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A Chance-Constrained Inverse DEA Approach under Managerial and Natural Disposability

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The traditional inverse data envelopment analysis (IDEA) models assess specific performance metrics in relation to changes in others, without taking into consideration the existence of random and undesirable outputs. This study presents a novel inverse DEA model with random and undesirable outputs, enabling the estimation of some random performance measures for changes of other random measures. The proposed chance-constrained inverse DEA model integrates both managerial and natural disposability constraints. By using the introduced approach, the estimation of natural disposable random inputs is presented for changes in random desirable outputs. Also, undesirable outputs are assessed for the perturbation of managerial disposable random inputs while the stochastic efficiency is maintained. The models are solved as linear problems, with a numerical example provided to illustrate their application. The findings indicate that this approach is effective for evaluating efficiency and performance metrics in scenarios involving random and undesirable outputs.

***Keywords**: Inverse DEA, Chance-Constrained DEA, Undesirable Outputs, Managerial Disposability.*

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1. Introduction

Calculating performance metrics is essential for organizations to evaluate how effectively they are utilizing their resources to achieve desired outcomes. However, when dealing with complex systems involving random and undesirable outputs, traditional estimation techniques may not yield accurate results. In such cases, utilizing an alternative inverse data envelopment analysis (IDEA) approach that considers both random and undesirable outputs can be advantageous.

IDEA typically focuses on evaluating performance factors based on desirable outputs and specific measures [10, 17]. Jahanshahloo et al. [9] estimated performance measures in the presence of certain undesirable indicators. Wegener and Amin [16] provided an alternative inverse DEA model to minimize greenhouse gas emissions. Lu et al. [12] extended an inverse DEA approach with undesirable output and frontier changes. Asadi et al. [2] developed an inverse multi-period free disposal hull model in the presence of undesirable outputs. Taher et al. [15] estimated the change levels of undesirable outputs and desirable inputs while interval variables are presented.

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Also, some studies such as [7, 8, 18] provided inverse DEA approaches with imprecise data. Ghomi et al. [7] have introduced an IDEA approach that includes desirable and random measures. Chance-constrained DEA techniques [4, 5, 13] have been used to assess the stochastic efficiency of entities with random data. Nevertheless, there is a scarcity of inverse DEA models to estimate performance metrics when there are undesirable outputs and random factors.

Therefore, this paper goes further by introducing an inverse DEA method that incorporates both random and undesirable outputs. By taking into account both random and undesirable outputs in performance assessments, organizations gain a more realistic understanding of their efficiency. Accurately estimating performance metrics with this method enables organizations to pinpoint inefficiencies or issues related to random factors, guiding decision-makers in optimizing resource allocation, reducing waste, and managing risks. Thus, this research proposes an alternative inverse chance-constrained DEA model that incorporates both undesirable outputs and random measures. The proposed techniques are based on managerial and natural disposability concepts. A practical example is provided to illustrate the introduced technique.

In sum, the main contribution of this study is threefold: first, providing an alternative inverse DEA approach with random and undesirable measures. Second, estimating natural disposable inputs for the alteration of desirable outputs retaining stochastic performance in the presence of undesirable outputs and random factors, and third, assessing random undesirable outputs for the perturbation of managerial disposable random inputs while ensuring efficiency is maintained.

The structure of this paper is organized as follows: Basic concepts and preliminaries are presented in Section 2. Chance-constrained Inverse DEA approaches under managerial and natural disposability are given in Section 3. An illustrative example is provided in Section 4 for clarification. Conclusions are displayed in Section 5.

2. Preliminaries

In this section, basic and primary definitions and approaches, including managerial and natural disposability, chance-constrained DEA and inverse DEA, are explained

2.1. Managerial and natural disposability concepts

Managerial disposability refers to the ability of managers to make decisions and take actions that are in the best interest of the organization. This concept emphasizes the importance of effective leadership and decision-making in achieving organizational goals and objectives. Managers who are able to effectively utilize resources, and make sound decisions are considered to have high managerial ability. Truthfully, inputs are increased to decrease undesirable outputs and increase desirable outputs under the managerial disposability assumption.

On the other hand, natural disposability refers to the idea that resources are limited and must be used wisely to ensure their sustainability. This concept emphasizes the importance of conserving natural resources and minimizing waste in order to protect the environment and ensure long-term viability. Organizations that prioritize natural disposability may implement sustainable practices, such as recycling, energy conservation, and waste reduction, to minimize their impact on the environment. Actually, entities attempt to decrease inputs to reduce undesirable outputs while it is attempted to increase desirable outputs.

Overall, both managerial and natural disposability concepts emphasize the importance of responsible resource management and decision-making to achieve organizational success while also considering the long-term impact on the environment. By integrating these concepts into their operations, organizations can strive to achieve a balance between economic prosperity and environmental sustainability. [14]

2.2. Chance-constrained DEA

Chance-constrained DEA is a DEA method that takes into account uncertainty in the input and output data. In traditional DEA, all input and output data are assumed to be deterministic and known with certainty. However, in many real-world situations, there is uncertainty in the data due to various factors such as measurement errors, sampling variability, and random fluctuations.

Chance-constrained DEA allows for the inclusion of this uncertainty in the analysis by incorporating probabilistic constraints on the input and output data. The goal is to determine the efficiency of entities under uncertainty, taking into account the likelihood of different scenarios.

By considering uncertainty in the input and output data, chance-constrained DEA provides a more robust and realistic assessment of the efficiency of DMUs. It allows decision-makers to account for the risk and variability in the data when evaluating the performance of organizations or entities. This can help in making more informed decisions and identifying areas for improvement in a more accurate and reliable manner. [4, 5, 13].

2.3. Inverse DEA

Inverse DEA is a method used to determine the input (output) levels required to reach a target level of output (input). The process of Inverse DEA involves specifying a target level of output (input) and then using mathematical optimization techniques to calculate the optimal input (output) levels that would be required to achieve this target while the efficiency values are preserved or improved. This can be useful for decision-makers who want to improve the efficiency of their operations by identifying the most effective use of resources (the optimal level of production). Actually, in Inverse DEA, the goal is to identify the optimal input and output levels that would lead to a specific target efficiency score.

By identifying the optimal input (output) levels required for each DMU to achieve a desired level of output (input), managers can make informed decisions about resource allocation, target setting and process improvement.

Overall, inverse DEA is a valuable tool for decision-makers to set performance targets, identify areas for improvement, and optimize resource allocation to enhance overall efficiency. [6, 10, 17]

3. Inverse Chance-Constrained DEA with Environmental Measures

Assume there are n DMUs, $DMU_j\,(j=1,\dots,n)$ with m natural disposable inputs $\tilde{x}_{ij}\,(i=1,\dots,m)$, K managerial disposable inputs $\tilde{z}_{kj}\,(k=1,\dots,K)$, s desirable outputs $\tilde{y}_{rj}\,(r=1,\dots,s)$ and C undesirable outputs $\tilde{b}_{cj}\,(c=1,\dots,C)$. The purpose is responding to the following questions:

- What is the smallest change that needs to be made to the natural disposable random inputs in order to achieve the desired changes in the desirable random outputs while the stochastic efficiency is consistent?
- What is the minimum change of undesirable outputs for the changes of managerial disposable random inputs where the stochastic efficiency value is preserved?

For this purpose, the procedure shown in Figure 1 and described below can be used.

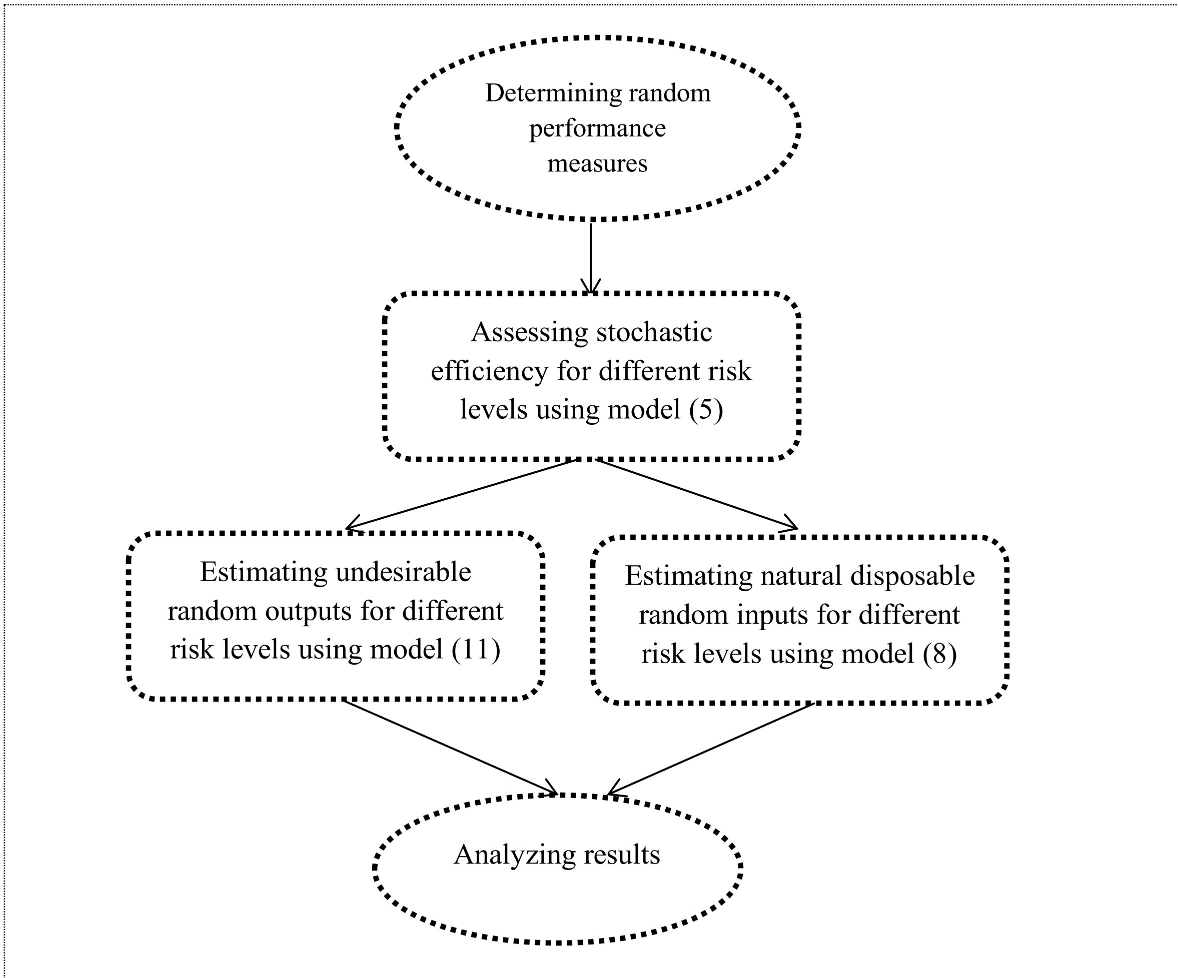

Figure 1. Procedure

3.1. Chance-constrained DEA with Undesirable Outputs

The following chance-constrained DEA model is suggested to assess the stochastic environmental efficiency of DMUs:

$$
\begin{aligned}
& e^{\alpha *} = Min \ \ \theta_\alpha \\
& s.t. \quad p\left\{\sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} \le \theta_\alpha \tilde{x}_{io}\right\} \ge 1-\alpha, \ \ i=1,\dots,m, \\
& \qquad p\left\{\sum_{j=1}^{n} \lambda_j \tilde{z}_{kj} \ge \tilde{z}_{ko}\right\} \ge 1-\alpha, \ \ k=1,\dots,K, \\
& \qquad p\left\{\sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} \ge \tilde{y}_{ro}\right\} \ge 1-\alpha, \ r=1,\dots,s, \\
& \qquad p\left\{\sum_{j=1}^{n} \lambda_j \tilde{b}_{cj} \le \theta_\alpha \tilde{b}_{co}\right\} \ge 1-\alpha, c=1,\dots,C, \\
& \qquad \lambda_j \ge 0, \ j=1,\dots,n.
\end{aligned}
\tag{1}
$$

In which α is the risk level and p shows the probability.

Assuming x_{io} and $\sigma_i^o(\lambda_j, \theta)$ as the average value of $\tilde{x}_{io}$ and the standard error of $\sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} - \theta_\alpha \tilde{x}_{io}$, it can be accomplished that

$$
\sum_{j=1}^{n} \lambda_j x_{ij} - \theta_\alpha x_{io} \le \Phi^{-1}(\alpha)\sigma_i^o(\lambda_j, \theta_\alpha) \tag{2}
$$

in which Φ is the standard normal distribution and Φ^{-1} is its inverse. In the same manner, additional restrictions within model (1) can be restated. Therefore, model (1) is converted into model (3).

$$
\begin{aligned}
& e^{\alpha *} = Min \ \ \theta_\alpha \\
& s.t. \ \ \sum_{j=1}^{n} \lambda_j x_{ij} - \theta_\alpha x_{io} \le \Phi^{-1}(\alpha)\sigma_i^o(\lambda_j, \theta_\alpha), i=1,\dots,m, \\
& \qquad y_{ro} - \sum_{j=1}^{n} \lambda_j y_{rj} \le \Phi^{-1}(\alpha)\sigma_r^o(\lambda_j), r=1,\dots,s, \\
& \qquad z_{ko} - \sum_{j=1}^{n} \lambda_j z_{kj} \le \Phi^{-1}(\alpha)\sigma_k^o(\lambda_j), k=1,\dots,K, \\
& \qquad \sum_{j=1}^{n} \lambda_j b_{cj} - \theta_\alpha b_{co} \le \Phi^{-1}(\alpha)\sigma_c^o(\lambda_j, \theta_\alpha), c=1,\dots,C, \\
& \qquad \lambda_j \ge 0.
\end{aligned}
\tag{3}
$$

Now, it is supposed that all stochastic performance measures are identified individually by single factors, i.e.

$$\tilde{x}_{ij} = x_{ij} + a_{ij}\gamma \ ,$$
$$\tilde{y}_{rj} = y_{rj} + d_{rj}\eta \ ,$$
$$\tilde{z}_{kj} = z_{kj} + e_{kj}\sigma \ ,$$
$$\tilde{b}_{cj} = b_{cj} + f_{cj}\upsilon \ ,$$

that x_{ij}, y_{rj}, z_{kj} and b_{cj} are the average values of $\tilde{x}_{ij}$, $\tilde{y}_{rj}$, $\tilde{z}_{kj}$ and $\tilde{b}_{cj}$. Besides, a_{ij}, d_{rj}, e_{kj} and f_{cj} display the standard errors and γ, η, σ and υ are imagined to be independent random variables with standard normal distributions. This assumption has been considered to transform model (3) to a linear problem and has been utilized in economics for a long period of time. Readers can refer to Li and Huang [11] for finding out more. According to Cooper et al. [4], it is deemed that all measures, x_{ij}, y_{rj}, z_{kj}, b_{cj}, a_{ij}, d_{rj}, e_{kj} and f_{cj} are non-negative.

Therefore, model (3) is replaced with model (4).

$$\begin{aligned}
& e^{\alpha *} = Min \ \ \theta_\alpha \\
& s.t. \sum_{j=1}^{n} \lambda_j x_{ij} - \theta_\alpha x_{io} \le \Phi^{-1}(\alpha) \left| \sum_{j=1}^{n} \lambda_j a_{ij} - \theta_\alpha a_{io} \right|, \ \ i = 1,\dots,m, \\
& y_{ro} - \sum_{j=1}^{n} \lambda_j y_{rj} \le \Phi^{-1}(\alpha) \left| \sum_{j=1}^{n} \lambda_j d_{rj} - d_{ro} \right|, \quad r = 1,\dots,s, \\
& z_{ko} - \sum_{j=1}^{n} \lambda_j z_{kj} \le \Phi^{-1}(\alpha) \left| \sum_{j=1}^{n} \lambda_j e_{kj} - e_{ko} \right|, \ \ k = 1,\dots,K, \\
& \sum_{j=1}^{n} \lambda_j b_{cj} - \theta_\alpha b_{co} \le \Phi^{-1}(\alpha) \left| \sum_{j=1}^{n} \lambda_j f_{cj} - \theta_\alpha f_{co} \right|, \ \ c = 1,\dots,C, \\
& \lambda_j \ge 0.
\end{aligned} \tag{4}$$

In this stage, the goal programming theory is used to transform the non-linear problem (4) into a linear model. Thus, by taking into account

$$\left| \sum_{j=1}^{n} \lambda_j a_{ij} - \theta_\alpha a_{io} \right| = p_i^+ + p_i^-, \ \ i = 1,\dots,m,$$
$$\sum_{j=1}^{n} \lambda_j a_{ij} - \theta_\alpha a_{io} = p_i^+ - p_i^-, \ \ i = 1,\dots,m,$$
$$p_i^+ p_i^- = 0, \ \ i = 1,\dots,m,$$

$$\left|\sum_{j=1}^{n}\lambda_j d_{rj}-d_{ro}\right|=q_r^{+}+q_r^{-},\quad r=1,\dots,s,$$
$$\sum_{j=1}^{n}\lambda_j d_{rj}-d_{ro}=q_r^{+}-q_r^{-},\quad r=1,\dots,s,$$
$$q_r^{+}q_r^{-}=0,\quad r=1,\dots,s,$$
$$\left|\sum_{j=1}^{n}\lambda_j e_{kj}-e_{ko}\right|=u_k^{+}+u_k^{-},\quad k=1,\dots,K,$$
$$\sum_{j=1}^{n}\lambda_j e_{kj}-e_{ko}=u_k^{+}-u_k^{-},\quad k=1,\dots,K,$$
$$u_k^{+}u_k^{-}=0,\quad k=1,\dots,K,$$
$$\left|\sum_{j=1}^{n}\lambda_j f_{cj}-\theta_\alpha f_{co}\right|=g_c^{+}+g_c^{-},\quad c=1,\dots,C,$$
$$\sum_{j=1}^{n}\lambda_j f_{cj}-\theta_\alpha f_{co}=g_c^{+}-g_c^{-},\quad c=1,\dots,C,$$
$$g_c^{+}g_c^{-}=0,\quad c=1,\dots,C,$$

we have

$$\begin{aligned}
&e^{\alpha *}=Min\quad \theta_\alpha\\
&s.t.\sum_{j=1}^{N}\lambda_j x_{ij}-\Phi^{-1}(\alpha)\sigma(p_i^{+}+p_i^{-})\le\theta_\alpha x_{io},\ i=1,\dots,m,\\
&\sum_{j=1}^{N}\lambda_j a_{ij}-\theta_\alpha a_{io}=p_i^{+}-p_i^{-},\ i=1,\dots,m,\\
&\sum_{j=1}^{N}\lambda_j y_{rj}+\Phi^{-1}(\alpha)\sigma(q_r^{+}+q_r^{-})\ge y_{ro},\quad r=1,\dots,s,\\
&\sum_{j=1}^{N}\lambda_j d_{rj}-d_{ro}=q_r^{+}-q_r^{-},\quad r=1,\dots,s,\\
&\sum_{j=1}^{N}\lambda_j z_{kj}+\Phi^{-1}(\alpha)\sigma(u_{kt}^{+}+u_{kt}^{-})\ge u_{ko},\ k=1,\dots,K,\\
&\sum_{j=1}^{N}\lambda_j e_{kj}-e_{ko}=u_k^{+}-u_k^{-},\ k=1,\dots,K,\\
&\sum_{j=1}^{N}\lambda_j b_{cj}-\Phi^{-1}(\alpha)\sigma(g_c^{+}+g_c^{-})\le\theta_\alpha b_{co},\ c=1,\dots,C,\\
&\sum_{j=1}^{N}\lambda_j f_{cj}-\theta_\alpha f_{co}=g_c^{+}-g_c^{-},\ c=1,\dots,C,\\
&p_i^{+}p_i^{-}=0,\\
&q_r^{+}q_r^{-}=0,\\
&u_k^{+}u_k^{-}=0,\\
&g_c^{+}g_c^{-}=0,\\
&\lambda_j\ge 0,\ j=1,\dots,n,\\
&p_i^{+},p_i^{-},q_r^{+},q_r^{-},u_k^{+},u_k^{-},g_c^{+},g_c^{-}\ge 0,\forall i,r,k,c.
\end{aligned}\tag{5}$$

Following [1], we can ignore $p_i^+ p_i^- = 0,\ q_r^+ q_r^- = 0,\quad u_k^+ u_k^- = 0$ and $g_c^+ g_c^- = 0$ in calculating model (5) and solve model (5) as a linear problem.

Definition 2.1. If $e^{\alpha^*} = 1$ in model (5), the entity under examination is called stochastic efficient for the risk level α. Otherwise, it is stochastic inefficient.

Notice that, in practical scenarios, the predetermined level α represents the level of risk a planner is willing to accept in terms of violating chance constraints. A lower α indicates a higher level of confidence in the DMU being assessed and a lower level of risk for the planner. Typically, a planner's confidence is high, so it is assumed that α is less than or equal to 0.5.

3.2. Estimating Natural Disposable Random Inputs

In this part, the minimum changes of natural disposable random inputs are addressed for the perturbations of random desirable outputs while the stochastic efficiency is preserved for the level α. Accordingly, the subsequent chance-constrained multi-objective problem is provided:

$$
\begin{aligned}
& Min\ \ \tilde{\omega}_{io} \\
& s.t.\ p\left\{\sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} \le \theta_\alpha^* \tilde{\omega}_{io}\right\} \ge 1-\alpha,\ \ i = 1,\dots,m, \\
& \quad p\left\{\sum_{j=1}^{n} \lambda_j \tilde{z}_{kj} \ge \tilde{z}_{ko}\right\} \ge 1-\alpha,\ \ k = 1,\dots,K, \\
& \quad p\left\{\sum_{j=1}^{n} \lambda_j \tilde{y}_{rj} \ge \tilde{\beta}_{ro}\right\} \ge 1-\alpha,\quad r = 1,\dots,s, \\
& \quad p\left\{\sum_{j=1}^{n} \lambda_j \tilde{b}_{cj} \le \theta_\alpha^* \tilde{b}_{co}\right\} \ge 1-\alpha, c = 1,\dots,C, \\
& \qquad \lambda_j \ge 0,\ j = 1,\dots,n.
\end{aligned}
\tag{6}
$$

The new random desirable outputs are shown by $\tilde{\beta}_o = \tilde{Y}_o + \Delta\tilde{Y}_o$, $\Delta\tilde{Y}_o \ge 0, \Delta\tilde{Y}_o \ne 0$ and the projection of natural disposable inputs is denoted by $\tilde{\omega}_{io} = \tilde{x}_{io} + \Delta\tilde{x}_{io}$. Model (6) can be reformulated into the following problem as explained in the previous section.

$$\begin{aligned}
&Min\ \ \tilde{\omega}_{io}\\
&s.t. \sum_{j=1}^{n} \lambda_j x_{ij} - \theta_\alpha^* \omega_{io} \le \Phi^{-1}(\alpha)\sigma_i^o(\lambda_j), i = 1,\ldots,m,\\
&\quad \beta_{ro} - \sum_{j=1}^{n} \lambda_j y_{rj} \le \Phi^{-1}(\alpha)\sigma_r^o(\lambda_j), \quad r = 1,\ldots,s,\\
&\quad z_{ko} - \sum_{j=1}^{n} \lambda_j z_{kj} \le \Phi^{-1}(\alpha)\sigma_k^o(\lambda_j), k = 1,\ldots,K,\\
&\quad \sum_{j=1}^{n} \lambda_j b_{cj} - \theta_\alpha^* b_{co} \le \Phi^{-1}(\alpha)\sigma_c^o(\lambda_j), c = 1,\ldots,C,\\
&\qquad \lambda_j \ge 0.
\end{aligned} \tag{7}$$

In models (6) and (7), θ_α^* is the optimal value obtained from (5). As previously mentioned, it is supposed that all random performance measures are determined individually by single factors. Therefore, model (7) can be substituted with model (8) by considering this aspect and the goal programming theory, i.e.

$$\begin{aligned}
&\left| \sum_{j=1}^{n} \lambda_j a_{ij} - \theta_\alpha^* h_{io} \right| = p_i'^{+} + p_i'^{-},\ i = 1,\ldots,m,\\
&\sum_{j=1}^{n} \lambda_j a_{ij} - \theta_\alpha^* h_{io} = p_i'^{+} - p_i'^{-},\ i = 1,\ldots,m,\\
&p_i'^{+} p_i'^{-} = 0,\ i = 1,\ldots,m,\\
&\left| \sum_{j=1}^{n} \lambda_j d_{rj} - l_{ro} \right| = v_r^+ + v_r^-, \quad r = 1,\ldots,s,\\
&\sum_{j=1}^{n} \lambda_j d_{rj} - l_{ro} = v_r^+ - v_r^-, \quad r = 1,\ldots,s,\\
&v_r^+ v_r^- = 0, \quad r = 1,\ldots,s,\\
&\left| \sum_{j=1}^{n} \lambda_j e_{kj} - e_{ko} \right| = u_k^+ + u_k^-, \quad k = 1,\ldots,K,\\
&\sum_{j=1}^{n} \lambda_j e_{kj} - e_{ko} = u_k^+ - u_k^-, \quad k = 1,\ldots,K,\\
&u_k^+ u_k^- = 0, \quad k = 1,\ldots,K,\\
&\left| \sum_{j=1}^{n} \lambda_j f_{cj} - \theta_\alpha^* f_{co} \right| = g_c^+ + g_c^-,\ c = 1,\ldots,C,\\
&\sum_{j=1}^{n} \lambda_j f_{cj} - \theta_\alpha^* f_{co} = g_c^+ - g_c^-,\ c = 1,\ldots,C,\\
&g_c^+ g_c^- = 0,\ c = 1,\ldots,C,
\end{aligned}$$

$$
\begin{aligned}
& Min \quad \sum_i \omega_{io} \\
& s.t. \sum_{j=1}^{N} \lambda_j x_{ij} - \Phi^{-1}(\alpha)\sigma(p_i'^{+} - p_i'^{-}) \le \theta_\alpha^* \omega_{io}, \ i = 1,\dots,m, \\
& \sum_{j=1}^{N} \lambda_j a_{ij} - \theta_\alpha^* h_{io} = p_i'^{+} - p_i'^{-}, \ i = 1,\dots,m, \\
& \sum_{j=1}^{N} \lambda_j y_{rj} + \Phi^{-1}(\alpha)\sigma(v_r^{+} - v_r^{-}) \ge \beta_{ro}, \quad r = 1,\dots,s, \\
& \sum_{j=1}^{N} \lambda_j d_{rj} - l_{ro} = v_r^{+} - v_r^{-}, \quad r = 1,\dots,s, \\
& \sum_{j=1}^{N} \lambda_j z_{kj} + \Phi^{-1}(\alpha)\sigma(u_k^{+} - u_k^{-}) \ge z_{ko}, \ k = 1,\dots,K, \\
& \sum_{j=1}^{N} \lambda_j e_{kj} - e_{ko} = u_k^{+} - u_k^{-}, \ k = 1,\dots,K, \\
& \sum_{j=1}^{N} \lambda_j b_{cj} - \Phi^{-1}(\alpha)\sigma(g_c^{+} - g_c^{-}) \le \theta_\alpha^* b_{co}, \ c = 1,\dots,C, \\
& \sum_{j=1}^{N} \lambda_j f_{cj} - \theta_\alpha^* f_{co} = g_c^{+} - g_c^{-}, \ c = 1,\dots,C, \\
& \quad p_i'^{+} p_i'^{-} = 0, \\
& \quad v_r^{+} v_r^{-} = 0, \\
& \quad u_k^{+} u_k^{-} = 0, \\
& \quad g_c^{+} g_c^{-} = 0, \\
& \quad \lambda_j \ge 0, \ j = 1,\dots,n, \\
& p_i'^{+}, p_i'^{-}, v_r^{+}, v_r^{-}, u_k^{+}, u_k^{-}, g_c^{+}, g_c^{-} \ge 0, \forall i, r, k, c.
\end{aligned}
\tag{8}
$$

Notice that by following Charnes and Cooper [3], stochastic variables in objective functions are addressed using their expected values and optimizing the expected values of $\tilde{\omega}_{io}$ shown by ω_{io}, and also the weighted sum approach is used to transform the multi objective problem into one objective model. Furthermore, $p_i'^{+} p_i'^{-} = 0, v_r^{+} v_r^{-} = 0, u_k^{+} u_k^{-} = 0,$ and $g_c^{+} g_c^{-} = 0$ can be ignored by following [1].

3.3. Estimating Undesirable Outputs

This section addresses how to find the minimum random undesirable outputs for the perturbation of managerial disposable random inputs, while maintaining stochastic efficiency at a certain level α. The new managerial disposable random inputs are shown by $\tilde{\rho}_o = \tilde{Z}_o + \Delta\tilde{Z}_o, \ \ \Delta\tilde{Z}_o \ge 0, \Delta\tilde{Z}_o \ne 0$ and the targets of undesirable outputs are shown by $\tilde{\pi}_{co} = \tilde{B}_{co} + \Delta\tilde{B}_{co}$. Consequently, the following problem of multi-objective optimization under chance constraints is introduced:

$$
\begin{aligned}
& Min\ \ \tilde{\pi}_{co} \\
& s.t.\ p\left\{\sum_{j=1}^{n}\lambda_j\tilde{x}_{ij} \le \theta_\alpha^*\tilde{x}_{io}\right\} \ge 1-\alpha,\ \ i=1,\dots,m, \\
& \quad p\left\{\sum_{j=1}^{n}\lambda_j\tilde{z}_{kj} \ge \tilde{\rho}_{ko}\right\} \ge 1-\alpha,\ \ k=1,\dots,K, \\
& \quad p\left\{\sum_{j=1}^{n}\lambda_j\tilde{y}_{rj} \ge \tilde{y}_{ro}\right\} \ge 1-\alpha,\quad r=1,\dots,s, \\
& \quad p\left\{\sum_{j=1}^{n}\lambda_j\tilde{b}_{cj} \le \theta_\alpha^*\tilde{\pi}_{co}\right\} \ge 1-\alpha, c=1,\dots,C, \\
& \qquad \lambda_j \ge 0,\ j=1,\dots,n.
\end{aligned}
\tag{9}
$$

Model (9) can be restructured into the next problem as outlined in the preceding section.

$$
\begin{aligned}
& Min\ \ \tilde{\pi}_{io} \\
& s.t. \sum_{j=1}^{n}\lambda_j x_{ij} - \theta_\alpha^* x_{io} \le \Phi^{-1}(\alpha)\sigma_i^o(\lambda_j), i=1,\dots,m, \\
& \quad y_{ro} - \sum_{j=1}^{n}\lambda_j y_{rj} \le \Phi^{-1}(\alpha)\sigma_r^o(\lambda_j),\quad r=1,\dots,s, \\
& \quad \rho_{ko} - \sum_{j=1}^{n}\lambda_j z_{kj} \le \Phi^{-1}(\alpha)\sigma_k^o(\lambda_j), k=1,\dots,K, \\
& \quad \sum_{j=1}^{n}\lambda_j b_{cj} - \theta_\alpha^*\pi_{co} \le \Phi^{-1}(\alpha)\sigma_c^o(\lambda_j), c=1,\dots,C, \\
& \qquad \lambda_j \ge 0.
\end{aligned}
\tag{10}
$$

In model (10), the optimal value θ_α^* is achieved from model (5). It has been previously stated that each random performance measure is influenced by a single factor. Taking into account this consideration and the principles of goal programming theory, namely,

$$
\begin{aligned}
& \left|\sum_{j=1}^{n}\lambda_j a_{ij} - \theta_\alpha^* a_{io}\right| = p_i^+ + p_i^-,\ \ i=1,\dots,m, \\
& \sum_{j=1}^{n}\lambda_j a_{ij} - \theta_\alpha^* a_{io} = p_i^+ - p_i^-,\ \ i=1,\dots,m, \\
& p_i^+ p_i^- = 0,\ \ i=1,\dots,m,
\end{aligned}
$$

$$\left|\sum_{j=1}^{n}\lambda_j d_{rj}-d_{ro}\right|=q_r^{+}+q_r^{-},\quad r=1,\ldots,s,$$
$$\sum_{j=1}^{n}\lambda_j d_{rj}-d_{ro}=q_r^{+}-q_r^{-},\quad r=1,\ldots,s,$$
$$q_r^{+}q_r^{-}=0,\quad r=1,\ldots,s,$$
$$\left|\sum_{j=1}^{n}\lambda_j e_{kj}-l_{ko}\right|=u_k'^{+}+u_k'^{-},\quad k=1,\ldots,K,$$
$$\sum_{j=1}^{n}\lambda_j e_{kj}-l_{ko}=u_k'^{+}-u_k'^{-},\quad k=1,\ldots,K,$$
$$u_k'^{+}u_k'^{-}=0,\quad k=1,\ldots,K,$$
$$\left|\sum_{j=1}^{n}\lambda_j f_{cj}-\theta_\alpha^{*}h_{co}\right|=g_c'^{+}+g_c'^{-},\quad c=1,\ldots,C,$$
$$\sum_{j=1}^{n}\lambda_j f_{cj}-\theta_\alpha^{*}h_{co}=g_c'^{+}-g_c'^{-},\quad c=1,\ldots,C,$$
$$g_c'^{+}g_c'^{-}=0,\quad c=1,\ldots,C,$$

Model (10) can be substituted with model (11).

$$
\begin{aligned}
Min\quad & \sum_i \pi_{io} \\
s.t.\; & \sum_{j=1}^{N}\lambda_j x_{ij}-\Phi^{-1}(\alpha)\sigma(p_i^{+}-p_i^{-})\le\theta_\alpha^{*}x_{io},\ i=1,\ldots,m,\\
& \sum_{j=1}^{N}\lambda_j a_{ij}-\theta_\alpha^{*}a_{io}=p_i^{+}-p_i^{-},\ i=1,\ldots,m,\\
& \sum_{j=1}^{N}\lambda_j y_{rj}+\Phi^{-1}(\alpha)\sigma(q_r^{+}-q_r^{-})\ge y_{ro},\quad r=1,\ldots,s,\\
& \sum_{j=1}^{N}\lambda_j d_{rj}-d_{ro}=q_r^{+}-q_r^{-},\quad r=1,\ldots,s,\\
& \sum_{j=1}^{N}\lambda_j z_{kj}+\Phi^{-1}(\alpha)\sigma(u_k'^{+}-u_k'^{-})\ge\rho_{ko},\ k=1,\ldots,K,\\
& \sum_{j=1}^{N}\lambda_j e_{kj}-l_{ko}=u_k'^{+}-u_k'^{-},\ k=1,\ldots,K,\\
& \sum_{j=1}^{N}\lambda_j b_{cj}-\Phi^{-1}(\alpha)\sigma(g_c'^{+}-g_c'^{-})\le\theta_\alpha^{*}\pi_{co},\ c=1,\ldots,C,\\
& \sum_{j=1}^{N}\lambda_j f_{cj}-\theta_\alpha^{*}h_{co}=g_c'^{+}-g_c'^{-},\ c=1,\ldots,C,\\
& p_i^{+}p_i^{-}=0,\\
& q_r^{+}q_r^{-}=0,\\
& u_k'^{+}u_k'^{-}=0,\\
& g_c'^{+}g_c'^{-}=0,\\
& \lambda_j\ge 0,\ j=1,\ldots,n,\\
& p_i^{+},p_i^{-},q_r^{+},q_r^{-},u_k'^{+},u_k'^{-},g_c'^{+},g_c'^{-}\ge 0,\forall i,r,k,c.
\end{aligned}
\tag{11}
$$

4. An Illustrative Example

It is assumed that there are 10 areas with two inputs, costs and fossil fuel consumption (FFC), one desirable output, GDP, and one undesirable output, CO2 emissions. Costs are considered as inputs that can be managed, while fossil fuel consumption is seen as a natural disposable input. Dataset is presented in Table 1.

Table 1. Dataset

Area	FFC		Costs		GDP		CO2	
	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.
1	15	4.9	68.3	9.5	213.8	19.5	3.7	1.6
2	16.1	1.7	29.7	3.85	375.2	12.85	3.9	2.1
3	7.4	4.9	21.5	5.3	490.8	5.3	1.8	1.12
4	6	3.6	44.1	3.7	172.9	8.7	1.5	1.2
5	36.2	9.1	93.3	4.2	434.5	74.2	8.3	5.62
6	3.5	6.5	95.6	6.49	534.2	26.49	2.9	0.9
7	17.1	2.1	25.3	5.53	444.4	35.53	4.5	0.7
8	13.4	3.3	86.7	6.61	534.6	36.61	4.8	0.2
9	35.3	8.1	21.1	6.12	338.1	16.12	8	0.07
10	6	3.2	51.9	2.44	110.1	16.44	1.7	0.65

In order to calculate stochastic efficiencies and the impact of increasing GDP by 2% on fossil fuel consumption, models (5) and (8) are used at different risk levels (0.01, 0.1, 0.2, and 0.5). The results are presented in Table 2, indicating that efficiency scores are non-increasing as the risk level rises. Areas 3 and 6 are determined as stochastic efficient in all risk levels under examination. Area 9 is the most stochastic inefficient area in all levels considered. Managers of this area should pay more attention to their performance.

Additionally, the average values of FFC for areas 1, 2, 5, 7 and 10 for risk levels 0.01, 0.1, 0.2, and 0.5 are displayed in Table 2. For Area 1 and the risk level 0.01, the average value of FFC from 15 should be reach 6.97 concerning the increase of GDP by 2%. Also, the new amount of FFC is obtained 7.04 for the risk levels 0.1 and 0.2. For $\alpha = 0.5$, the average value of FFC in Area 1 is obtained 4.47.

Furthermore, the costs are increased by 2% and the changes of CO2 emissions are addressed while the stochastic efficiency scores are maintained. The last 10 rows of Table 2 show the findings. For illustration, consider Area 3. For levels 0.1, 0.2 and 0.5, the amount of CO2 emission has no change while the problem is infeasible for the risk level 0.01. Also, the new average amounts of CO2 emission for Area 9 considering the different levels 0.01, 0.1, 0.2 and 0.5 are 3.78, 4.79, 5.53 and 8.03, respectively. Similarly, the results of other areas can be investigated. As can be seen, in some areas, the changes in CO2 emissions and FFC cannot be attributed to a 2% increase in managerial disposable input, cost and desirable output, GDP, due to the surrounding circumstances and constraints.

Therefore, managers should pay attention to the stochastic efficiencies and strive to improve their performance, especially in areas that are identified as stochastic inefficient across all risk levels. Also, it is important for managers to consider the impact of increasing GDP by 2% on fossil fuel consumption. They should be aware that changes in this variable may not always be directly proportional to the increase in GDP due to surrounding circumstances and constraints.

The proposed inverse DEA model with random and undesirable measures can be used to estimate natural disposable random inputs for the alterations of random desirable outputs. Moreover, it can be applied to assess undesirable outputs for the perturbation of managerial disposable random inputs. This model can help in analyzing efficiency and performance measures in situations where undesirable outputs and random factors are presented. The limitations surrounding changes in CO2 emissions and fossil fuel consumption due to the increase in costs and GDP should be paid attention. They should consider these factors when making decisions and setting performance targets.

Table 2. Results

Area	Efficiency			
	0.01	0.1	0.2	0.5
1	0.66	0.62	0.61	0.56
2	0.85	0.53	0.45	0.4
3	1	1	1	1
4	1	1	1	0.89
5	0.69	0.46	0.41	0.34
6	1	1	1	1
7	0.75	0.61	0.54	0.38
8	0.99	0.81	0.73	0.58
9	0.39	0.29	0.25	0.17
10	1	0.97	0.96	0.93
area	New FFC	New FFC	New FFC	New FFC
	0.01	0.1	0.2	0.5
1	6.97	7.04	7.04	4.47
2	7.85	8.68	10.37	IF
5	23.92	25.36	IF	10.02
7	4.22	10.72	12.31	IF
10	IF	2.75	2.75	2.05
area	New CO2	New CO2	New CO2	New CO2
	0.01	0.1	0.2	0.5
1	3.66	3.69	3.68	3.77
2	1.89	3.08	3.58	3.92
3	IF	1.81	1.81	1.81
4	1.41	1.39	1.38	1.53
5	4.44	6.48	7.23	8.47
6	IF	IF	IF	IF
7	3.42	3.37	3.54	4.52
8	2.94	3.53	3.89	4.84
9	3.78	4.79	5.53	8.03
10	1.7	1.7	1.7	1.73

IF: Infeasible

5. Conclusions

In many real-world examinations, random and undesirable measures are presented. Accordingly, the purpose of this research was two-fold. The first purpose was estimating natural disposable inputs for the change of desirable outputs in the presence of undesirable outputs and random factors while the stochastic performance is preserved. The second purpose was addressing random undesirable outputs for the perturbation of managerial disposable random inputs while maintaining efficiency. To achieve this, the study introduced a chance-constrained DEA model that considers both managerial and natural disposability, along with chance-constrained inverse DEA frameworks. These approaches were applied in an illustrative example to demonstrate their effectiveness. The results show the proposed techniques are beneficial to estimate some random measures for the changes of others under managerial and natural disposability.

The study suggests that this technique can be further developed to assess performance measures in situations involving fuzzy and stochastic variables simultaneously. Additionally, the framework can be expanded to account for cases where flexible and non-discretionary measures are included.

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