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Optimal Sample Size in Type-II Progressive Censoring Using a Bayesian Prediction Approach

E. Basiri[1], S.M.T.K. MirMostafaee[2,*]

This paper considers the progressively Type-II censoring and determines the optimal sample size using a Bayesian prediction approach. To this end, two criteria, namely the Bayes risk function of the point predictor for a future progressively censored order statistic and the designing cost of the experiment are considered. In the Bayesian prediction, the general entropy loss function is applied. We find the optimal sample size such that the Bayes risk function and the cost of the experiment do not exceed two pre-fixed values. To show the usefulness of the results, some numerical computations are presented.

***Keywords**: optimization problem, general entropy loss function, Bayes risk function, prediction.*

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1. Introduction

The scheme of progressive Type-II censoring is of importance in life-testing experiments. It allows the experimenter to remove units from a life test at various stages during the experiment. Suppose N units are placed on a lifetime test. At the first failure time, R_1 surviving items are randomly withdrawn from the test. At the second failure time, R_2 surviving items are selected at random and taken out of the experiment, and so on. Finally, at the time of the n-th failure, the remaining R_n objects are removed, where $R_n = N - n - \sum_{i=1}^{n-1} R_i$.
If the failure times are based on an absolutely continuous cumulative distribution function (CDF) $F_\theta(.)$ and probability density function (PDF) $f_\theta(.)$, where θ is the model parameter, and $X_{i:n:N}^{\tilde{R}}$ denotes the i-th failure time, for $1 \le i \le n$, then the random variables $X_{1:n:N}^{\tilde{R}}, \cdots, X_{n:n:N}^{\tilde{R}}$ are called progressively Type-II censored order statistics (PCOs) based on the censoring scheme $\tilde{R} = (R_1, \cdots, R_n)$, where $N = n + \sum_{i=1}^{n} R_i$. Then the joint PDF of $X_{1:n:N}^{\tilde{R}}, \cdots, X_{n:n:N}^{\tilde{R}}$ is

$$f_{X_{1:n:N}^{\tilde{R}}, \cdots, X_{n:n:N}^{\tilde{R}}}(x_1, \cdots, x_n) = C^{\tilde{R}} \prod_{i=1}^{n} (1 - F_\theta(x_i))^{R_i} f_\theta(x_i), \quad (1)$$

in which $C^{\tilde{R}} = \prod_{i=1}^{n} \left(N - \sum_{j=1}^{i-1} R_j - i + 1\right)$, with $\sum_{j=1}^{0} R_j \equiv 0$.

Also, the marginal PDF of $X_{i:n:N}^{\tilde{R}}$, for $1 \le i \le n$, is given by

* Corresponding Author.
[1] Department of Mathematics and Applications, Kosar University of Bojnord, Bojnord, Iran, Email: elhambasiri@kub.ac.ir.
[2] Department of Statistics, University of Mazandaran, Babolsar, Iran, Email: m.mirmostafaee@umz.ac.ir.

$$f_{X^{\tilde{R}}_{i:n:N}}(x) = c^{\tilde{R}}_{i-1} \sum_{j=1}^{i} a^{\tilde{R}}_{j,i} \left(1 - F_\theta(x)\right)^{\gamma^{\tilde{R}}_j - 1} f_\theta(x), \tag{2}$$

where

$$\gamma^{\tilde{R}}_j = N - j + 1 - \sum\nolimits_{l=1}^{j-1} R_l = n - j + 1 + \sum\nolimits_{l=j}^{n} R_l,$$

$$c^{\tilde{R}}_{i-1} = \prod_{j=1}^{i} \gamma^{\tilde{R}}_j, \qquad 1 \le i \le n,$$

$$a^{\tilde{R}}_{j,i} = \prod_{\substack{l=1 \\ l \ne j}}^{i} \frac{1}{\gamma^{\tilde{R}}_l - \gamma^{\tilde{R}}_j}, \quad 1 \le j \le i \le n.$$

The properties and applications of progressive censoring are well-known. For a detailed discussion of progressive censoring, we refer the reader to Balakrishnan and Aggarwala [1], Balakrishnan [2], Balakrishnan and Cramer [3] and the references contained therein.

In a parameter estimation problem, perhaps the most popular loss function is the squared error loss function, which can be easily justified on the grounds of minimum variance unbiased estimation. However, one property of this loss function is that it is symmetric and gives equal weights to overestimation and underestimation of the same magnitude. This property may not be applicable in some real-life situations. A number of asymmetric loss functions are available in the statistical literature. Basu and Ebrahimi [4] defined a modified LINEX loss function. A suitable alternative to the modified LINEX loss function is the general entropy loss function, defined as (see Calabria and Pulcini [5] and Calabria and Pulcini [6])

$$L(\hat{\theta}, \theta) \propto \left(\frac{\hat{\theta}}{\theta}\right)^{\delta} - \delta \, ln\left(\frac{\hat{\theta}}{\theta}\right) - 1, \tag{3}$$

where $\delta \neq 0$ and $\hat{\theta}$ is an estimator of θ.

This loss function is a simple generalization of the entropy loss function used by several authors, where the shape parameter δ is taken to be equal to 1. It may be noted that when $\delta > 0$, a positive error causes more serious consequences than a negative error. The Bayes estimator of θ under the general entropy loss function is given by (provided that it exists)

$$\hat{\theta}_{GE} = \left[E\left(\theta^{-\delta} | data\right)\right]^{\frac{-1}{\delta}}. \tag{4}$$

In many actuarial studies, when enough information is available, predictive inference for unobserved events may be the main interest. The prediction problems have received considerable attention in the literature, and both frequentist and Bayesian inferential methods have been developed in this regard.

Optimization in censoring schemes is one of the issues that has been studied so far by many researchers. See, for example, Ahmadi et al. [7] and Basiri and Asgharzadeh [8].

In this paper, we intend to find the optimal values for the sample size using the Bayes risk and cost functions in progressive Type-II censoring from the one-parameter exponential distribution in a Bayesian prediction problem.

2. Main Results

Throughout this paper, we assume that $X_{1:n:N}^{\tilde{R}}, \cdots, X_{n:n:N}^{\tilde{R}}$ are the progressively Type-II censored order statistics with the censoring scheme $\tilde{R} = (R_1, \cdots, R_n)$ from a sample of size N of independent and identically distributed (iid) continuous random variables from the one-parameter exponential distribution, denoted by $Exp(\theta)$. The PDF and CDF of $Exp(\theta)$ are, respectively, given by

$$f_\theta(x) = \theta \mathrm{e}^{-\theta x}, \quad \text{and} \quad F_\theta(x) = 1 - \mathrm{e}^{-\theta x}. \tag{5}$$

Besides, we suppose that $Y_{i:m:M}^{\tilde{S}}$, $1 \leq i \leq m$, represents the *i*-th progressively Type-II censored order statistic with the censoring scheme $\tilde{S} = (S_1, \cdots, S_m)$ in a future independent sample of size M from the same distribution. For notational simplicity, hereafter, we will use X_i instead of $X_{i:n:N}^{\tilde{R}}$ and Y_i for $Y_{i:m:M}^{\tilde{S}}$.

In this paper, n is considered an unknown value, which should be determined. We define the set of admissible values for n as

$$\zeta_n = \{n \in \mathbb{N} | 1 \leq n \leq N\}.$$

First, we recall a well-known property of PCOs that the *progressively* Type-II censored spacings as (see, for example, Balakrishnan and Aggarwala [1])

$$Z_i = \gamma_i^{\tilde{R}}(X_i - X_{i-1}), \qquad 1 \leq i \leq n, \;\; \text{with} \;\; X_0 \equiv 0,$$

are iid random variables, each of them has the one-parameter exponential distribution, $Exp(\theta)$. So, the random variable $T = \sum_{i=1}^{n}(1 + R_i)X_i = \sum_{i=1}^{n} Z_i$ has a gamma distribution with parameters n and θ, denoted by $\Gamma(n, \theta)$.

On the one hand, from (1) and (5), the likelihood function of θ can be written as

$$L(\theta) = C^{\tilde{R}} \theta^n \mathrm{e}^{-\theta t},$$

where t is the observed value of T. So, considering the prior for θ as

$$\pi(\theta) = \frac{b^a}{\Gamma(a)} \theta^{a-1} \mathrm{e}^{-b\theta}, \quad \theta > 0, a, b > 0,$$

leads to the posterior PDF for θ as

$$\pi(\theta|x_1,\cdots,x_n)=\frac{(b+t)^{a+n}}{\Gamma(a+n)}\theta^{a+n-1}\mathrm{e}^{-\theta(b+t)},$$

where $x_1,\cdots,x_n$ are the observed values of $X_1,\cdots,X_n$ and $\Gamma(.)$ is the complete gamma function.

On the other hand, from (2) and (5), the PDF of Y_i can be expressed as

$$f_{Y_i}(y)=\theta c_{i-1}^{\tilde{S}}\sum_{j=1}^{i}a_{j,i}^{\tilde{S}}\,\mathrm{e}^{-\theta\gamma_j^{\tilde{S}}y},\quad y>0. \tag{6}$$

So, the predictive density function for Y_i takes the form

$$\begin{aligned}f_{Y_i}^*(y|x_1,\cdots,x_n)&=c_{i-1}^{\tilde{S}}\sum_{j=1}^{i}a_{j,i}^{\tilde{S}}\frac{(b+t)^{a+n}}{\Gamma(a+n)}\int_0^\infty\theta^{a+n}\mathrm{e}^{-\theta(b+t+\gamma_j^{\tilde{S}}y)}\mathrm{d}\theta\\&=c_{i-1}^{\tilde{S}}(b+t)^{a+n}(a+n)\sum_{j=1}^{i}\frac{a_{j,i}^{\tilde{S}}}{(b+t+\gamma_j^{\tilde{S}}y)^{a+n+1}}.\end{aligned}$$

Under the general entropy loss function given in (3) and based on (4), the point predictor for Y_i is

$$\hat{Y}_{i;GE}=\left[E\left(Y_i^{-\delta}|X_1,\cdots,X_n\right)\right]^{\frac{-1}{\delta}},$$

where (assuming $-(n+a)<\delta<1$)

$$\begin{aligned}E\left(Y_i^{-\delta}|X_1,\cdots,X_n\right)&=c_{i-1}^{\tilde{S}}\sum_{j=1}^{i}a_{j,i}^{\tilde{S}}\frac{(b+T)^{a+n}}{\Gamma(a+n)}\left\{\int_0^\infty\theta^{a+n}\,\mathrm{e}^{-\theta(b+T)}\left[\int_0^\infty y^{-\delta}\mathrm{e}^{-\theta\gamma_j^{\tilde{S}}y}dy\right]\mathrm{d}\theta\right\}\\&=c_{i-1}^{\tilde{S}}\sum_{j=1}^{i}a_{j,i}^{\tilde{S}}\frac{(b+T)^{a+n}\Gamma(1-\delta)}{\Gamma(a+n)(\gamma_j^{\tilde{S}})^{1-\delta}}\left\{\int_0^\infty\theta^{a+n+\delta-1}\,\mathrm{e}^{-\theta(b+T)}\mathrm{d}\theta\right\}\\&=\frac{\Gamma(a+n+\delta)\Gamma(1-\delta)}{\Gamma(a+n)(b+T)^{\delta}}c_{i-1}^{\tilde{S}}\sum_{j=1}^{i}\frac{a_{j,i}^{\tilde{S}}}{(\gamma_j^{\tilde{S}})^{1-\delta}}\\&=\frac{\Gamma(a+n+\delta)}{\Gamma(a+n)(b+T)^{\delta}}A_1\left(i,\delta,\tilde{S}\right),\quad\text{say.}\end{aligned} \tag{7}$$

Since in many real applications, no prior knowledge is available about the distribution of θ, we may take $a=b=0$, i.e. Jeffrey's non-informative prior for θ. So, we have

$$E\left(Y_i^{-\delta}|X_1,\cdots,X_n\right)=\frac{\Gamma(n+\delta)}{(n-1)!T^{\delta}}A_1\left(i,\delta,\tilde{S}\right).$$

Then, the point predictor for Y_i is

$$\hat{Y}_{i;GE} = T\,\Lambda(i,\delta,n,\tilde{S}), \tag{8}$$

in which

$$\Lambda(i,\delta,n,\tilde{S}) = \left[\frac{\Gamma(n+\delta)}{(n-1)!} A_1\big(i,\delta,\tilde{S}\big)\right]^{-\frac{1}{\delta}}.$$

We can consider the risk function of the obtained point predictor of Y_i, which is computed to be

$$\mathrm{R}\big(Y_i,\hat{Y}_{i;GE}\big) = E_\theta\big[\mathrm{L}\big(Y_i,\hat{Y}_{i;GE}\big)\big].$$

Based on (3), we obtain

$$\begin{aligned}\mathrm{R}\big(Y_i,\hat{Y}_{i;GE}\big) &= E_\theta\left[\left(\frac{\hat{Y}_{i;GE}}{Y_i}\right)^{\delta} - \delta\ln\left(\frac{\hat{Y}_{i;GE}}{Y_i}\right)\right] - 1\\ &= E_\theta\left[\hat{Y}_{i;GE}{}^{\delta}\right] E_\theta\left[\frac{1}{Y_i{}^{\delta}}\right] - \delta\, E_\theta\big[\ln\big(\hat{Y}_{i;GE}\big)\big] + \delta\, E_\theta[\ln(Y_i)] - 1,\end{aligned} \tag{9}$$

where the last equality is obtained due to the independence of $\hat{Y}_{i;GE}$ and Y_i.

On the other hand, from (8), we can find

$$\begin{aligned}E_\theta\left[\hat{Y}_{i;GE}{}^{\delta}\right] &= \Lambda\big(i,\delta,n,\tilde{S}\big)^{\delta} E_\theta(T^{\delta})\\ &= \frac{1}{\theta^{\delta}}\Lambda\big(i,\delta,n,\tilde{S}\big)^{\delta}\frac{\Gamma(n+\delta)}{(n-1)!}\\ &= \frac{1}{\theta^{\delta}}\left(\Gamma(1-\delta)c_{i-1}^{\tilde{S}}\sum_{j=1}^{i}\frac{a_{j,i}^{\tilde{S}}}{(\gamma_j^{\tilde{S}})^{1-\delta}}\right)^{-1}\\ &= \frac{1}{\theta^{\delta}}\Big(A_1\big(i,\delta,\tilde{S}\big)\Big)^{-1}.\end{aligned} \tag{10}$$

Also, from (6), we have

$$\begin{aligned}E_\theta\left[\frac{1}{Y_i{}^{\delta}}\right] &= \theta c_{i-1}^{\tilde{S}}\sum_{j=1}^{i} a_{j,i}^{\tilde{S}}\int_0^{\infty} y^{-\delta} \mathrm{e}^{-\theta\gamma_j^{\tilde{S}} y}\,\mathrm{d}y\\ &= \theta^{\delta}\Gamma(1-\delta)c_{i-1}^{\tilde{S}}\sum_{j=1}^{i}\frac{a_{j,i}^{\tilde{S}}}{(\gamma_j^{\tilde{S}})^{1-\delta}}\\ &= \theta^{\delta} A_1\big(i,\delta,\tilde{S}\big).\end{aligned} \tag{11}$$

Moreover, we can obtain (see Gradshteyn and Ryzhik [9], p. 573)

$$E_\theta\left[\ln\left(\hat{Y}_{i;GE}\right)\right] = E_\theta[\ln(T)] + \ln\left(\Lambda(i,\delta,n,\tilde{S})\right)$$
$$= \int_0^\infty \ln(t)\frac{\theta^n}{(n-1)!}t^{n-1}e^{-\theta t}dt + \ln\left(\Lambda(i,\delta,n,\tilde{S})\right)$$
$$= \Psi(n) - \ln(\theta) + \ln\left(\Lambda(i,\delta,n,\tilde{S})\right), \quad (12)$$

where $\Psi(x) = \frac{\Gamma\prime(x)}{\Gamma(x)}$ is the Euler psi function or digamma function.

Finally, from the fact that $c_{i-1}^{\tilde{S}}\sum_{j=1}^{i}\frac{a_{j,i}^{\tilde{S}}}{\gamma_j^{\tilde{S}}} = 1$, we find (see Gradshteyn and Ryzhik [9], p. 571)

$$E_\theta[\ln(Y_i)] = \theta c_{i-1}^{\tilde{S}}\sum_{j=1}^{i}a_{j,i}^{\tilde{S}}\int_0^\infty \ln(y)e^{-\theta\gamma_j^{\tilde{S}}y}\,dy$$
$$= -c_{i-1}^{\tilde{S}}\sum_{j=1}^{i}\frac{a_{j,i}^{\tilde{S}}}{\gamma_j^{\tilde{S}}}\left(\gamma + \ln\left(\theta\gamma_j^{\tilde{S}}\right)\right) \quad (13)$$
$$= -\gamma - \ln(\theta) - c_{i-1}^{\tilde{S}}\sum_{j=1}^{i}\frac{a_{j,i}^{\tilde{S}}}{\gamma_j^{\tilde{S}}}\ln\left(\gamma_j^{\tilde{S}}\right)$$
$$= -\gamma - \ln(\theta) - A_2\left(i,\tilde{S}\right). \quad \text{say,}$$

in which γ is the Euler constant.

Based on Equations (9)-(13), we have

$$\mathrm{R}\left(Y_i,\hat{Y}_{i;GE}\right) = -\delta\left[\Psi(n) + \ln\left(\Lambda(i,\delta,n,\tilde{S})\right) + \gamma + A_2\left(i,\tilde{S}\right)\right]$$
$$= -\delta\Psi(n) + \ln\left(\frac{\Gamma(n+\delta)}{(n-1)!}\right) + \ln\left(A_1\left(i,\delta,\tilde{S}\right)\right) - \delta\gamma - \delta A_2\left(i,\tilde{S}\right),$$

which is free of θ, as we expected. Thus, the risk function and the Bayes risk function are the same.

We have computed the values of $\mathrm{R}\left(Y_i,\hat{Y}_{i;GE}\right)$. Table 1 shows some values of $\mathrm{R}\left(Y_i,\hat{Y}_{i;GE}\right)$ for different choices of i and n when $\delta = -0.5, 0.5$, $m = 10$, $N = 20$, $M = 20$ and $\widetilde{S_1} = (10,0,\cdots,0)$, $\widetilde{S_2} = (0,\cdots,0,10)$ and $\widetilde{S_3} = (1,\cdots,1)$. From Table 1, one can observe that $\mathrm{R}\left(Y_i,\hat{Y}_{i;GE}\right)$ is a decreasing function of n and i, when all other parameters are kept fixed. Moreover, $\mathrm{R}\left(Y_1,\hat{Y}_{1;GE}\right)$, namely the risk function of the first predictor does not depend on the censoring scheme.

In this paper, as the second criterion, we consider a cost function as follows

$$C(n) = p_0 + p_uN + p_tT,$$

where p_0, p_u and p_t are the sampling set-up cost or any other related cost involved in sampling, the cost per unit, and the cost related to the test time, respectively. Since T is a random variable whose distribution is $\Gamma(n,\theta)$, we can consider the expected value of the cost function, which is given by

$$E_\theta[C(n)] = p_0 + p_uN + p_t\frac{n}{\theta}. \quad (14)$$

Table 1. The values of $\mathrm{R}\left(Y_i, \hat{Y}_{i;GE}\right)$.

			N			
δ	$\tilde{S}$	i	5	10	15	20
-0.5	$\widetilde{S_1}$	1	0.1965	0.1812	0.1765	0.1742
		3	0.0792	0.0639	0.0592	0.0570
		5	0.0579	0.0425	0.0379	0.0356
		7	0.0500	0.0346	0.0300	0.0277
	$\widetilde{S_2}$	1	0.1965	0.1812	0.1765	0.1742
		3	0.0752	0.0598	0.0552	0.0529
		5	0.0555	0.0402	0.0355	0.0332
		7	0.0476	0.0323	0.0276	0.0254
	$\widetilde{S_3}$	1	0.1965	0.1812	0.1765	0.1742
		3	0.0754	0.0600	0.0554	0.0531
		5	0.0561	0.0407	0.0361	0.0338
		7	0.0489	0.0336	0.0289	0.0267
0.5	$\widetilde{S_1}$	1	0.3104	0.2966	0.2922	0.2901
		3	0.0839	0.0701	0.0657	0.0636
		5	0.0580	0.0442	0.0398	0.0377
		7	0.0490	0.0352	0.0308	0.0287
	$\widetilde{S_2}$	1	0.3104	0.2966	0.2922	0.2901
		3	0.0796	0.0659	0.0615	0.0593
		5	0.0555	0.0417	0.0373	0.0352
		7	0.0466	0.0328	0.0284	0.0262
	$\widetilde{S_3}$	1	0.3104	0.2966	0.2922	0.2901
		3	0.0798	0.0661	0.0617	0.0595
		5	0.0561	0.0423	0.0379	0.0357
		7	0.0479	0.0341	0.0297	0.0275

From (14), we can see that $E_\theta[C(n)]$ depends on the unknown parameter θ, and therefore it can be replaced by its preliminary estimator based on past experiments and pre-information. In Table 2, we present some values of $E_\theta[C(n)]$ for different choices of n, when $\theta = 1, p_0 = 1, p_u = 0.5,\ p_t = 1$ and $N = 20$. From Table 2, we observe that the values of $E_\theta[C(n)]$ increase as n increases, as we expected.

Table 2. The values of $E_\theta[C(n)]$

n	1	5	10	15	20
$E_\theta[C(n)]$	12	16	21	26	31

In the sequel, the aim is to find the optimal value for n under the optimization problem $\mathrm{R}\left(Y_i, \hat{Y}_{i;GE}\right) \leq R^*$ and $E_\theta[C(n)] \leq B^*$, where R^* and B^* are pre-fixed values.

On the one hand, $\mathrm{R}\left(Y_i, \hat{Y}_{i;GE}\right) \leq R^*$ is equivalent to saying that $n \geq n_0$, where n_0 is the solution of the following equation

$$-\delta\Psi(n) + \ln\left(\frac{\Gamma(n+\delta)}{(n-1)!}\right) = R^* - \ln\left(A_1\big(i,\delta,\tilde{S}\big)\right) + \delta\gamma + \delta A_2\big(i,\tilde{S}\big),$$

where $A_1(\cdot,\cdot,\cdot)$ and $A_2(\cdot,\cdot)$ are defined in (7) and (13), respectively.

On the other hand, from (14), we have

$$E_\theta[C(n)] \le B^* \Leftrightarrow n \le \theta(B^* - p_0 - p_u N)/p_t.$$

So, we readily find that the optimal value for n, say n_{opt}, satisfies the following inequality

$$\lfloor n_0 \rfloor \le n_{opt} \le \min\left\{N, \left\lfloor \frac{\theta(B^* - p_0 - p_u N)}{p_t} \right\rfloor\right\},$$

where $\lfloor\cdot\rfloor$ is the floor function.

Table 3 presents some values of n_{opt} by considering different values of i when $B^* = 20$, $p_0 = 1$, $p_u = 0.5$, $p_t = 1$, $m = 10$, $N = 20$, $M = 20$, $\theta = 1$ and $\widetilde{S_1} = (10,0,\cdots,0)$, $\widetilde{S_2} = (0,\cdots,0,10)$ and $\widetilde{S_3} = (1,\cdots,1)$ for $\delta = -0.5$, $R^* = 0.2$ and $\delta = 0.5$, $R^* = 0.3$.

Table 3. The values of n_{opt} **(**a dash (-) means that there is no n_{opt}**)**

δ	$\tilde{S}$	B^*	R^*	i	n_{opt}
-0.5	$\widetilde{S_1}$	20	0.2	1	{5,…,9}
				3	{2,…,9}
				5	{2,…,9}
				7	{2,…,9}
	$\widetilde{S_2}$	20	0.2	1	{5,…,9}
				3	{2,…,9}
				5	{2,…,9}
				7	{2,…,9}
	$\widetilde{S_3}$	20	0.2	1	{5,…,9}
				3	{2,…,9}
				5	{2,…,9}
				7	{2,…,9}
0.5	$\widetilde{S_1}$	20	0.3	1	-
				3	{2,…,9}
				5	{1,…,9}
				7	{1,…,9}
	$\widetilde{S_2}$	20	0.3	1	-
				3	{2,…,9}
				5	{1,…,9}
				7	{1,…,9}
	$\widetilde{S_3}$	20	0.3	1	{9}
				3	{1,…,9}
				5	{1,…,9}
				7	{1,…,9}

Figure 1 presents the values of $E_\theta[C(n)]$ for $\theta = 1$, $p_0 = 1$, $p_u = 0.5$, $p_t = 1$ and $N = 20$, when $B^* = 20$. Figures 2 and 3 show the values of $\mathrm{R}(Y_i, \hat{Y}_{i;GE})$ for different choices of *i* and *n* for $\delta = -0.5$, $R^* = 0.2$ and $\delta = 0.5$, $R^* = 0.3$, respectively, when $m = 10$, $N = 20$, $M = 20$, and for $\widetilde{S_1} = (10,0,\cdots,0)$, $\widetilde{S_2} = (0,\cdots,0,10)$ and $\widetilde{S_3} = (1,\cdots,1)$.

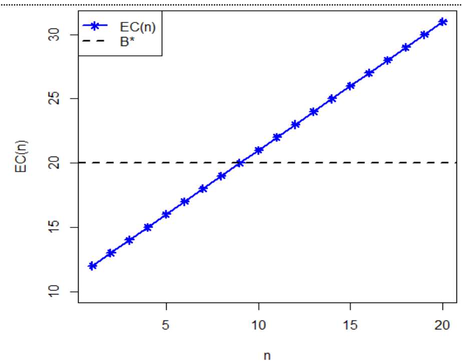

Figure 1. The values of $E_\theta[C(n)]$ for $\theta = 1$, $p_0 = 1$, $p_u = 0.5$, $p_t = 1$ and $N = 20$, when $B^* = 20$.

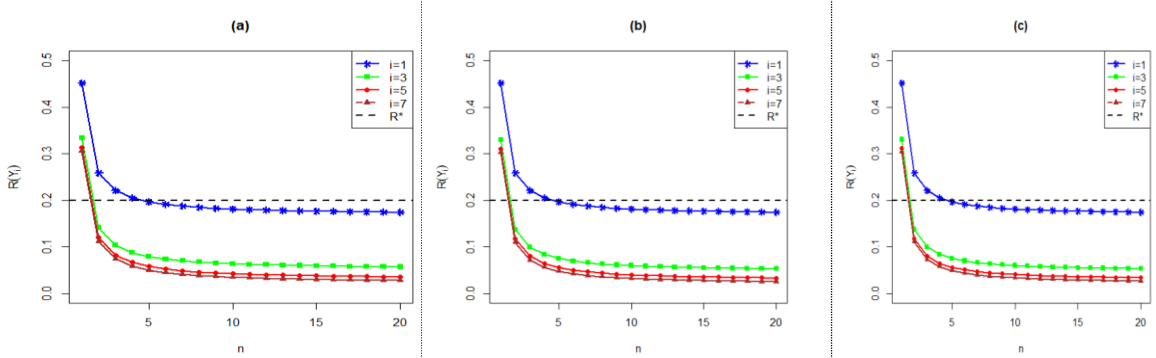

Figure 2. The values of $\mathrm{R}(Y_i, \hat{Y}_{i;GE})$ for different choices of *i* and *n* when $\delta = -0.5$, $m = 10$, $N = 20$, $M = 20$, and $R^* = 0.2$, when (a): $\widetilde{S_1} = (10,0,\cdots,0)$, (b): $\widetilde{S_2} = (0,\cdots,0,10)$ and (c): $\widetilde{S_3} = (1,\cdots,1)$.

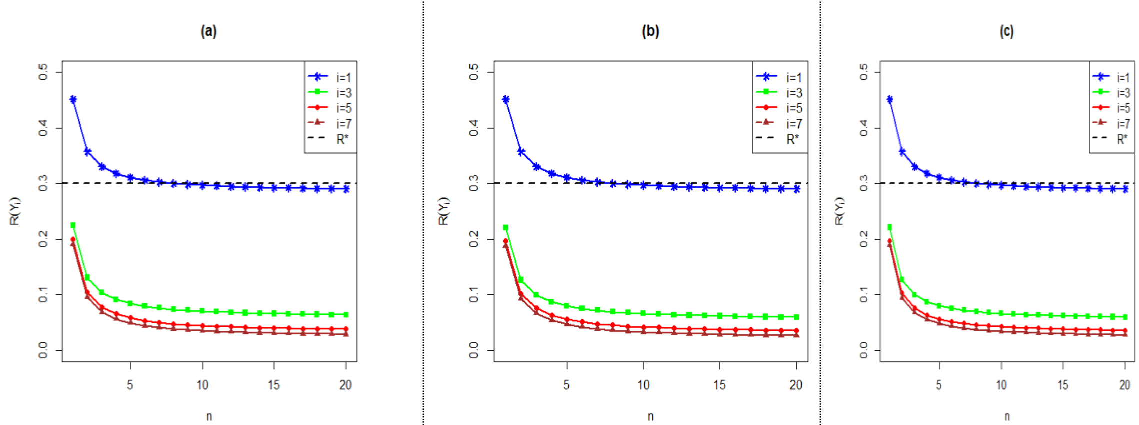

Figure 3. The values of $\mathrm{R}(Y_i, \hat{Y}_{i;GE})$ for different choices of *i* and *n* when $\delta = 0.5$, $m = 10$, $N = 20$, $M = 20$, and $R^* = 0.3$, when (a): $\widetilde{S_1} = (10,0,\cdots,0)$, (b): $\widetilde{S_2} = (0,\cdots,0,10)$ and (c): $\widetilde{S_3} = (1,\cdots,1)$.

3. Concluding Remarks

This paper focused on the problem of Bayesian prediction of a future progressively Type-II censored order statistic under an asymmetric loss function. Two main criteria are applied to determining the optimal values of *n*, the number of failures to be observed in the informative sample. One of them is the Bayes risk function, and the other is the cost function. The values of optimized *n* are computed and tabulated for selected cases. All the computations in this paper were done using the statistical software R [10].

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