DMUs with undesirable outputs with a three-step process in fuzzy random environments can be used Naseri and NikSafat method [30]. Pourmahmoud and Gholam Azad presented a hybrid method based on logistic regression DEA [34]. In real-world problems, there are some DMUs that their data is not deterministic, but it is grey. In fact, the grey system theory was first proposed by Deng[10]. This theory can solve non-deterministic decision-making problems with partial and limited information. [20]. In recent years, it has been used in linear programming problems [9] and evaluation [37]. Grey data does not have an exact value, but takes its possible value from a certain interval or set of numbers [21]. The three-parameter interval grey number is one of the types of grey numbers existing in grey decision problems. Luo [24] first introduced the three-parameter interval grey number and presented a multi-indicator decision-making method with three-parameter interval grey information. In recent years, studies have been conducted on multi-criteria decision-making problems with three-parameter interval grey values [23, 47]. The important characteristic of three-parameter interval grey number is that in addition to having a known lower limit and upper limit , its most probable value, i.e., the "center of gravity" point, is known. When evaluating DMUs with three-parameter interval grey data, inputs and outputs can be selected from the center of gravity of this data, in this case the results are more reliable for managers. The base of the classical DEA models is the improvement of relative performance of the unit being evaluated by decreasing the consumption of input and increasing output production. Sometimes some outputs of DMU are in a situation in which their production must be decreased rather than being increased. Accordingly, there may be undesirable outputs such as pollution and the wastes that must be decreased in order to make improvement in the efficiency of the products [1,14, ,33]. Omrani et al evaluated the sustainable efficiency of DMUs with undesirable outputs [31]. Tu et al. investigated the environmental efficiency of China's cement industry based on the DEA model of undesirable output [45]. To evaluate the efficiency of the units being evaluated with the presence of undesirable factors the model presented by Seiford et al [38] can be used. If the undesirable factors in DMUs are in the form of integers, we can use DEA model presented by Chen et al [7]. Guo and Wu offered a method for stable and unique ranking of DMUs by considering undesirable outputs [16]. Amir Teimouri et al, in the presence of undesirable factors, calculated the relative efficiency of DMUs by means of developing CCR model, increasing the level of undesirable inputs and decreasing the level of undesirable outputs [2]. Zhou et al presented a method based on exponential transformation of undesirable outputs into desirable outputs. They used classical models to calculate environmental efficiency [46]. Toloo and Hančlová used multivariate measures in DEA in the presence of undesirable outputs [44]. Pishgar-Komleh et al, presented a different method to calculate the efficiency. Their model combined DEA and life cycle assessment [32]. Jahanshahloo et al used undesirable factors simultaneously in non-radial DEA models [18]. Hadi Venche et al calculated the efficiency of each DMU in the presence of undesirable inputs and undesirable outputs simultaneously [17]. Shephard provides a method to deal with undesirable and desirable outputs by making use of the principle of weak disposability for the first time [39, 40]. Aiming at controlling undesirable outputs using the principle of weak disposability, Fare and Grosskopf applied a contraction factor to all DMUs [15]. Referring to the principle of weak accessibility, Kuosmanen proposed a non-parametric formulation and considered a separate contraction factor for each DMU [19]. Madadi et al presented a DEA model for resource allocation under weak disposability with the aim of reducing energy consumption and CO2 pollution [25]. In order to check the environmental efficiency under weak disposability, the super-efficiency model of Taleb et al can be used [41]. Yang and Pollitt presented three technologies based on which some models were made. These models were according to the assumptions of strong and weak disposability along with the presence of undesirable outputs while assessing the efficiency measurements [49]. In this paper, a hybrid model with strong and weak disposability principles is proposed to evaluate DMUs in the presence of three-parameter grey undesirable factors. According to the center of gravity of these numbers, by using the proposed method, the desirable and undesirable outputs will increase and decrease respectively. Therefore, the result of using this method is more reliable for managers.

This paper has been organized as follows: section 2 reviews the basic concepts being used in this paper, section 3 will propose the method to calculate the efficiency of units in the presence of the center of gravity of three-parameter interval grey undesirable factors, In section 4, the practical example of twenty public and private hospitals will be discussed and finally section 5 will conclude the paper and further research will be discussed.

2. Basic concepts

Let that the grey number of three-parameter interval is represented by the symbol $A(\otimes) \in [\underline{a},\ \tilde{a},\ \bar{a}],\ \underline{a} \le \tilde{a} \le \bar{a}$ where $\underline{a}$ is the lower limit, $\tilde{a}$ is the center of gravity (the number with the most possibility in the interval), and $\bar{a}$ is the upper limit [22]. Suppose there is n DMU_j $(j = 1, 2, \dots n)$ with the center of gravity of the inputs and outputs of the three-parameter interval grey $(\tilde{X}_j, \tilde{Y}_j)$ so that the center of gravity of the inputs $\tilde{X}_j \in R_+^m$, $(j = 1, 2, \dots n)$ are used to produce the center of gravity of the output $\tilde{Y}_j \in R_+^s$, $(j = 1, 2, \dots n)$. The set of feasible activities is called the production possibility set (PPS) and is described as follows [5,15]:

$$PPS = \left\{ (\tilde{X}, \tilde{Y})^T \in R_+^m \times R_+^s : \ : \tilde{Y} \ge 0 \ can\ be\ produced\ \tilde{X} \ge 0 \right\}.$$

Vectors the center of gravity of the inputs and the output is stated as follows:

$$I(\tilde{Y}) = \left\{ \tilde{X} \,\middle|\, \tilde{X} \ge 0 \ \ is\ able\ to\ produced\ \tilde{Y} \right\},$$

$$O(\tilde{X}) = \left\{ \tilde{Y} \,\middle|\, \tilde{Y} \ge 0 \ \ can\ be\ produced\ by\ \tilde{X} \right\}.$$

For the set of production possibilities, Banker et al [5] stated axioms the following:

Principle 1. Inclusion of observations

$$\forall\left(j=1,2,\dots n\right) \; \left(\tilde{X}_j,\tilde{Y}_j\right)\in PPS$$

Principle 2. Convexity

$$\forall\ \left(\tilde{X},\tilde{Y}\right),\left(\overline{\tilde{X}},\overline{\tilde{Y}}\right)\in PPS\ ,\ \forall\ \lambda\in\left[0,1\right]\ ;\ \left(\lambda\tilde{X}+\left(1-\lambda\right)\overline{\tilde{X}},\lambda\tilde{Y}+\left(1-\lambda\right)\overline{\tilde{Y}}\right)\in PPS.$$

Principle 3. The constant return to scale

$$\forall\ \left(\tilde{X},\tilde{Y}\right)\in PPS,\ \forall\ \lambda\geq 0\ ;\ \left(\lambda\tilde{X},\lambda\tilde{Y}\right)\in PPS.$$

Principle 4. Strong and weak disposability the center of gravity of inputs and the center of gravity of the output

A. Strong disposability the center of gravity of inputs and the center of gravity of the outputs

The Strong disposability the center of gravity of inputs for each $\overline{\tilde{X}}\geq\tilde{X}$ is as follows:

$$\tilde{X}\in I^S(\tilde{Y})\ \Rightarrow\overline{\tilde{X}}\in I^S(\tilde{Y}).$$

The corresponding the center of gravity of inputs set is as follows:

$$I^S(\tilde{Y})=\left\{\tilde{X}:\sum_{j=1}^{n}\lambda_j\tilde{x}_j\leq\tilde{X},\ \sum_{j=1}^{n}\lambda_j\tilde{y}_j\geq\tilde{Y},\ \lambda_j\geq 0,\ j=1,\dots,n\right\}.$$

The Strong disposability the center of gravity of the outputs for each $\overline{\tilde{Y}}\leq\tilde{Y}$ is as follows:

$$\tilde{Y}\in O^S(\tilde{X})\ \Rightarrow\overline{\tilde{Y}}\in O^S(\tilde{X}).$$

The corresponding the center of gravity of the outputs set is as follows:

$$O^S(\tilde{X})=\left\{\tilde{Y}:\sum_{j=1}^{n}\lambda_j\tilde{x}_j\leq\tilde{X},\ \sum_{j=1}^{n}\lambda_j\tilde{y}_j\geq\tilde{Y},\ \lambda_j\geq 0,\ j=1,\dots,n\right\}.$$

B. Weak disposability the center of gravity of inputs and the center of gravity of the outputs

Principle of weak disposability based on the center of gravity of inputs:

$$\forall\ \alpha\geq 1;\ \ \tilde{X}\in I^W(\tilde{Y})\ \Rightarrow\alpha\tilde{X}\in I^W(\tilde{Y}).$$

Where the corresponding the center of gravity of inputs set is as follows:

$$I^W(\tilde{Y})=\left\{\tilde{X}:\sum_{j=1}^{n}\alpha\lambda_j\tilde{x}_j=\tilde{X},\ \sum_{j=1}^{n}\lambda_j\tilde{y}_j\geq\tilde{Y},\ \alpha\geq 1,\ \lambda_j\geq 0,\ j=1,..,n\right\}.$$

Principle of weak disposability based on the center of gravity of the outputs:

$$\forall\ \ 0\leq\beta\leq 1;\ \ \ \tilde{Y}\in O(\tilde{X})\Rightarrow\beta\tilde{Y}\in O^W(\tilde{X}).$$

Where the corresponding the center of gravity of the outputs set is as follows:

$$O^W(\tilde{X}) = \left\{ \tilde{Y} : \sum_{j=1}^{n} \lambda_j \tilde{x}_j \le \tilde{X}\,,\ \sum_{j=1}^{n} \beta\lambda_j \tilde{y}_j = \tilde{Y}\,,\ 0 \le \beta \le 1\,,\ \lambda_j \ge 0\,,\ j=1,..,n \right\}.$$

Principle5. Minimum interpolation.

The production possibility set is the smallest set that applies to the principles of the above axioms. Each of DEA models belong to a unique PPS and is made by a set of definite assumptions and principles. Banker et al. [4], based on the above five axioms, without considering the third principle of PPS under the return to variable scale stated as follows:

$$P_v(\tilde{X},\tilde{Y}) = \left\{ (\tilde{X},\tilde{Y}) \in R_+^m \times R_+^s : \sum_{j=1}^{n} \lambda_j \tilde{x}_j \le \tilde{X}, \sum_{j=1}^{n} \lambda_j \tilde{y}_j \ge \tilde{Y}, 1^T \lambda = 1,\ \lambda \ge 0 \right\}.$$

3. Proposed method

If we have the undesirable center of gravity of inputs among the center of gravity of inputs and if we have the undesirable center of gravity of the outputs among the center of gravity of outputs, then the center of gravity of inputs vector is represented as $\tilde{X} = (\tilde{X}^g, \tilde{X}^b)$ and the center of gravity of outputs vector in the form $\tilde{Y} = (\tilde{Y}^g, \tilde{Y}^b)$, the desirable center of gravity of inputs , the undesirable center of gravity of inputs , the desirable center of gravity of outputs and the undesirable center of gravity of outputs will be shown by $\tilde{X}^g$, $\tilde{X}^b$, $\tilde{Y}^g$, $\tilde{Y}^b$ respectively.

Based on the principles presented in section 2, production possibility set, DMUs including undesirable factors, each of which uses the desirable center of gravity of input $\tilde{X} = (\tilde{X}^g, \tilde{X}^b)^T$ to produce the center of gravity of outputs $\tilde{Y} = (\tilde{Y}^g, \tilde{Y}^b)^T$, is as follows:

$$\begin{aligned} PPS^u = \Big\{ & ((\tilde{X}^g, \tilde{X}^b)^T, (\tilde{Y}^g, \tilde{Y}^b)^T) \in R_+^m \times R_+^s : \\ & \textit{output vector } \tilde{Y} = (\tilde{Y}^g, \tilde{Y}^b)^T \textit{ can be produced by input vector } \tilde{X} = (\tilde{X}^g, \tilde{X}^b)^T \Big\}. \end{aligned}$$

If decision making units include undesirable factors, principle of weak disposability is considered as follows:

$$\begin{aligned} \Big\{ & \forall\ \alpha \ge 1 \text{ and } 0 \le \beta \le 1 : \\ & \left((\tilde{X}^g, \tilde{X}^b)^T, (\tilde{Y}^g, \tilde{Y}^b)^T \right) \in PPS^u \Rightarrow \left((\tilde{X}^g, \alpha\, \tilde{X}^b)^T, (\tilde{Y}^g, \beta\, \tilde{Y}^b)^T \right) \in PPS^u \Big\}. \end{aligned}$$

In the principle of Strong disposability, increasing the desirable center of gravity of outputs and decreasing of the desirable center of gravity of inputs are the aim of the decision making unit. That is, in this principle, a mount of produced the undesirable center of gravity of outputs is not very important and also the undesirable center of gravity of inputs are not used very much. While in Principle of weak disposability, decrease of the undesirable center of gravity of outputs and increase of the undesirable center of gravity of inputs are desired. Suppose that among the center of gravity of inputs of the decision making units, I_1 of the center of gravity of inputs are desirable,

$(\tilde{X}_i^g, i=1,..,I_1), m-I_1$ of the center of gravity of inputs are undesirable $(\tilde{X}_i^b, i=I_1+1,..,m)$, and also among their the center of gravity of outputs, R_1 of the center of gravity of outputs are desirable, $(\tilde{Y}_r^g, r=1,..,R_1), s-R_1$ of the center of gravity of outputs are undesirable$(\tilde{Y}_r^b, r=R_1+1,..,s)$. With these assumptions, the production possibility set with variable return to scale efficiency under strong and weak disposability of the center of gravity of inputs and the center of gravity of outputs in the presence of undesirable factors can be expressed as follows:

$$P_v^u\left((\tilde{X}^g,\tilde{X}^b)^T,(\tilde{Y}^g,\tilde{Y}^b)^T\right)=\Big\{\left((\tilde{X}^g,\tilde{X}^b)^T,(\tilde{Y}^g,\tilde{Y}^b)^T\right)\in R_+^m\times R_+^s:$$

$$\begin{aligned}
&\sum_{j=1}^n\lambda_j\tilde{x}_j^g\le\tilde{X}_i^g,\ \mathrm{i}=1,..,I_1,\\
&\alpha\sum_{j=1}^n\lambda_j\tilde{x}_j^b=\tilde{X}_i^b,\ \mathrm{i}=I_1+1,..,m,\\
&\sum_{j=1}^n\lambda_j\tilde{y}_j^g\ge\tilde{Y}_r^g,\ r=1,..,R_1,\\
&\beta\sum_{j=1}^n\lambda_j\tilde{y}_j^b=\tilde{Y}_r^b,\ r=R_1+1,..,s,\\
&\sum_{j=1}^n\lambda_j=1,\\
&\alpha\ge1,\ 0\le\beta\le1,\ \lambda_j\ge0,\ \mathrm{j}=1,...,\mathrm{n}\Big\}.
\end{aligned}\tag{1}$$

Based on the production possibility set (1), to evaluate the efficiency of the decision unit of the z unit from the input point of view the following model is presented:

$$
\begin{aligned}
\theta_Z^u:\ \min\quad & \theta \\
s.t.\quad & \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij}^{g} \le \theta \tilde{x}_{iZ}^{g},\ \mathrm{i}=1,..,I_1, \\
& \alpha \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij}^{b} = \theta \tilde{x}_{iZ}^{b},\ \mathrm{i}=I_1+1,..,m, \\
& \sum_{j=1}^{n} \lambda_j \tilde{y}_{rj}^{g} \ge \tilde{y}_{rZ}^{g},\ r=1,..,R_1, \\
& \beta \sum_{j=1}^{n} \lambda_j \tilde{y}_{rj}^{b} = \tilde{y}_{rZ}^{b},\ r=R_1+1,..,s, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \alpha \ge 1, \\
& 0 \le \beta \le 1, \\
& \lambda_j \ge 0,\ \mathrm{j}=1,...,\mathrm{n}.
\end{aligned} \tag{2}
$$

Model (2) is a nonlinear model, by changing the variables $\alpha\lambda_j = \lambda_j + \gamma_j$ and $\beta\lambda_j = \lambda_j - \delta_j$ $j=1,\ldots,n$, we can write the following form:

$$
\begin{aligned}
(\theta_Z^u)^* =\ \min\quad & \theta \\
s.t.\quad & \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij}^{g} \le \theta \tilde{x}_{iZ}^{g},\ \mathrm{i}=1,..,I_1, \\
& \sum_{j=1}^{n} (\lambda_j + \gamma_j) \tilde{x}_{ij}^{b} = \theta \tilde{x}_{iZ}^{b},\ \mathrm{i}=I_1+1,..,m, \\
& \sum_{j=1}^{n} \lambda_j \tilde{y}_{rj}^{g} \ge \tilde{y}_{rZ}^{g},\ r=1,..,R_1, \\
& \sum_{j=1}^{n} (\lambda_j - \delta_j) \tilde{y}_{rj}^{b} = \tilde{y}_{rZ}^{b},\ r=R_1+1,..,s, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j - \delta_j \ge 0,\ \mathrm{j}=1,...,\mathrm{n}, \\
& \lambda_j, \gamma_j, \delta_j \ge 0,\ \mathrm{j}=1,...,\mathrm{n}.
\end{aligned} \tag{3}
$$

Lemma. Models (2) and (3) are feasible.

Proof. Suppose $\theta=1,\ \lambda_z=1,\ \alpha=1,\ \beta=1,\ \lambda_j=0,\ j=1,\ldots,n,\ j\neq z$. It is easy to check that these values satisfies to model constraints (2), so it is a feasible solution of the model .thus, the model (2) is feasible. Similarly, it can be proved that for each $j=1,...,n,\ \theta=1,\ \lambda_z=1,\ \lambda_j=0,\ j\neq z,\ \gamma_j=0,$ $\delta_j=0,$ there is a feasible solution for model (3) .Thus, the model (3) is feasible.■

Theorem 1. For each feasible solution of model (2), there is a corresponding feasible solution of model (3).

proof .let $(\theta,\lambda_j,\alpha,\beta),\ j=1,...,n$ be a feasible solution to the model (2) , so that satisfies the constraint .If the change of variables $\alpha\lambda_j=\lambda_j+\gamma_j$ and $\beta\lambda_j=\lambda_j-\delta_j,\ j=1,...,n$ is considered as that $\lambda_j-\delta_j\geq 0,\ \ j=1,...,n$ is considered , then the solution $(\theta,\lambda_j,\gamma_j,\delta_j),\ j=1,...,n$ in the model constraints (3) is also the same then a feasible solution corresponds to model (3) .similarly , it can be proved that any feasible solution of the model (3) has a corresponding feasible solution of model (2). ■

Theorem 2. The optimal value of model (3) is interval of $(0,1]$. In other words

$$0<(\theta_Z^u)^*\leq 1$$

Proof. First, it is proved that θ is positive in every feasible solution. According proof by contradiction method, suppose that θ is not positive, from the first and second constraints, due to the non-negativity of $\tilde{x}_{ij}^g,\ \tilde{x}_{ij}^b,\ j=1,...,n$ and the variables $\lambda_j,\gamma_j,\ j=1,...,n$, it is enough to proof that $\theta\neq 0$. Suppose we have $\theta=0$ in model (3), we will have model (3)

$$\sum_{j=1}^{n}\lambda_j\tilde{x}_{ij}^g\leq 0,\ \ i=1,..,I_1,$$

$$\sum_{j=1}^{n}(\lambda_j+\gamma_j)\tilde{x}_{ij}^b=0,\ \ i=I_1+1,..,m,$$

According to $\tilde{x}_{ij}^g\geq 0,\mathrm{i}=1,..,I_1$ and $\tilde{x}_{ij}^b\geq 0,\mathrm{i}=I_1+1,..,m$, from the above relations we will have for $j=1,...,n{:}\ \lambda_j=0\ and\ \gamma_j=0.$ By placing $\lambda_j=0,\ j=1,...,n$ in the third and fourth constraints of model (3), we have:

$$\tilde{y}_{rZ}^g\leq 0,\ r=1,..,R_1 \text{ and } -\delta_j\tilde{y}_{rj}^b=\tilde{y}_{rZ}^b,\ r=R_1+1,..,s.$$

Given that there is at least one output with a positive value for the unit under evaluation, the above relations are a contradiction. Therefore $\theta>0$. To complete the proof is sufficient to reject $\theta>1$. Suppose $(\theta=1,\ \lambda_z=1,\ \lambda_j=0,j\neq z,j=1,...,n)$ is a feasible solution of model (3), on the other hand, because model (3) is a minimization problem, then always $\theta\leq 1$. ■

Definition 1 Strong efficient: The decision-making unit Z with $((\tilde{X}_Z^g,\tilde{X}_Z^b)^T,(\tilde{Y}_Z^g,\tilde{Y}_Z^b)^T)\in P_v^u\left((\tilde{X}^g,\tilde{X}^b)^T,(\tilde{Y}^g,\tilde{Y}^b)^T\right)$ is strongly efficient, when no other decision-making unit like K can be found in the form

$$((\tilde{X}_K^g,\tilde{X}_K^b)^T,(\tilde{Y}_K^g,\tilde{Y}_K^b)^T))\in P_V^u\left((\tilde{X}^g,\tilde{X}^b)^T,(\tilde{Y}^g,\tilde{Y}^b)^T\right),$$

so that

$$\tilde{X}_K^g \le \tilde{X}_z^g, \ \tilde{X}_K^g \ne \tilde{X}_z^g,$$
$$\tilde{X}_K^b \ge \tilde{X}_z^b, \ \tilde{X}_K^b \ne \tilde{X}_z^b,$$
$$\tilde{Y}_K^g \ge \tilde{Y}_z^g, \ \tilde{Y}_K^g \ne \tilde{Y}_z^g,$$
$$\tilde{Y}_K^b \le \tilde{Y}_z^b, \ \tilde{Y}_K^b \ne \tilde{Y}_z^b.$$

Definition 2 Weak efficient: The decision-making unit Z with $((\tilde{X}_Z^g, \tilde{X}_Z^b)^T, (\tilde{Y}_Z^g, \tilde{Y}_Z^b)^T) \in P_v^u\left((\tilde{X}^g, \tilde{X}^b)^T, (\tilde{Y}^g, \tilde{Y}^b)^T\right)$ is weakly efficient, when no other decision-making unit like K can be found in the form

$$((\tilde{X}_K^g, \tilde{X}_K^b)^T, (\tilde{Y}_K^g, \tilde{Y}_K^b)^T)) \in P_v^u\left((\tilde{X}^g, \tilde{X}^b)^T, (\tilde{Y}^g, \tilde{Y}^b)^T\right),$$

so that

$$\tilde{X}_K^g < \tilde{X}_z^g,$$
$$\tilde{X}_K^b > \tilde{X}_z^b,$$
$$\tilde{Y}_K^g > \tilde{Y}_z^g,$$
$$\tilde{Y}_K^b < \tilde{Y}_z^b.$$

By introducing the variables of slack $s_i^+, \ i = 1,..,I_1$ and $s_r^-, \ r = 1,..,R_1$, model (3) can be written as follows:

$$(\theta_Z^u)^* = \min \quad \theta - \left(\sum_{i=1}^{I_1} s_i^- + \sum_{r=1}^{R_1} s_r^+\right)$$
$$\begin{aligned}
s.t. \quad & \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij}^g + s_i^- = \theta \tilde{x}_{iZ}^g, \ \mathrm{i} = 1,\ldots, I_1, \qquad (4)\\
& \sum_{j=1}^{n} (\lambda_j + \gamma_j) \tilde{x}_{ij}^b = \theta \tilde{x}_{iZ}^b, \ \mathrm{i} = I_1 + 1,\ldots, m,\\
& \sum_{j=1}^{n} \lambda_j \tilde{y}_{rj}^g - s_r^+ = \tilde{y}_{rZ}^g, \ r = 1,\ldots, R_1,\\
& \sum_{j=1}^{n} (\lambda_j - \delta_j) \tilde{y}_{ij}^b = \tilde{y}_{rZ}^b, \ r = R_1 + 1,\ldots, s,\\
& \sum_{j=1}^{n} \lambda_j = 1,\\
& \lambda_j - \delta_j \ge 0, \ \mathrm{j=1,\ldots,n},\\
& s_i^- \ge 0, \ \mathrm{i} = 1,\ldots, I_1,\\
& s_r^+ \ge 0, \ r = R_1 + 1,\ldots, s,\\
& \lambda_j, \gamma_j, \delta_j \ge 0, \ \mathrm{j=1,\ldots,n}.
\end{aligned}$$

In model (4), when $(\theta_Z^u)^* = 1$ and in the optimal solution, the values of $s_i^+ = 0,\ i = 1,..,I_1$ and $s_r^- = 0,\ r = 1,..,R_1$ then the decision making unit Z is strong efficient. Also, if $(\theta_Z^u)^* = 1$ and in the optimal solution, at least one of the slack variables are opposite to zero, then the decision-making unit Z is weakly efficient.

4. Applications

In this section, we examined the proposed method on a practical example. The example data related to twenty public and private hospitals includes two inputs and two outputs in the form of three-parameter grey interval as listed in Table 1. The amount of money paid for the purchase and repair of hospital medical equipment in terms of billions Rial as a desirable input ($\tilde{x}_1^g$), the amount of donations from donors to the hospital in millions of Rial as undesirable input ($\tilde{x}_2^b$), The number of cancer patients discharged from the hospital in thousands of people as desired output ($\tilde{y}_2^b$) and the hospital's Demand from the contracting party's insurances in millions of Rial as undesirable output ($\tilde{y}_2^b$) has been considered.

Table 1. Input and output data of twenty hospitals

DMUs	$\tilde{x}_1^g$	$\tilde{x}_2^b$	$\tilde{y}_1^g$	$\tilde{y}_2^b$
1	[1.90, 3.62, 3.97]	[13215.10, 13412.65, 13550.20]	[0.01, 0.03, 0.15]	[0.17, 0.18, 0.20]
2	[4.22, 5.89, 613]	[5450.50, 5624.25, 6100.24]	[0.045, 0.06, 0.087]	[52.05, 55.04, 56.01]
3	[6.35, 8.29, 10.72]	[57155.14, 57367.59, 57527.80]	[3.89, 4.26, 5.45]	[88.85, 91.25, 95.78]
4	[3.17, 4.66, 5.46]	[4495.25, 4578.01, 5150.13]	[0.12, 0.16, 0.19]	[12.58, 12.89, 13.24]
5	[8.46, 9.13, 10.86]	[12558.18, 13033.26, ١٤١٥٨.17]	[0.07, 0.08, 0.09]	[1535. 05, 1675.09, 1722.15]
6	[9.16, 9.86, 10.27]	[5149.25, 5286.59, 6475.60]	[0.51, 0.54, 0.58]	[175.86, 180.40, 180.75]
7	[5.59, 6.40, 7.63]	[21895, 22603,2300]	[0.10, 0.11, 0.14]	[575.25, 682.70, 714.28]
8	[7,41, 8.76, 8.98]	[4250, 4141.22, 5105.30]	[٠.13, 0.15, 0.16]	[145.15, 159.41, 162.17]
9	[5.79, 6/21, 7.35]	[8500.49, 8677.18, 9145.50]	[0.09, 0.10, 0.12]	[345.21, 389.36, 412.41]
10	[2.24, 3.15, 3.65]	[3125.78, 3224.78, 3480.20]	0	[118.45, 124.36, 127.26]
11	[8, 19.54, 20.15]	[5950.15, 6628.34, 6930.45]	0	[130, 132.93, 135.11]
12	[4.18, 5.68, 6.20]	[2200.50, 2437.73, 2846.16]	0	[26.12, 28.83, 31.13]
13	[9.30, 9.70, 9.75]	[6750.25, 7693.37, 8115.30]	[0.14, 0.17, 0.18]	[365.25, 378.96, 382.45]
14	[10.50, 14.27, 14.80]	[5465.90, 5923.81, 6350.40]	0	[1315.78, 1335.77, 1398.81]
15	[9.25, 10.41, 11.30]	[23568.90, 23796.66, 24850.17]	[1, 1.03, 1.06]	[385.13, 401.51, 411.26]
16	[7.60, 8.91, 9.42]	[4890.85, 5132.71, 6756.13]	[0.87, 0.96, 1.01]	[23.46, 26.24, 27.43]
17	[1.80, 2.38, 3.40]	[10955.75, 11369.62, 13125.10]	0	[29.19, 30.35, 33.43]
18	[6.15, 6.95, 7.42]	[12225.83, 12385.47, 12760.95]	[0.03, 0.06, 0.1]	[99.16, 106.18, 110.78]
19	[4, 10.61, 11.10]	[10465.25, 11756.10, 12745.86]	[1.22, 1.28, 1.34]	15/38[14.25, 15.38, 16.62]
20	[3.50, 5.81, 6.10]	[950.45, 1019.29, 2150.40]	0	[8. 85, 9.67, 10.75]

Model (3) was implemented on the center of gravity of the data in Table (1)and the results are given in Table (2).

Table 2. The results of implementation the model (3)

DMUs	θ^*	λ^*	δ^*
1	0.7216	$\lambda_3^* = 0.0070,\ \lambda_{10}^* = 0.2474, \lambda_{17}^* = 0.7455$	$\delta_3^* = 0.0070,\ \delta_{10}^* = 0.2460,\ \delta_{17}^* = 0.7455$
2	0.5955	$\lambda_3^* = 0.0001,\ \lambda_{10}^* = 0.9379,\ \lambda_{16}^* = 0.0620$	$\delta_{10}^* = 0.5085$
3	1	$\lambda_3^* = 1$	0
4	0.849	$\lambda_3^* = 0.0075,\ \lambda_{10}^* = 0.8592,\ \lambda_{16}^* = 0.1333$	$\delta_3^* = 0.0075,\ \delta_{10}^* = 0.7837$
5	1	$\lambda_5^* = 1$	$\delta_5^* = 0.0586$
6	0.8063	$\lambda_{10}^* = 0.1867,\ \lambda_{14}^* = 0.1056 ,\ \lambda_{16}^* = 0.5625,\ \lambda_{20}^* = 0.1452$	0
7	0.8065	$\lambda_3^* = 0.0184,\ \lambda_5^* = 0.3959,\ \lambda_{17}^* = 0.5857$	0
8	0.6869	$\lambda_{10}^* = 0.3584,\ \lambda_{14}^* = 0.0800,\ \lambda_{16}^* = 0.1563,\ \lambda_{20}^* = 0.4054$	0
9	0.6901	$\lambda_3^* = 0.0199,\ \lambda_5^* = 0.1714 ,\lambda_{10}^* = 0.8073,\ \lambda_{16}^* = 0.0014$	0
10	1	$\lambda_{10}^* = 1$	0
11	0.2924	$\lambda_{10}^* = 0.2601,\ \lambda_{14}^* = 0.0705,\ \lambda_{20}^* = 0.6695$	0
12	0.8166	$\lambda_{10}^* = 0.4405,\ \lambda_{20}^* = 0.5595$	$\delta_{10}^* = 0.2521$
13	0.6101	$\lambda_5^* = 0.00850,\ \lambda_{10}^* = 0.6299,\ \lambda_{14}^* = 0.1151,\ \lambda_{16}^* = 0.17$	0
14	1	$\lambda_{14}^* = 1$	0
15	0.6479	$\lambda_3^* = 0.18010,\ \lambda_5^* = 0.1988,\ \lambda_{10}^* = 0.3641,\ \lambda_{16}^* = 0.2570$	0
16	1	$\lambda_{16}^* = 1$	0
17	1	$\lambda_{17}^* = 1$	0
18	0.4432	$\lambda_3^* = 0.0141,\ \lambda_{10}^* = 0.8015,\ \lambda_{17}^* = 0.1844$	$\delta_{10}^* = 0.0030$
19	0.8675	$\lambda_3^* = 0.0970,\ \lambda_{16}^* = 0.9030$	$\delta_3^* = 0.0970,\ \delta_{16}^* = 0.3169$
20	1	$\lambda_{20}^* = 1$	0

In the above table, we named 20 hospitals with units 1 to 20. The γ^* value for all DMUs is zero.

DMU_3, DMU_5, DMU_{10}, DMU_{14}, DMU_{16}, DMU_{17} and DMU_{20} are efficient. Therefore, it can be said that these seven hospitals have paid enough attention to the reduction of desirable inputs, the increase of undesirable inputs, the increase of desirable output and the reduction of undesirable output. Although DMU_5 is efficient, its undesirable output can be reduced by $\lambda_5^* - \delta_5^* = 0.9414$ and it would still be efficient. The rest of the units have lower performance than these seven units which means that they are inefficient. According to Figure 1, among the inefficient units, DMU_{19} have the maximum amount and DMU_{11} have the minimum amount. In other words, after the efficient units, DMU_{19} has the maximum number of discharged cancer patients and the minimum Demand from insurance companies. DMU_{11} had the worst performance among all hospitals. This unit should reduce the cost of purchasing and repairing its medical equipment to improve its conditions, since its desirable output is high compared to other hospitals. In Table 2, the reference units are specified which can be used to convert inefficient units into efficient ones. The efficiency value of the first unit is equal to 0.7216. DMU_3, DMU_{10}, DMU_{17} are specified as its reference units. If this unit wants to become an efficient unit, the linear combination of $\lambda_3^* = 0.0070$, $\lambda_{10}^* = 0.2474$, $\lambda_{17}^* = 0.7455$ values with the center of gravity of the inputs (desirable and undesirable) and the center of gravity of the desirable output of the reference units should be performed. For its undesirable output center of gravity, $\lambda_{10}^* - \delta_{10}^*$ must be multiplied by the DMU_{10} undesirable output center of gravity. Therefore, based on the combined model with strong and weak disposability principles, inefficient units using reference sets can increase their desirable outputs and decrease their undesirable outputs so that they become an efficient unit.

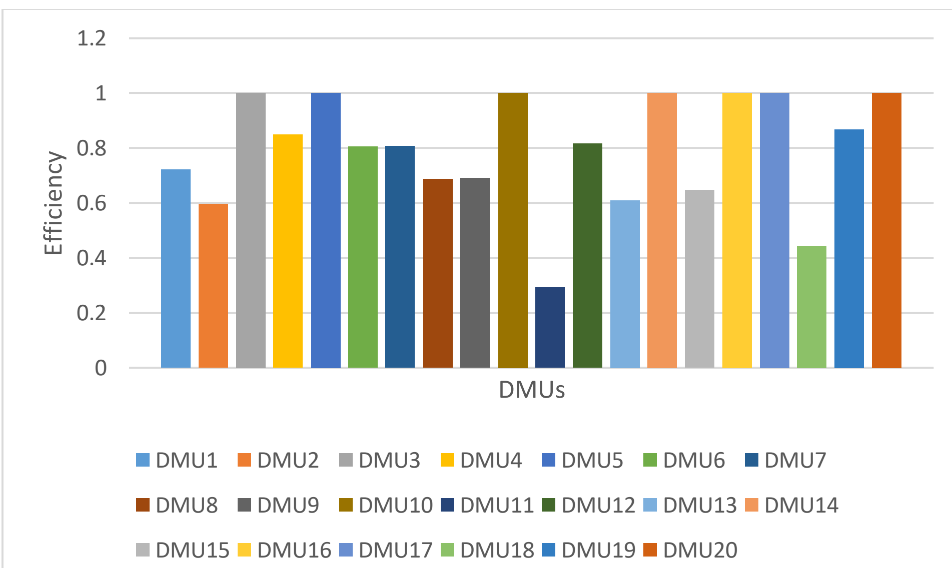

Fig. 1 Rank the DMUs

5. Conclusion

To evaluate the efficiency of DMUs with undesirable factors, it is recommended to use the principles of strong and weak availability. In this case, PPS will have differences. The center of gravity of three-parameter interval grey data is its most probable value. In this paper, based on two principles of strong and weak disposability, a linear programming model was presented to calculate the efficiency of unit evaluation under in the presence of the center of gravity of desirable and undesirable inputs and outputs. The proposed model was discussed on the data of twenty public and private hospitals which were used to determine the efficient and inefficient units. Based on the results of the model, efficient unit that must reduce their undesirable output level to remain efficient was introduced. For inefficient units, using reference units, necessary strategies were determined to improve their existing conditions. For further research, the proposed method can be developed for various industries in the form of network.

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