Some Results on Necessary Conditions for Two Quasidifferentiable Optimization Problems

Zun-Quan Xia ¹
Chun-Ling Song ²
Li-Wei Zhang ³

Two main results of necessary conditions of optimality for two kinds of problems, bilevel optimization and quasidifferentiable MPEC, are presented via Demyanov sum of quasidifferentials. The result that Lagrange multipliers are independent of the choices of quasidifferentials and supergradients is given.

Keywords: Nonsmooth optimization; quasidifferentiable function; bilevel optimization; MPEC; optimality conditions; Demyanov difference (sum).

1 – Introduction

We consider the following two classes of problems,

\[(P_1)\]
\[
\begin{align*}
\min & \quad \theta(x) = f(x, v_1(x), \ldots, v_m(x)) \\
\text{s.t.} & \quad v_i(x) = \max\{\varphi_i(x, y_i) : G_i(x, y_i) \leq 0, H_i(x, y_i) = 0\}, \quad i = 1, \ldots, m,
\end{align*}
\]

where \(f : R^{n+m} \to R^1\) is quasidifferentiable, \(x \in R^n, y_i \in R^{p_i}\), and

\[
\varphi_i, g_{ij}, h_{jk} \in C^2, \quad i = 1, \ldots, m, \quad j = 1, \ldots, p_i, k = p_i + 1, \ldots, q_i,
\]

\[
G_i(x, y_i) = (g_{i1}(x, y_i), \ldots, g_{ip_i}(x, y_i))^T,
\]

\[
H_i(x, y_i) = (h_{i(p_i+1)}(x, y_i), \ldots, h_{iq_i}(x, y_i))^T.
\]

¹Invited Paper
²CORA, Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, China, zqxiazhb@dlut.edu.cn.
³Department of Mathematics, School of Sciences, Foshang University, Foshang 528313, China, China, sohu@163.com.
⁴CORA, Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, China, lwzhang@dlut.edu.cn.
Some Results on Necessary Conditions

and

$$\begin{align*}
\min & \quad f(x, y) \\
\text{s. t.} & \quad z_j(x, y) \leq 0, \quad j = 1, \ldots, r, \\
& \quad y = (y_1, \ldots, y_m)^T \in S(x) = \prod_{i=1}^{m} S_i(x),
\end{align*}$$

(P2)

where $f, z_j : R^n \times R^{s_1} \times \cdots \times R^{s_m} \rightarrow R^1$, $j = 1, \ldots, r$, are quasidifferentiable, $x \in R^n, y_i \in S_i(x) \subseteq R^{s_i}$, and

$$\begin{align*}
\varphi_i, g_{ij}, h_{ik} & \in C^2, \quad i = 1, \ldots, m, j = 1, \ldots, p_i, k = p_i + 1, \ldots, q_i, \\
G_i(x, y_i) & = (g_{i1}(x, y_i), \ldots, g_{ip_i}(x, y_i))^T, \\
H_i(x, y_i) & = (h_{i(p_i+1)}(x, y_i), \ldots, h_{iq_i}(x, y_i))^T, \\
S_i(x) & = \text{arg max} \{ \varphi_i(x, y_i) | G_i(x, y_i) \leq 0, H_i(x, y_i) = 0 \}, i = 1, \ldots, m.
\end{align*}$$

A function $f : R^n \rightarrow R^1$ is said to be quasidifferentiable at $x$ in the sense of Demyanov and Rubinov (1980), if $f$ is directionally differentiable at $x \in R^n$ and there exists a pair of compact convex sets, $\partial f(x), \overline{\partial f(x)} \subseteq R^n$, such that

$$f'(x; d) = \max_{v \in \partial f(x)} \langle v, d \rangle + \min_{w \in \overline{\partial f(x)}} \langle w, d \rangle, \quad \forall d \in R^n;$$

see [2]. $\partial f(x) = [\partial f(x), \overline{\partial f(x)}]$ is called a quasidifferential of $f$ at $x$, $\partial f(x)$ and $\overline{\partial f(x)}$ are called subdifferential and superdifferential of $f$ at $x$, respectively. Elements of a subdifferential and a superdifferential are called subgradients and supergradients, respectively.

The following assumptions will be used in this paper for ensuring the quasidifferentiability of $v_i(\cdot), i = 1, \ldots, m$, at $x^0$ and the validity of optimality conditions. Define:

$$Y_i(x) = \text{Arg max} \{ \varphi_i(x, y_i) | G_i(x, y_i) \leq 0, H_i(x, y_i) = 0 \}, \quad i = 1, \ldots, m.$$

**Assumption 1.1.** (Uniform Boundedness) $Y_i(x)$ is uniformly bounded in some neighborhood of $x^0$; i.e., there exists a neighborhood $N_i$ of $x^0$ and a bounded set $T_i \subseteq R^{s_i}$ such that $Y_i(x) \subseteq T_i$, for any $x \in N_i$.

**Assumption 1.2.** (M-F Constraint Qualification) For every $y^0_i \in Y_i(x^0)$, the lower problem,

$$\begin{align*}
\max & \quad \varphi_i(x, y_i) \\
\text{s. t.} & \quad G_i(x, y_i) \leq 0, H_i(x, y_i) = 0, \quad i = 1, \ldots, m,
\end{align*}$$

satisfies M-F constraint qualification:

1. The vectors $\nabla_y g_{ij}(x^0, y^0_i), j = p_i + 1, \ldots, q_i$, are linearly independent.

2. There exists a $w_i \in R^{s_i}$ satisfying,

$$\begin{align*}
w_i^T \nabla_y g_{ij}(x^0, y^0_i) & < 0, \quad \forall j \in \{ \nu | g_{\nu}(x^0, y^0_i) = 0, \nu = 1, \ldots, p \}, \\
w_i^T \nabla_y h_{ij}(x^0, y^0_i) & = 0, \quad j = p + 1, \ldots, q.
\end{align*}$$
Assumption 1.3. (Second Order Sufficient Conditions) For any $\alpha_i \in A_i(x^0, y_i^0)$ and $d \neq 0$ satisfying the following conditions,

$$\nabla y_i g_{ij}(x^0, y_i^0)^T d = 0, \quad \forall j \in J(\alpha_i) = \{ j \mid \alpha_{ij} > 0, j = 1, \cdots, p \},$$

$$\nabla y_i h_{ij}(x^0, y_i^0)^T d = 0, \quad j = p + 1, \cdots, q,$$

one has,

$$d^T \nabla^2 y_i L_i(x^0, y_i^0, \alpha_i)d > 0,$$

where,

$$A_i(x^0, y_i^0) = \left\{ \alpha \in \mathbb{R}^p \mid \begin{array}{l}
\nabla y_i L_i(x^0, y_i^0, \alpha) = 0 \\
\alpha_j \geq 0, \\
\alpha_{ij} g_{ij}(x^0, y_i^0) = 0, \\
\alpha_{j} h_{ij}, \\
\end{array} \right\}. \quad (7)$$

$$L_i(x, y_i, \alpha_i) = \varphi_i(x, y_i) + \sum_{k=1}^{p_i} \alpha_k g_{ik} + \sum_{k=p_i+1}^{q_i} \alpha_k h_{ik}, \quad (8)$$

$$\alpha_i = (\alpha_1, \cdots, \alpha_{p_i}, \alpha_{p_i+1}, \cdots, \alpha_{q_i}). \quad (9)$$

Here, co$C$ denotes the convex hull of $C$. In the next section, necessary conditions for problem $(P_1)$, i.e., for a class of quasidifferentiable bilevel optimization, are given, and necessary conditions for problem $(P_2)$, i.e., for a class of quasidifferentiable MPEC problems, are presented in Section 3.

2 – The Case of $(P_1)$

For every $y_i^0 \in Y_i(x^0)$, the set of Lagrange multiplier vectors of lower level problem is nonempty if and only if M-F constraint qualification holds at $y_i^0$; see [6, 10]. Moreover, the following theorem holds.

Theorem 2.1. [1, 7, 8, 11, 12, 13] Suppose that the M-F constraint qualification and second order sufficient conditions hold for every $y_i^0 \in Y_i(x^0)$. Then, $v_i(\cdot)$ is (local) Lipschitzian, directionally differentiable and

$$v'_i(x^0; d) = \sup_{y_i \in Y_i(x^0)} \inf_{\alpha \in A_i(x^0, y_i)} d^T \nabla L_i(x^0, y_i, \alpha). \quad (1)$$

If for every $i$, $i = 1, \cdots, m$, $Y_i(x^0)$ is finite, i.e., $Y_i(x^0) = \{ y_i^1, \cdots, y_i^M \}$, then (1) can be written as:

$$v'_i(x^0; d) = \max\{ \langle d, e \rangle \mid e \in C_i^1 \} - \max\{ \langle d, e \rangle \mid e \in C_i^2 \}, \quad (2)$$

i.e., $v_i(\cdot)$ is quasidifferentiable at $x^0$, and $[C_i^1, -C_i^2]$ is a quasidifferential of $v_i$ at $x^0$, where,
Some Results on Necessary Conditions

\[ C_i^1 = \text{co}(\bigcup_{l=1}^{\beta_i} (\sum_{\mu \neq \lambda} B_{l,i}^\mu)), \quad C_i^2 = \sum_{l=1}^{\beta_i} B_{l,i}, \]

\[ B_i^l = -\text{co}(\bigcup_{\alpha \in A(x^0, y^l_i)} \{\nabla_x L_i(x^0, y^l_i, \alpha)\}), \quad \forall l = 1, \ldots, \beta_i, \]

\[ y_i^l \in R^{S_i}, \quad \forall j = 1, \ldots, \beta_i. \]

In the remainder of the paper, we assume that for every \( i, \ i = 1, \ldots, m, \ Y_i(x^0) \) is finite. Consider problem (\( P_1 \)). Since \( v(\cdot) \) is Lipschitzian, it is uniformly directionally differentiable. By the quasidifferential calculus of composition functions [3], one has that \( \theta \) is quasidifferentiable at \( x^0 \), and \([\partial \theta(x^0), \bar{\partial} \theta(x^0)]\) is formulated by:

\[
\partial \theta(x^0) = \{ w \mid w = (v^{(1)}, \ldots, v^{(n)}) + \sum_{l=1}^{n+m} [v^{(i)}(\lambda_l + \mu_i) - v^{(i)}(\lambda_i - v^{(i)}(\mu_l)),
\]

\[ u = (v^{(1)}, \ldots, v^{(n+m)}) \in \delta f(y^0), \lambda_i \in C_{i-n}, \mu_i \in -C_{i-n} \}, \]

\[
\bar{\partial} \theta(x^0) = \{ w \mid w = (\bar{v}^{(1)}, \ldots, \bar{v}^{(n)}) + \sum_{l=1}^{n+m} [u^{(i)}(\lambda_l + \mu_i) + u^{(i)}(\lambda_i - u^{(i)}(\mu_l)),
\]

\[ u = (u^{(1)}, \ldots, u^{(n+m)}) \in \bar{\delta} f(y^0), \lambda_i \in C_{i-n}, \mu_i \in -C_{i-n} \}, \]

\[ v' \leq v \leq v'', \ v' \leq 0, \ v'' \geq 0, \ y^0 = (x^0, v_1(x^0), \ldots, v_m(x^0))^T, \]

\[ y^{(i)} = v^{(i)} - u^{(i)}, \bar{v}^{(i)} = u^{(i)} + u^{(i)}, i = 1, \ldots, n. \]

Let \( A, B \subseteq R^n \) be convex compact. The Demyanov difference of \( A \) and \( B \), our basic operation here, is defined by:

\[ A^\perp B = \text{elco}\{\nabla \delta^*(h \mid A) - \nabla \delta^*(h \mid B) \mid h \in T\}, \]

where, \( T = \{ h \in R^n \mid \nabla \delta^* (\cdot \mid A)(h) \text{ and } \nabla \delta^* (\cdot \mid B)(h) \text{ exist} \}. \) The form of Demyanov difference, \( \partial f(x) \sim (-\partial f(x)) \), will play the main role and is also denoted by \( \partial^+ f(x) \).

**Lemma 2.1.** [4] Let \( f : R^n \rightarrow R^1 \) be quasidifferentiable. If \( x^0 \in \arg \min_{x \in R^n} f(x) \), then \( 0 \in \partial^+ f(x^0) \).

The following theorem can be obtained in terms of Lemma 2.1.

**Theorem 2.2.** Suppose Assumptions 1, 2 and 3 hold, and for \( i, \ i = 1, \ldots, m, \ Y_i(x^0) \) is finite. If \( x^0 \) is a minimizer of (\( P_1 \)), then \( 0 \in \partial^+ \theta(x^0) \).

In what follows, assume that \( \theta \) is a maximal function, i.e.,

\[
\min \theta(x) = \max \{ v_1(x), \ldots, v_m(x) \}
\]

s.t.

\[ v_i(x) = \max \{ \varphi_i(x, y_i) \mid G_i(x, y_i) \leq 0, H_i(x, y_i) = 0 \}, \quad i = 1, \ldots, m. \]

One has from the quasidifferential calculus of maximal functions that the quasidifferential of \( \theta(\cdot) \) at \( x^0 \) is given by:

\[ \partial \theta(x^0) = \text{co} \bigcup_{k \in R(x^0)} (C_k^1 + \sum_{i \in R(x) \setminus \{k\}} C_i^2), \]

\[ \bar{\partial} \theta(x^0) = -\sum_{i \in R(x^0)} C_i^2, \]

where, \( R(x^0) = \{ i \in 1 : m \mid \theta(x) = v_i(x^0) \} \).
Lemma 2.2. [3] Let $A, B \subseteq \mathbb{R}^n$ be convex compact. Then, $(A + B)^\circ - B = A$. □

Lemma 2.3. Let $A_i, B \subseteq \mathbb{R}^n$, $i = 1, \cdots, m$, be convex compact. Then, $\text{co}(\bigcup_{i=1}^m A_i)^\circ - B \subseteq \text{co} \bigcup_{i=1}^m (A_i - B)$. □

**Proof:** straightforward.

**Theorem 2.3.** Suppose Assumptions 1, 2 and 3 hold, and for $i$, $i = 1, \cdots, m$, $Y_i(x^0)$ is finite. If $x^0$ is a minimizer of (3), then there exists a finite number of $\alpha \in A_0(x^0, y_k^l) \subseteq A(x^0, y_k^l), l = 1, \cdots, \beta, k, k \in R(x^0)$ and $\lambda(\alpha, l, k) \geq 0$, such that

$$\sum_{\alpha \in A_0(x^0, y_k^l), l = 1, \cdots, \beta, k \in R(x^0)} \lambda(\alpha, l, k) \nabla_x L_k(x^0, y_k^l, \alpha) = 0.$$ 

**Proof:** One has from Theorem 2.2 that $0 \in \partial^+ \theta(x^0)$. Compute $\partial^+ \theta(x^0)$,

$$\partial^+ \theta(x^0) = \partial \theta(x^0) - (\partial \theta(x^0)) = \text{co}[\bigcup_{k \in R(x^0)} (C_k^1 + \sum_{t \in R(x^0)} C_t^2)]^\circ - (\sum_{t \in R(x^0)} C_t^2) \subseteq \text{co}\{\{C_k^1 + \sum_{t \in R(x^0)} C_t^2\}^\circ - \sum_{t \in R(x^0)} C_t^2 \mid k \in R(x^0)\} \text{ (Lemma 2.3)}$$

$$= \text{co}\{C_k^1 - C_k^2 \mid k \in R(x^0)\} \text{ (Lemma 2.2)}$$

while,

$$C_k^1 - C_k^2 = \text{co}(\bigcup_{i=1}^{\beta_k} \sum_{v \neq k} B_v^i) - \sum_{l=1}^{\beta_k} B_l^i \subseteq \text{co}\bigcup_{i=1}^{\beta_k} (\sum_{v \neq k} B_v^i - \sum_{l=1}^{\beta_k} B_l^i) \text{ (Lemma 2.3)}$$

$$= \text{co}\bigcup_{i=1}^{\beta_k} (0 - B_k^i) \text{ (Lemma 2.2)}$$

$$= -\text{co}\bigcup_{i=1}^{\beta_k} B_k^i.$$ 

As a consequence, one has:

$$0 \in \text{co}\{-\text{co}\bigcup_{i=1}^{\beta_k} B_k^i \mid k \in R(x^0)\}$$

$$= \text{co}\{\bigcup_{i=1}^{\beta_k} (\bigcup_{\alpha \in A(x^0, y_k^l)} \{\nabla_x L_k(x^0, y_k^l, \alpha)\}) \mid k \in R(x^0)\}$$

$$= \text{co}\{\nabla_x L_k(x^0, y_k^l, \alpha) \mid \alpha \in A(x^0, y_k^l), l = 1, \cdots, \beta, k \in R(x^0)\},$$

that is, there exists a finite number of $\lambda(\alpha, l, k) \geq 0$, such that

$$\sum_{\alpha \in A_0(x^0, y_k^l), l = 1, \cdots, \beta, k \in R(x^0)} \lambda(\alpha, l, k) \nabla_x L_k(x^0, y_k^l, \alpha) = 0,$$

where, $A_0(x^0, y_k^l)$ is a finite subset of $A(x^0, y_k^l)$. □
Some Results on Necessary Conditions

For \((P_1)\), if \(\varphi_i, i = 1, \cdots, m\), are strictly concave, \(g_{ij}, i = 1, \cdots, m, j = 1, \cdots, p_i\), are convex, \(h_{ik}, i = 1, \cdots, m, k = p_i + 1, \cdots, q_i\), are affine, and \(\text{Slate}\) constraint qualification holds, then (1) holds and the solution of lower level problem (1) is unique, that is, \(Y_i(x^0) = \{y_i^0\}\); see [9]. Therefore, one has:

\[
v'_i(x^0; d) = \inf_{\alpha \in A_i(x^0, y_i^0)} d^T \nabla_x L_i(x^0, y_i^0, \alpha).
\]

In other words, \(v_i\) is superdifferentiable at \(x^0\), and \(\partial v_i(x^0) = \text{co}\{\nabla_x L(x^0, y_i^0, \alpha) \mid \alpha \in A_i(x^0, y_i^0)\}\). By the quasidifferential calculus of composition functions, the quasidifferential of \(\vartheta(\cdot)\) at \(x^0\) is formulated as:

\[
\vartheta(x^0) = \{w \mid w = (v^{(1)}, \cdots, v^{(n)}) + \sum_{i=n+1}^{n+m} (v^{(i)} - v^{(i)} (\mu_i, \mu_i \in B_{i-n})
\}

\[
\partial \vartheta(x^0) = \{w \mid w = (v^{(1)}, \cdots, v^{(n)}) + \sum_{i=n+1}^{n+m} (u^{(i)} + v^{(i)} (\mu_i, \mu_i \in B_{i-n})
\}
\]

where,

\[
v' \leq v \leq v', v' \leq 0, v'' \geq 0, y^0 = (x^0, y_1(x^0), \cdots, y_m(x^0))^T,
\]

\[
v^{(i)} = v^{(i)} (\mu_i), \quad v^{(i)} = v^{(i)} + v^{(i)}
\]

\[
B_{i-n} = \text{co}\{\nabla_x L_{i-n}(x^0, y_{i-n}, \alpha) \mid \alpha \in A_{i-n}(x^0, y_{i-n})\}\}

Similarly, if \(f\) is a maximal function, then we have the following theorem.

**Theorem 2.4.** Assume that \(\varphi_i, i = 1, \cdots, m\), are strictly concave, \(g_{ij}, i = 1, \cdots, m, j = 1, \cdots, p_i\), are convex, \(h_{ik}, i = 1, \cdots, m, k = p_i + 1, \cdots, q_i\), are affine, and \(\text{Slate}\) constraint qualification holds. If \(x^0\) is a minimizer of (3), then there exists a finite number of \(\lambda(\alpha, k) \geq 0\), such that

\[
\sum_{\alpha \in A_0(x^0, y_0^0), k \in R(x^0)} \lambda(\alpha, k) \nabla_x L_k(x^0, y_k^0, \alpha) = 0,
\]

where, \(A_0(x^0, y_0^0)\) is a finite subset of \(A(x^0, y_0^0)\), and \(Y_k(x^0) = \{y_k^0\}\).

**Proof:** By Theorem 2.2 one has that \(0 \in \partial^+ \vartheta(x^0)\). We only need to compute \(\partial^+ \vartheta(x^0)\),

\[
\partial^+ \vartheta(x^0) = \partial^+ \vartheta(x^0)^\perp (-\partial^+ \vartheta(x^0))
\]

\[
= \text{co}\{\forall k \in R(x^0) (\partial v_k(x^0) - \sum_{i \in R(x^0)} \partial v_i(x^0))\}^\perp (-\sum_{k \in R(x^0)} \partial v_k(x^0))
\]

\[
= \text{co}\{\forall k \in R(x^0) (\partial v_k(x^0) - \sum_{i \in R(x^0)} \partial v_i(x^0))\}^\perp (-\sum_{k \in R(x^0)} \partial v_k(x^0)) \mid k \in R(x^0)\}
\]

\[
= \text{co}\{\partial v_k(x^0)^\perp (-\partial v_k(x^0)) \mid k \in R(x^0)\}\}
\]

\[
= \text{co}\{\nabla_x L(x^0, y_k^0, \alpha) \mid k \in R(x^0), \alpha \in A_k(x^0, y_k^0)\}\}.
\]
Hence, it follows from the definition of convex hull that there exists a finite number of 
\( \lambda(\alpha, \kappa) \geq 0 \), such that

\[
\sum_{\alpha \in A_0(x^0, y^0_k), \kappa \in R(x^0)} \lambda(\alpha, \kappa) \nabla_x L_k(x^0, y^0_k, \alpha) = 0,
\]

where, \( A_0(x^0, y^0_k) \) is a finite subset of \( A(x^0, y^0_k) \), and \( Y_k(x^0) = \{ y^0_k \} \).

\[\square\]

3 – The Case of (P₂)

Here, we consider problem (P₂), which is equivalent to the following problem,

\[
\begin{align*}
\min & \quad f(x, y) \\
\text{s. t.} & \quad z_j(x, y) \leq 0, \quad j = 1, \cdots, r, \\
& \quad v_i(x) \leq \varphi_i(x, y_i), \quad i = 1, \cdots, m, \\
& \quad G_i(x, y_i) \leq 0, \quad i = 1, \cdots, m, \\
& \quad H_i(x, y_i) = 0, \quad i = 1, \cdots, m,
\end{align*}
\]

(1)

where, \( v_i(x) = \max \{ \varphi_i(x, y_i) \mid G_i(x, y_i) \leq 0, H_i(x, y_i) = 0 \} \).

**Lemma 3.1.** Let \( A \subseteq \mathbb{R}^n, B \subseteq \mathbb{R}^m, d = (d_1, d_2) \in \mathbb{R}^n \times \mathbb{R}^m \). Then, \( G_d(A \times B) = G_{d_1}(A) \times G_{d_2}(B) \), where \( G_d(A) \) denotes the maximal face of \( A \) determined by \( d \).

**Proof:** According to the definition of maximal face one has that \((\overline{x}, \overline{y}) \in G_d(A \times B)\) if and only if

\[
\langle (\overline{x}, \overline{y}), (d_1, d_2) \rangle = \max_{(x, y) \in A \times B} \langle (x, y), (d_1, d_2) \rangle,
\]

that is,

\[
\langle \overline{x}, d_1 \rangle + \langle \overline{y}, d_2 \rangle = \max_{x \in A} \langle x, d_1 \rangle + \max_{y \in B} \langle y, d_2 \rangle.
\]

Therefore,

\[
\max_{x \in A} \langle x - \overline{x}, d_1 \rangle + \max_{y \in B} \langle y - \overline{y}, d_2 \rangle = 0. \tag{2}
\]

Since \((\overline{x}, \overline{y}) \in G_d(A \times B)\), we have \((\overline{x}, \overline{y}) \in A \times B\), that is, \( \overline{x} \in A \) and \( \overline{y} \in B \). Hence,

\[
\max_{x \in A} \langle x - \overline{x}, d_1 \rangle \geq 0
\]

and

\[
\max_{y \in B} \langle y - \overline{y}, d_2 \rangle \geq 0.
\]

Using the last two inequalities in (2), one has:

\[
\max_{x \in A} \langle x - \overline{x}, d_1 \rangle = 0
\]
and
\[
\max_{y \in B} (y - \bar{y}, d_2) = 0,
\]
that is,
\[
\langle \bar{x}, d_1 \rangle = \max_{x \in A} (x, d_1), \quad \langle \bar{y}, d_2 \rangle = \max_{y \in B} (y, d_2).
\]
This leads to \((\bar{x}, \bar{y}) \in G_{d_1}(A) \times G_{d_2}(B)\).

**Corollary 3.1.** Let \(A \subseteq R^n, B \subseteq R^m\) and \(d = (d_1, d_2) \in R^n \times R^m\). If \(\delta^*(\cdot \mid A \times B)\) is differentiable at \(d\), then \(\delta^*(\cdot \mid A)\) is differentiable at \(d_1\) and \(\delta^*(\cdot \mid B)\) is differentiable at \(d_2\). Moreover,
\[
\nabla \delta^*(d \mid A \times B) = \nabla \delta^*(d_1 \mid A) \times \nabla \delta^*(d_2 \mid B).
\]

**Lemma 3.2.** Let \(T \subseteq R^n \times R^m\) be a full measure. Then, \(P_{R^n}(T)\) and \(P_{R^m}(T)\) are full measures with respect to \(R^n\) and \(R^m\), respectively, where \(P_X(T)\) denotes the projection of \(T\) onto \(X\).

**Proof:** By contradiction, assume that \(P_{R^n}(T)\) is not a full measure subset of \(R^n\). Then, there exists \(A \subseteq R^n\), not a zero measure, and \(A \subseteq R^n \setminus P_{R^n}(T)\). Therefore, \(A \times R^m \subseteq R^n \times R^m\) is not a zero measure, which contradicts the fact that \(A \times R^m \subseteq (R^n \times R^m) \setminus T\) and \(T\) is a full measure with respect to \(R^n \times R^m\). Hence, \(P_{R^n}(T)\) is a full measure with respect to \(R^n\). In a similar way, we can prove that \(P_{R^m}(T)\) is a full measure with respect to \(R^m\).

**Lemma 3.3.** Let \(A, C \subseteq R^n, B, D \subseteq R^m\) be convex compact. Then, \((A \times B)^{-} (C \times D) \subseteq (A^{-}C) \times (B^{-}D)\).

**Proof:** Let \(T = \{d \in R^n \times R^m \mid \delta^*(\cdot \mid A \times B)\) and \(\delta^*(\cdot \mid C \times D)\) are differentiable at \(d\}\). One has from the definition of Demyanov difference, Corollary 3.1 and Lemma 3.2,
\[
(A \times B)^{-} (C \times D)
\]
\[
= \text{clco} \{ \nabla \delta^*(d_1 \mid A \times B) - \nabla \delta^*(d_2 \mid C \times D) \mid d \in T \}
\]
\[
= \text{clco} \{ \nabla \delta^*(d_1 \mid A) \times \nabla \delta^*(d_2 \mid B) - \nabla \delta^*(d_1 \mid C) \times \nabla \delta^*(d_2 \mid D) \mid d \in T \}
\]
\[
\subseteq \text{clco} \left\{ \nabla \delta^*(d_1 \mid A) \times \nabla \delta^*(d_2 \mid B) - \nabla \delta^*(d_1 \mid C) \times \nabla \delta^*(d_2 \mid D) \mid d \in P_{R^n}(T) \right\}
\]
\[
= \text{clco} \{ \nabla \delta^*(d_1 \mid A) - \nabla \delta^*(d_1 \mid C) \mid d_1 \in P_{R^n}(T) \times \}
\]
\[
\text{clco} \{ \nabla \delta^*(d_2 \mid B) - \nabla \delta^*(d_2 \mid D) \mid d_2 \in P_{R^m}(T) \}
\]
\[
= (A^{-}C) \times (B^{-}D).
\]
Theorem 3.1. Suppose Assumptions 1, 2 and 3 hold, and for \( i, i = 1, \ldots, m, Y_i(x^0) \) is finite. If \( x^0 \) is a minimizer of (P2), then there exist \( \lambda_j \geq 0, j = 0, \ldots, r, \mu_i \geq 0, i = 1, \ldots, m, \nu_{ij} \geq 0, i = 1, \ldots, m, j = 1, \ldots, p_i, \) and \( \omega_{ik}, i = 1, \ldots, m, k = p_i + 1, \ldots, q_i, \) not all zero, such that

\[
0 \in \lambda_0 \partial_f f(x^0, y^0) + \sum_{j=1}^{r} \lambda_j \partial_z z_j(x^0, y^0) + \sum_{i=1}^{m} \mu_i \partial_\alpha A(x^0, y_i^0, \alpha) \big| \alpha \in \mathcal{A}(x^0, y_i^0, l = 1, \ldots, \beta_i) - \nabla \varphi_{ix}(x^0, y_i^0) + \sum_{i=1}^{m} \sum_{j=1}^{p_i} \nu_{ij} \nabla g_i(x^0, y_i^0) + \sum_{i=1}^{m} \sum_{k=p_i+1}^{q_i} \omega_{ik} \nabla h_i(x^0, y_i^0),
\]

where, \( y^0 = (y_1^0, \ldots, y_m^0)^T. \)

**Proof:** Problem (1) is equivalent to:

\[
\begin{align*}
\min & \quad f(x, y) \\
\text{s.t.} & \quad z_j(x, y) \leq 0, \quad j = 1, \ldots, r, \\
& \quad v_i(x) - \varphi_i(x, y_i) \leq 0, \quad i = 1, \ldots, m, \\
& \quad G_i(x, y_i) \leq 0, \quad i = 1, \ldots, m, \\
& \quad H_i(x, y_i) = 0, \quad i = 1, \ldots, m.
\end{align*}
\]

(3)

If \( x^0 \) is a minimizer of (P2), then there exists \( y^0 \in \mathbb{R}^m \) such that \((x^0, y^0)\) is a minimizer of (3). Consider \( v_i \) as a function of \((x, y)\). We have from Theorem 2.1 that

\[
\frac{\partial v_i(x^0, y^0)}{\partial x^0} = C_i^1 \times 0_m,
\]

\[
\frac{\partial v_i(x^0, y^0)}{\partial y^0} = -C_i^2 \times 0_m.
\]

Similarly, we have that \( v_i - \varphi_i \) is quasidifferentiable at \((x^0, y^0)\) and

\[
\frac{\partial (v_i - \varphi_i)(x^0, y^0)}{\partial x^0} = (C_i^1 - \nabla \varphi_{ix}(x^0, y_i^0)) \times \begin{bmatrix} 0 \times \cdots \times 0 \\ - \nabla \varphi_{iy_i}(x^0, y_i^0) \times 0 \times \cdots \times 0 \end{bmatrix}.
\]

By virtue of the necessary conditions of constrained quasidifferentiable optimization due to Gao [5], there exist \( \lambda_j \geq 0, j = 0, \ldots, r, \mu_i \geq 0, i = 1, \ldots, m, \nu_{ij} \geq 0, i = 1, \ldots, m, j = 1, \ldots, p_i, \) and \( \omega_{ik}, i = 1, \ldots, m, k = p_i + 1, \ldots, q_i, \) not all zero, such that

\[
0 \in \lambda_0 \partial_f f(x^0, y^0) + \sum_{j=1}^{r} \lambda_j \partial_z z_j(x^0, y^0) + \sum_{i=1}^{m} \mu_i \partial_\alpha A(x^0, y_i^0, \alpha) \big| \alpha \in \mathcal{A}(x^0, y_i^0, l = 1, \ldots, \beta_i) - \nabla \varphi_{ix}(x^0, y_i^0) + \sum_{i=1}^{m} \sum_{j=1}^{p_i} \nu_{ij} \nabla g_i(x^0, y_i^0) + \sum_{i=1}^{m} \sum_{k=p_i+1}^{q_i} \omega_{ik} \nabla h_i(x^0, y_i^0),
\]

(4)
Computing \( (C_1^1 - \nabla \varphi_{i_k}(x_0^0, y_0^0)) - C_1^2 \), one has:

\[
(C_1^1 - \nabla \varphi_{i_k}(x_0^0, y_0^0)) - C_1^2 \\
= -\nabla \varphi_{i_k}(x_0^0, y_0^0) + (C_1^1 - C_1^2) \\
= -\nabla \varphi_{i_k}(x_0^0, y_0^0) - \co \bigcup_{i=1}^{s_1} B_i \\
= -\nabla \varphi_{i_k}(x_0^0, y_0^0) - \co \bigcup_{i=1}^{s_1} \left\{ \nabla L_i(x_0^0, y_0^0) \big| \alpha \in A(x_0^0, y_0^0) \right\} \\
= -\nabla \varphi_{i_k}(x_0^0, y_0^0) + \co \bigcup_{i=1}^{s_1} \nabla L_i(x_0^0, y_0^0) \big| \alpha \in A(x_0^0, y_0^0), l = 1, \ldots, \beta_i \}.
\]

Combining the above formula with (4), the conclusion is obtained from Lemma 3.3. □

If in (P_2), \( f, z_j, j = 1, \ldots, r \), are differentiable, then the necessary conditions given in Theorem 3.1 turns to qualities.

**Corollary 3.2.** Suppose that \( f, z_j, j = 1, \ldots, r \), are differentiable in (P_2). Assumptions 1, 2 and 3 hold, and for \( i, j = 1, \ldots, m \), \( Y_i(x_0^0) \) is finite. If \( x_0^0 \) is a minimizer of (P_2), then there exist \( \lambda_j \geq 0, j = 0, \ldots, r \), \( \mu_i \geq 0, i = 1, \ldots, m \), \( \nu_{ij} \geq 0, j = 1, \ldots, p_i \) and \( \omega_{ik}, i = 1, \ldots, m, k = p_i + 1, \ldots, q_i \), not all zero, such that

\[
0_n = \lambda_0 \nabla f(x_0^0, y_0^0) + \sum_{j=1}^{r} \lambda_j \nabla z_{j_0}(x_0^0, y_0^0) + \\
\sum_{i=1}^{m} \mu_i \bigcup_{i=1}^{s_1} \nabla L_i(x_0^0, y_0^0, \alpha) \big| \alpha \in A(x_0^0, y_0^0), l = 1, \ldots, \beta_i \} - \nabla \varphi_{i_k}(x_0^0, y_0^0) + \\
\sum_{i=1}^{m} \sum_{j=1}^{p_i} \nu_{ij} \nabla g_{ij}(x_0^0, y_0^0) + \sum_{i=1}^{m} \sum_{k=p_i + 1}^{q_i} \omega_{ik} \nabla h_{ik}(x_0^0, y_0^0),
\]

\[
0_{s_i} = \lambda_0 \nabla f_{y_i}(x_0^0, y_0^0) + \sum_{j=1}^{r} \lambda_j \nabla z_{j_{y_i}}(x_0^0, y_0^0) + \sum_{i=1}^{m} \mu_i \left( -\nabla \varphi_{i_k}(x_0^0, y_0^0) + \sum_{i=1}^{m} \sum_{j=1}^{p_i} \nu_{ij} \nabla g_{ij}(x_0^0, y_0^0) + \sum_{i=1}^{m} \sum_{k=p_i + 1}^{q_i} \omega_{ik} \nabla h_{ik}(x_0^0, y_0^0),
\]

where, \( y_0^0 = (y_{10}^0, \ldots, y_{m0}^0)^T \). □

**Acknowledgements**

This work was supported by the Foundation of Ph.D. Units of the Ministry of Education of China (20020141013), National Science Foundations of China (10471015) and (10771026).

The authors are also greatly indebted to the anonymous referee for his valuable comments.

**References**


