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A Faster Algorithm to Diagnose the Solvability of a Three-Pair Network with Common Bottleneck Links

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A three-pair network is represented by a directed acyclic graph with two node sets $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$, where each source node $s_i \in S$ generates a message X_i and each terminal node $t_i \in T$ requests the corresponding message X_i. This paper presents an $O(|E|^2)$-time algorithm for determining the solvability of three-pair networks with common bottleneck links, where $|E|$ denotes the number of links in the network. The proposed algorithm is based on a region decomposition approach and improves the computational complexity of the previous polynomial-time algorithm, which required $O(|E|^3)$ time [4].

Keywords: *Network coding, region decomposition method, three-pair networks.*

1. Introduction

Network information flow (NIF) studies how information moves through a communication network and how this movement can be optimized [2]. It studies how information can be transmitted through a network and what the maximum achievable rates are. Traditionally, it assumed that intermediate nodes simply store and forward packets. Network Coding extends the NIF model [12]. Instead of only forwarding packets, intermediate nodes are allowed to encode (combine) multiple packets before transmitting them. This additional capability can increase throughput, improve robustness, and reduce the number of transmissions in some network scenarios.

A communication network $G = (V, E, S, T)$ consists of a finite directed acyclic graph $G = (V, E)$, where (V) is the set of nodes, (E) is the set of directed links (edges), $S \subset V$ is the set of source nodes and $T \subset V$ is the set of sink nodes. A three-pair network is a communication network with source node set $S = \{s_1, s_2, s_3\}$ and sink node set $T = \{t_1, t_2, t_3\}$, where each source $s_i \in S$ generates a message $X_i \in F$, and each sink node $t_i \in T$ requests the corresponding message X_i. Here, F denotes a finite field. For simplicity, each link $e \in E$ is assumed to be delay-free and capable of transmitting one symbol per network use. We further assume that $s_i \neq s_j$ and $t_i \neq t_j$, for each $i \neq j$. A three-pair network is solvable if each source s_i can send a unit rate of information flow to t_i, for each $i \in \{1, 2, 3\}$.

A sufficient and necessary condition for diagnosing the solvability of three-pair networks with common bottleneck links is presented in [4]. For each two edges a and b in E, the method of [4] has an iteration. In each iteration, the method determines whether the set of two edges $\{a, b\}$ is a cut. Using the search algorithm [3], each iteration runs in $O(|E|)$-time. Since there are at most $|E|^2$ pairs

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of $\{a, b\}$, the running time of [4] is $O(|E|^3)$ (note that we show that bottleneck links are computed in $O(|E|^2)$-time).

The region decomposition method [14, 15, 16] has been found to be efficient for analyzing network structure, which was very successful in the $3s/nt$ sum networks (a network with three sources and n terminals such that each source s_i generates a message $X_i \in F$ and each terminal wants to obtain the sum $\sum_{i=1}^{3} X_i$) [10, 16], two-unicast networks with rate (1,2) [14] and two-multicast networks [15,17]. Song et al. [15] defined a unique graph, which is called *the basic region graph*. A network coding problem on graph G can be converted to a coding problem on the basic region graph, which has a much simpler topological structure than graph G.

A new sufficient and necessary condition for the solvability of single rate two-pair networks is presented by [7] using the region decomposition method. The method presents an $O(|E|)$-time algorithm to find a bottleneck link in the single rate two-pair networks. Xu et al. [19] focus on networks with two pairs of sources and sinks and derive bounds on and exact values of two functions relevant to encoding complexity. In [8], an $O(|E|)$-time version of Wang and Shroff's characterization [18] is presented, which diagnoses the solvability of networks with two unit-rate multicast sessions. The method presented in [8] is the region decomposition method. Recently, [9] proposes a new sufficient and necessary condition to determine the solvability of single rate n-pair networks with common bottleneck links and presents an algorithm to the single rate n-pair networks, which runs in $O(|V||E|^2)$-time.

This paper presents an $O(|E|^2)$ algorithm to diagnose the solvability of three-pair networks using the region decomposition method and the basic region graph. The sufficient and necessary condition of [4] has seven parts. The algorithm presents a region explanation for each part in the basic region graph. Our paper consists of four sections in addition to the Introduction section. Section 2 briefly introduce some definitions and notations about three-pair networks and the region decomposition of a graph. Some properties of the region decomposition for a three-pair network are presented in Section 3. Section 4 describes the sufficient and necessary conditions of [4] in the basic region graph, and gives an algorithm to diagnose the solvability of three-pair networks with common bottleneck links. Finally, Section 5 concludes the paper.

2. Preliminaries

Consider a communication network $G = (V, E, S, T)$. For any link $e = (u, v) \in E$, node $u \in V$ is called the tail of e and node $v \in V$ is called the head of e. Tail and head of $e = (u, v)$ are denoted by $u = tail(e)$ and $v = head(e)$, respectively. Moreover, we call e an incoming link of v and an outgoing link of u. For two links $e, e' \in E$, we call e an incoming link of e' (or e' an outgoing link of e) if $tail(e') = head(e)$. For each $e \in E$, the set of incoming links of e *is* denoted by $In(e)$. A $u - v$ path Q is a series of ordered edges $e_1, e_2, \ldots, e_n$ such that $u = tail(e_1)$, $v = head(e_n)$ and $head(e_i) = tail(e_{i+1})$, for $i = 1, 2, \cdots, n - 1$. Each link $e \in E$ has a nonnegative capacity $c(e)$. We assume $c(e) = 1$ for each link $e \in E$.

A *point-to-point network* is a communication network with source set $S = \{s\}$ and sink set $T = \{t\}$. Let $G = (V, E, \{s\}, \{t\})$ be a point-to-point network and $V = W \cup \overline{W}$ be a vertex partition of it such that $s \in W$ and $t \in \overline{W}$. An $s - t$ *cut* is a collection of all the edges from W to $\overline{W}$. For an $s - t$ cut C, the capacity of C is defined as $\sum_{e \in C} c(e)$. For all $s - t$ cuts, the minimum cut capacity is the minimum of the cut capacities. A *minimum cut* is a cut that achieves this minimum capacity.

Consider a three-pair network $G = (V, E, S = \{s_1, s_2, s_3\}, T = \{t_1, t_2, t_3\})$. We assume that each source s_i has an imaginary incoming link, called X_i source link $s(i)$, and each terminal t_j has an imaginary outgoing link, called terminal link $t(j)$. Note that the source links have no tail and the terminal links have no head. For the sake of convenience, if $e \in E$ is not a source link (resp., terminal link), we call e a non-source link (resp. non-terminal link). Also, we assume that each non-source non-terminal link e of G is on a path from some source to some terminal, otherwise, link e is removed from G. We always suppose there exists at least one path from s_i to t_i, for each $i = 1, 2, 3$, otherwise, the given three-pair network is unsolvable.

2.1. Solvability of the three-pair networks

This section briefly explains the necessary and sufficient condition in [4] for diagnosing the solvability of three-pair networks.

Definition 2.1 *[5]. For a point-to-point network $G = (V, E, \{s\}, \{t\})$, the A-set of G is defined as the union of all its minimum cuts.* For a communication network $G = (V, E, S, T)$, there are $|S| \times |T|$ point-to-point networks. For a given $s_i \in S$ and $t_j \in T$, there is a point-to-point network $G_{i,j} = (V, E, \{s_i\}, \{t_j\})$.

Definition 2.2 *[5]. For a point to point network $G_{i,j} = (V, E, \{s_i\}, \{t_j\})$, the $A_{i,j}$-set of $G_{i,j}$ is defined as the union of all its $s_i - t_j$ minimum cuts and denoted by $A_{i,j}$.*

Definition 2.3 *[4]. For a three-pair network $G = (V, E, S = \{s_1, s_2, s_3\}, T = \{t_1, t_2, t_3\})$, the bottleneck links are defined as follows.*

$$A(1,2,3) = A_{1,1} \cap A_{2,2} \cap A_{3,3}.$$

In this paper, three-pair networks with common bottleneck links are considered which implies $A(1,2,3) \neq \emptyset$ (definitions of bottleneck links for the case of two sources and two sinks presented in [3]).

Definition 2.4 *[4]. For each $i, j \in \{1,2,3\}$, set $A_{i,j}^{(2)}$ is defined as follows.*

$$A_{i,j}^{(2)} = \{\{a,b\}: \{a,b\}\ is\ an\ s_i - t_j\ cut\ of\ G_{i,j}\ and\ a,b \notin A_{i,j}\},$$

where a and b are two edges of E.

The next theorem presents the necessary and sufficient conditions for diagnosing the solvability of three-pair networks.

Theorem 2.1 *[4]. Three-pair Network G is unsolvable if and only if there exist distinct $i, j, k \in \{1,2,3\}$ and two edges $a, b \in E$ such that one of the following statements holds:*
(1) $A_{i,j} \cap A(1,2,3) \neq \emptyset$.
(2) $\exists a \in A_{i,i} \cap A_{j,j} \cap A_{i,k}$ *and* $b \in A_{k,k}$ *such that* $\{a,b\} \in A_{j,i}^{(2)} \cap A_{j,k}^{(2)}$.
(3) $\exists a \in A_{i,i} \cap A_{k,k} \cap A_{j,k}$ *and* $b \in A_{j,j}$ *such that* $\{a,b\} \in A_{j,i}^{(2)} \cap A_{k,i}^{(2)}$.
(4) $\exists a \in A_{i,i} \cap A_{j,j} \cap A_{i,k}$ *and* $b \in A_{k,k} \cap A_{j,i}$ *such that* $\{a,b\} \in A_{j,k}^{(2)}$.
(5) $\exists a \in A_{j,j} \cap A_{i,k}$ *and* $b \in A_{i,i} \cap A_{k,k} \cap A_{j,i}$ *such that* $\{a,b\} \in A_{j,k}^{(2)}$.
(6) $\exists a \in A_{i,i} \cap A_{i,j}$ *and* $b \in A_{j,j}$ *such that* $\{a,b\} \in A_{k,i}^{(2)} \cap A_{k,j}^{(2)} \cap A_{k,k}^{(2)}$.

(7) $\exists a \in A_{j,j} \cap A_{k,j}$ *and* $b \in A_{k,k}$ *such that* $\{a,b\} \in A_{i,i}^{(2)} \cap A_{j,i}^{(2)} \cap A_{k,i}^{(2)}$.

In the following, there are examples of three-pair networks.

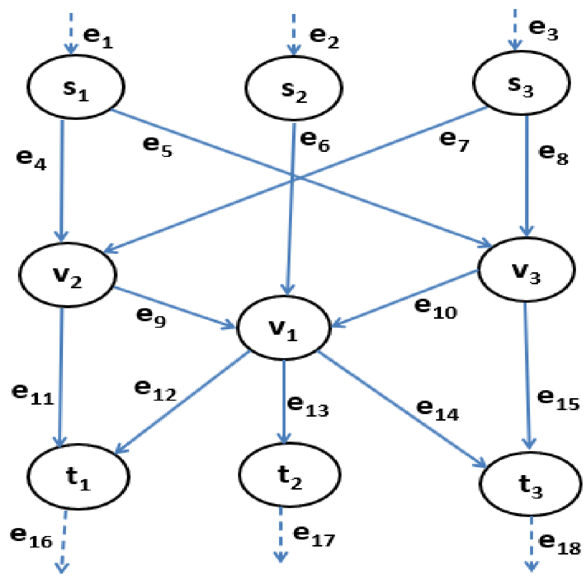

Figure 1. An example of a three-pair network with $A(1,2,3) = \emptyset$.

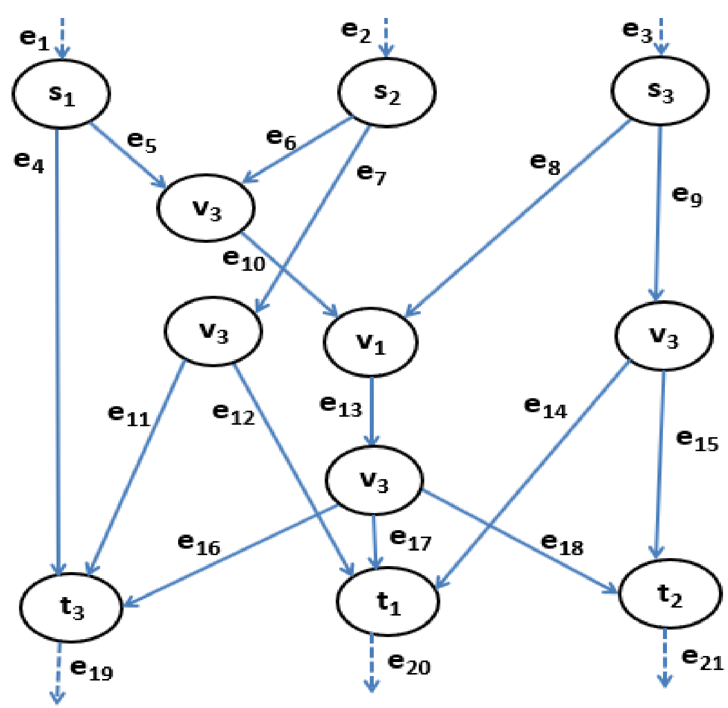

Figure 2. An example of the three-pair network G_1 with $A(1,2,3) = \{e_{13}\} \neq \emptyset$ and $A_{1,2} \cap A(1,2,3) = \{e_{13}\} \neq \emptyset$. Thus, this network satisfies Condition (1) of Theorem 2.1 and is unsolvable.

2.2. The region decomposition

In this section, we briefly introduce some definitions and notations about the region decomposition of a given graph.

Definition 2.5. *[16]. Let R be a non-empty subset of E. It is called a region of G if there exists an edge $e_l \in R$ such that for any $e \in R \setminus \{e_l\}$, R contains an incoming link of e. If E is partitioned into mutually disjoint regions, say $R_1, R_2, \cdots, R_N$, then the set $D = \{R_1, R_2, \cdots, R_N\}$ is a region decomposition of G.*

The edge e_l in Definition 2.5 is called the leader of R and is denoted as $e_l = \mathrm{lead(R)}$. A region R is called the X_i source region (or a source region for short) if lead(R) is the X_i source link, also R is called an X_j terminal region if R contains a terminal link X_j. The X_i source region and X_j terminal region are denoted by S_i and T_j , respectively. If R is neither a source region nor a terminal region, then R is called a coding region. If R is not a source region, we call R a non-source region.

Example 2.1. Consider the network G_1 in Fig. 2. We can be verified that three links e_1, e_2 and e_3 are the X_1, X_2, X_3 source links and three links e_{19}, e_{20}, e_{21} are the X_3, X_1, X_2 terminal links. Also, by Definition 2.5, $R = \{e_7, e_{11}\}$ is a region of G_1 with $\mathrm{lead(R)} = e_7$. Moreover, the subset $\{e_{11}, e_{12}\}$ is not a region because none of the incoming links e_{11} and e_{12} belong to this subset. The region decomposition D of G_1 is presented as $D = \{S_1, S_2, S_3, R_1, T_1, T_2, T_3\}$ where $S_1 = \{e_1, e_4, e_5\}$,$S_2 = \{e_2, e_6, e_7, e_{11}, e_{12}\}$,$S_3 = \{e_3, e_8, e_9, e_{14}, e_{15}\}$, $T_1 = \{e_{20}\}$, $T_2 = \{e_{21}\}$, $R_1 = \{e_{10}, e_{13}, e_{16}, e_{17}, e_{18}\}$ and $T_3 = \{e_{19}\}$.

Definition 2.6. *[16]. Let D be a region decomposition of G. The region graph of D is a directed, simple graph with vertex set D and edge set $\mathcal{E}_D$, where $\mathcal{E}_D$ is the set of all ordered pairs (R', R) such that R' contains an incoming link of $lead(R)$.*

We use $RG(D)$ to denote the region graph of D, i.e., $RG(D) = (D, \mathcal{E}_D)$. If (R', R) is an edge of $RG(D)$, we call R' a parent of R (R a child of R'). For $R \in D$, we use $In(R)$ to denote the set of parents of R in $RG(D)$. In general, a graph may have many region decompositions, so a graph may have many region graphs. For $R, R' \in D$, a path in $RG(D)$ from R' to R ($R' - R$ path) is a sequence of regions $\{R_0 = R', R_1, \cdots, R_k = R\}$ such that R_{i-1} is a parent of R_i for each $i \in \{1, \cdots, k\}$. We denote $R' \to R$ if there is a path from R' to R, and $R' \nrightarrow R$ otherwise. In particular, we have $R \to R$ for all $R \in D$. An $R' - R$ path Q is also denoted by $Q_{R' \to R}$.

Consider path Q in $RG(D)$, region $R \in D$ and subset D' of D. Since each path in $RG(D)$ is a sequence of regions, $R \in Q$ and $Q \cap D'$ are well defined. Let $Q = R_1, R_2, \ldots, R_k$ and $Q' = R'_1$, $R'_2, \ldots, R'_k$ be two paths such that $R_k = R'_1$, then path $Q \cup Q'$ can be defined as the sequence of regions $R_1, R_2, \ldots, R_k, R'_2, \ldots, R'_k$. For path Q in $RG(D)$ and subset D' of D, $Q \cap D'$contains regions that Q passes through them and are belong to D'.

By Definition 2.6, the region graph $RG(D)$ of the region decomposition D in Example 2.1 is depicted as follows:

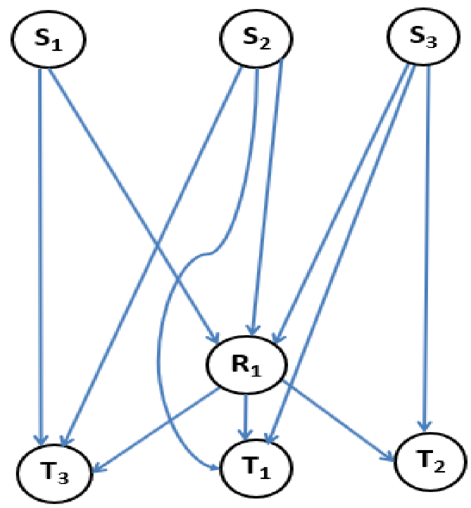

Figure 3. The region graph $RG(D)$ where D is given in Example 2.1.

Definition 2.7. *Let D^{**} be a region decomposition of G. D^{**} is called a basic region decomposition of G if*
(1) For any $R \in D^{**}$ and any $e \in R\backslash\{lead(R)\}$, $In(e) \subseteq R$.
(2) Each non-source region R in D^{**} has at least two parents in $RG(D^{**})$.

Example 2.2. For the network G_1 in Fig. 2, let $S_1 = \{e_1, e_4, e_5\}$, $S_2 = \{e_2, e_6, e_7, e_{11}, e_{12}\}$, $S_3 = \{e_3, e_8, e_9, e_{14}, e_{15}\}$, $R_1 = \{e_{10}\}$, $R_2 = \{e_{13}, e_{16}, e_{17}, e_{18}\}$, $T_1 = \{e_{20}\}$, $T_2 = \{e_{21}\}$ and $T_3 = \{e_{19}\}$. Then, $D^{**} = \{S_1, S_2, S_3, R_1, R_2, T_1, T_2, T_3\}$ is the basic region decomposition of G_1. Also, the basic region graph $RG(D^{**})$ is presented in Fig. 4.

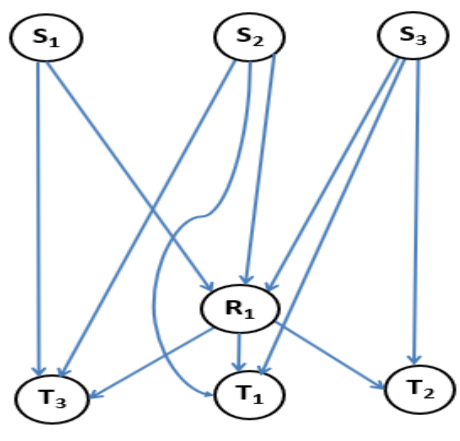

Figure 4. The basic region graph $RG(D^{**})$ where D^{**} is given in Example 2.2.

Notations D^{**} and $RG(D^{**})$ are used to denote the basic region decomposition and basic region graph of G, respectively. Network G is acyclic and finite, which means $RG(D^{**})$ is acyclic and finite. Network G is solvable if and only if $RG(D^{**})$ is feasible.

Definition 2.8. *[14]. Let θ be a subset of D^{**}. The super region $reg(\theta)$ is a subset of D^{**} which is defined recursively as follows:*
(1) $\theta \subseteq reg(\theta)$.
(2) For all $R \in D^{**}$, if $In(R) \subseteq reg(\theta)$, then $R \in reg(\theta)$.

Define $reg^0(\theta) = reg(\theta)\backslash\theta$. Moreover, if $\theta = \{R_1, \dots, R_k\}$, then denote $reg(\theta) = reg(R_1, \dots, R_k)$ and $reg^0(\theta) = reg^0(R_1, \dots, R_k)$.

Example 2.3. Consider the network G_1 and the basic region graph $RG(D^{**})$ of it in Fig. 4. Let $\theta = \{S_1, S_2\}$, then $reg(\theta) = \{S_1, S_2, R_1\}$ and $reg^0(\theta) = \{R_1\}$.

Lemma 2.1. *[14]. Suppose θ and θ' are two subsets of D^{**}. The following hold.*
(1) If $R \in reg(\theta)\backslash reg(\theta')$, then there exists a region $R' \in \theta\backslash\theta'$ and a path $Q_{R'\to R} \subseteq reg(\theta)\backslash reg(\theta')$.
(2) Suppose $\{R, R', R''\} \subseteq D^{**}$ and $R \in reg^0(R', R'')$. There exists a path
$$Q_{R''\to R} \subseteq reg(R', R'')\backslash\{R'\}.$$

The next lemma is a result of the last definitions.

Lemma 2.2. *Suppose that R and R' are two different non-source regions of D^{**} and θ is a subset of regions of D^{**}. Then*
(1) For $i \in \{1, 2, \dots, k\}$, if each $S_i - R$ path intersects set θ, then $R \in reg(\theta)$, where k is the number of sources.
(2) If $R' \to R$, then there exists at least one path from $S_i \in \{S_1, S_2, \dots, S_k\}$ to R that does not pass through R'.

Proof:
(1) For the sake of contradiction, suppose that $R \notin reg(\theta)$. Hence, region R has a parent R_1 where $R_1 \notin reg(\theta)$. Also, R_1 has a parent R_2 such that $R_2 \notin reg(\theta)$. And so on. Since $RG(D^{**})$ is finite, we can eventually find a region $R_l = S_i$, for $i \in \{1, 2, \dots, k\}$, such that R_l is a parent of R_{l-1} and $R_l \notin reg(\theta)$. Thus, path $R_l - R_{l-1} - \cdots - R_1 - R$ is an $S_i - R$ path which does not intersect θ, a contradiction (see Fig. 5).
(2) Since R is a non-source region, it has at least two parents. Therefore, it has a parent $R'' \neq R'$. Also, R'' has at least two parents in $RG(D^{**})$,
so, there exists a region R_1 such that $R_1 \neq R'$ and R_1 is a parent of R''. Since $RG(D^{**})$ is finite, we can finally find a path $R_l - R_{l-1} - \cdots - R_1 - R'' - R$, such that $R_l \in \{S_1, S_2, \dots, S_k\}$ and $R_i \neq R'$ for $i = 1, \dots, l$.

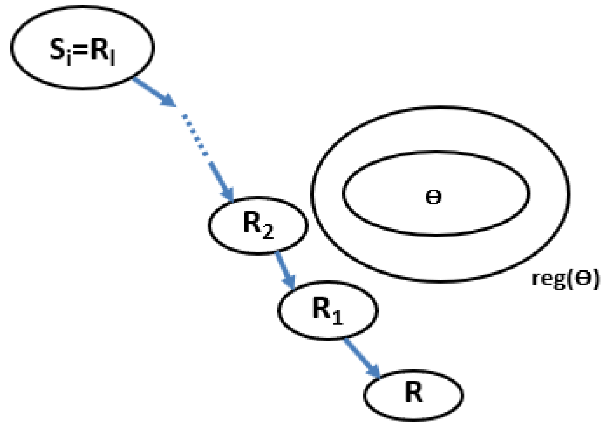

Figure 5. An illustration of the proof of Part (1) of Lemma 2.2.

3. Properties of the region decomposition for a three-pair network

This section establishes some properties of the region decomposition for a three-pair network.

Lemma 3.1. *For* $i, j \in \{1,2,3\}$ *and region* R *of* $RG(D^{**})$*,*

(1) $lead(R) \in A_{i.j}$ *if each* $S_i - T_j$ *path passes through* R*.*

(2) $\{lead(R_1), lead(R_2)\}$ *is an* $s_i - t_j$ *cut in* G *if each* $S_i - T_j$ *path passes through* R_1 *or* R_2 *in* $RG(D^{**})$*.*

Proof:

The claims follow directly from Definitions 2.2 and 2.5.

Lemma 3.2 *For* $j \in \{1,2,3\}$*, let* R*,* R' *and* R'' *be three distinct non-source regions of* $RG(D^{**})$ *such that* $R \in reg^0(R', S_j)$ *and* $R'' \in reg(R, S_j)$*. Then*

(1) For $i \in \{1,2,3\} \backslash \{j\}$*, each* $S_i - R$ *path* Q *passes through* R'*.*

(2) For $i \in \{1,2,3\} \backslash \{j\}$*, each* $S_i - R''$ *path* Q' *passes through* R*.*

(3) For $i \in \{1,2,3\} \backslash \{j\}$*, each* $S_i - R''$ *path* Q'' *passes through* R'*.*

Proof:

(1) For the sake of contradiction, suppose that Q is an $S_i - R$ path such that $R' \notin Q$, for $i \in \{1,2,3\} \backslash \{j\}$. Each source region has no incoming link, so $R \notin reg^0(R', S_j)$, which is a contradiction.

(2) The claim is proved in a similar way to (1).

(3) By Parts (1) and (2), the claim is concluded.

The next lemma concludes that there exists a unique region in the region decomposition consisting of all bottleneck links, which plays very important role in other results of this paper.

Lemma 3.3. *(1) There exists a region R^* in D^{**} such that $A(1,2,3) \subseteq R^*$.*

(2) $lead(R^*) \in A(1,2,3)$.

Proof:

(1) For the sake of contradiction, suppose that bottleneck links belong to different regions in D^{**}. Then there exist two different regions R and R' such that $A(1,2,3) \cap R \neq \emptyset$ and $A(1,2,3) \cap R' \neq \emptyset$. We have the following cases:

Case 1: There exists a path between R and R'.

Without loss of generality, we assume $R \rightarrow R'$, which means $R' \nrightarrow R$ (since the basic region graph $RG(D^{**})$ is acyclic). Region R' has a parent R'_1 such that $R'_1 \neq R$. Also, R'_1 has at least two parents in $RG(D^{**})$, so, there exists a region R'_2 such that $R'_2 \neq R$ and R'_2 is a parent of R'_1. Since $RG(D^{**})$ is finite, we can finally find a path $Q_1: R'_k - \cdots - R'_2 - R'_1 - R'$, such that $R'_k = S_i$, where $i \in \{1,2,3\}$ and $R'_l \neq R$, for $l = 1, \dots, k$. Region R' contains at least one bottleneck link, so, there exists a $R' - T_i$ path Q_2. By $R' \nrightarrow R$, we conclude that path Q_2 does not pass through R. Hence $Q_1 \cup Q_2$ is an $S_i - T_i$ path, which does not pass through R. This is a contradiction, because region R contains at least one bottleneck link.

Case 2: There exists no path between R and R'.

Region R has at least one bottleneck link, so, there exists a path Q_i from S_i to T_i, which passes through R. Path Q_i does not pass through R', because there is no path between R and R', which is a contradiction.

(2) Each incoming link of R^* must pass through $lead(R^*)$, hence by Part (1), we get $lead(R^*) \in A(1,2,3)$.

We call the region R^* as the bottleneck region of the basic region decomposition. For example, region R_2 in Fig. 4 is a bottleneck region. By Fig. 2, we have $lead(R_2) = lead(R^*) = e_{13} \in A(1,2,3)$ and $A(1,2,3) \subseteq R^*$. By Lemma 3.3, each $S_i - T_i$ path passes through R^*. In a similar way to [16], we define the sets Ω_j and Λ_j as follows.

Definition 3.1. *For each $j \in \{1,2,3\}$,*

(1) Ω_j is the set of all $R \in D^{**}$ such that $R \rightarrow T_j$ and $R \nrightarrow T_{j'}$ for each $j' \neq j$.

(2) We define

$$\Pi \triangleq D^{**} \setminus \cup_{j=1}^{3} \Omega_j.$$

(3) Also Λ_j is defined as the set of all $R' \in \Pi$ such that R' has a child $R \in \Omega_j$.

Example 3.1. For the basic region graph RG(D^{**}) in Fig. 4, Definition 3.1 gives $\Omega_1 = \{T_1\}$, $\Omega_2 = \{T_2\}$, $\Omega_3 = \{T_3\}$, $\Pi = \{S_1, S_2, S_3, R_1, R_2\}$, $\Lambda_1 = \{S_2, S_3, R_2\}$, $\Lambda_2 = \{S_3, R_2\}$ and $\Lambda_3 = \{S_1, S_2, R_2\}$.

The following remark follows directly from the definitions.

Remark 3.1. *If network G does not satisfy Condition (1) of Theorem 2.1, then*

(a) $T_j \in \Omega_j$ *, for each* $j \in \{1, 2, 3\}$.

(b) $R^* \notin \Omega_j$ *, for each* $j \in \{1, 2, 3\}$.

(c) $S_i \to T_j$ *, for each* $i, j \in \{1, 2, 3\}$. *Also,* $S_i \notin \Omega_j$ *, for each* $i, j \in \{1, 2, 3\}$.

(d) $\Lambda_j \cap \Omega_i = \emptyset$*, for each* $i, j \in \{1, 2, 3\}$.

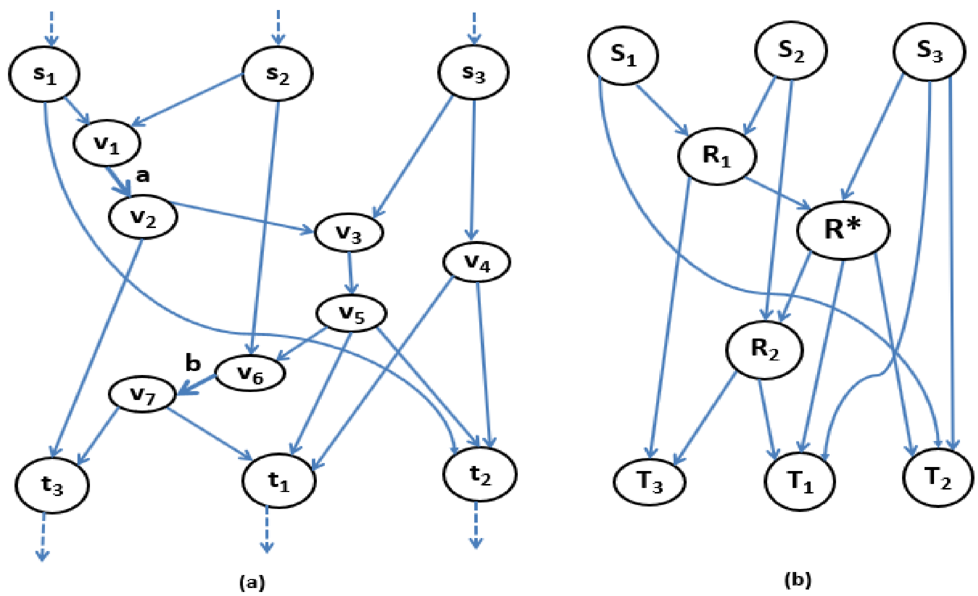

Figure 6. (a) An illustration of the three-pair network G_2 that does not satisfy Condition (1) of Theorem 2.1. This network satisfies Condition (2) of Theorem 2.1. (b) The basic region graph of G_2.

The next two lemmas present some properties of regions when network G does not satisfy Condition (1) of Theorem 2.1. These lemmas play a very important role in the next sections. If network G does not satisfy Condition (1) of Theorem 2.1, then $A_{i,j} \cap A(1,2,3) = \emptyset$ for each distinct $i, j, k \in \{1, 2, 3\}$. For example, Fig. 6 shows a three-pair network G_2 and its basic region graph. Network G_2, in Part (a) of Fig. 6, does not satisfy Condition (1) of Theorem 2.1 because $A(1,2,3) = \{v_3, v_5\}$ and $A_{i,j} \cap A(1,2,3) = \emptyset$, for each distinct $i, j, k \in \{1, 2, 3\}$. Moreover, G_2 satisfies Condition (2) of Theorem 2.1 because there are two edges a and b (depicted in G_2) such that $a \in A_{1,1} \cap A_{2,2} \cap A_{1,3}$, $b \in A_{3,3}$ and $\{a, b\} \in A_{2,1}^{(2)} \cap A_{2,3}^{(2)}$.

Lemma 3.4. *Suppose that network G does not satisfy Condition (1) of Theorem 2.1 and let R and* R' *be two non-source regions of its basic region decomposition. Then,* *(1) Suppose that* $R^* \in reg^0(R, S_j)$*, for some* $j \in \{1,2,3\}$*. If, for* $i \in \{1,2,3\}\backslash\{j\}$*, path* Q *is an* $S_i - T_j$ *path that does not pass through* R*, then* Q *does not pass through* R^**.*

(2) For each $j \in \{1,2,3\}$*, if* $R^* \neq R' \in \Lambda_j$ *, then, there exists a* $R' - T_j$ *path* Q'_j *that passes through* Ω_j *, and does not pass through* R^**.*

(3) For $j \in \{1,2,3\}$*, if* $R^* \neq R' \in \Lambda_j$ *and* Q'_j *is an* $S_j - R'$ *path, then* Q'_j *passes through* R^**.*

(4) For each $i, j \in \{1,2,3\}$*, if* Q *is an* $S_i - T_j$ *path, then* $Q \cap \Lambda_j \neq \emptyset$*.*

Proof:

(1) For the sake of contradiction, suppose that $R^* \in Q$. Path Q is an $S_i - T_j$ path that does not pass through R, so, the subpath of Q from S_i to R^* is an $S_i - R^*$ path that does not pass through R, which is contradiction.

(2) By $R' \in \Lambda_j$, region R' has a child $R_1 \in \Omega_j$. Moreover, by $R_1 \in \Omega_j$, there exists a $R_1 - T_j$ path Q_j such that each region of Q_j belongs to Ω_j . Thus, path $Q'_j = (R', R_1) \cup Qj$ is a $R' - T_j$ path that passes only through regions of Ω_j . Also, by Remark 3.1(b), path Q'_j does not pass through R^*.

(3) For the sake of contradiction, suppose that Q'_j is an $S_j - R'$ path that does not pass through R^***.** By $R' \in \Lambda_j$ and the last part of this lemma, there exists a $R' - T_j$ path Q_j that does not pass through R^*. Therefore $Q_j \cup Q'_j$ is an $S_j - T_j$ path that does not pass through R^*, which is a contradiction.

(4) Let Q be an $S_i - T_j$ path. By Remark 3.1(a), we get $T_j \in \Omega_j$, which means $Q \cap \Omega_j \neq \emptyset$. Let R be the first region in Q such that $R \in \Omega_j$ (note, R can be T_j). The parent of R in path Q belongs to Λ_j (note, by Remark 3.1(c), there exists at least one region in path Q such that it belongs to Λ_j).

Example 3.2. Consider the depicted network in Part (b) of Fig. 6. We see that $\mathrm{R}^* \in \mathrm{reg}^0(\mathrm{R}_1, \mathrm{S}_3)$. Moreover, Q: $\mathrm{S}_2, \mathrm{R}_2, \mathrm{T}_3$ is an $\mathrm{S}_2 - \mathrm{T}_3$ path that does not pass through both of the regions R_1 and R^*, which explains Part (1) of Lemma 3.4. On the other hand, by Definition 3.1, we have $\Lambda_3 = \{\mathrm{R}_1, \mathrm{R}_2\}$. Also, there is an $\mathrm{R}_2 - \mathrm{T}_3$ path that does not pass through the bottleneck region R^*.

Lemma 3.5. *Let* G *be a three-pair network with common bottleneck links that does not satisfy Condition (1) of Theorem 2.1 and* $RG(D^{**})$ *be the basic region graph of* G*. Then* $|\Lambda_j| \geq 2$*, for* $j =$ 1,2,3.

Proof : By $A(1,2,3) \neq \emptyset$, there exists an $S_i - T_j$ path Q, for each $i, j \in \{1,2,3\}$. Then, by Lemma 3.4(4), we have $Q \cap \Lambda_j \neq \emptyset$, which means $|\Lambda_j| \neq 0$. Let $R_1 \in Q \cap \Lambda_j$. By Lemma 2.2(2), there exists at least one path Q' from a region $S_i \in \{S_1, S_2, S_3\}$ to T_j that does not pass through R_1. By Lemma 3.4(4), we have $Q' \cap \Lambda_j \neq \emptyset$. Hence, there exists a region $R_2 \in Q' \cap \Lambda_j$ such that $R_1 \neq R_2$, because path Q' does not pass through R_1. Therefore, by $\{R_1, R_2\} \subseteq \Lambda_j$, we get $|\Lambda_j| \geq 2$.

3.1. Properties of unsolvable three-pair networks

By Theorem 2.1, in some unsolvable three-pair networks that do not satisfy Condition (1) of Theorem 2.1, there are edges a and b such that $a \in A_{i,i} \cap A_{j,j} \cap A_{i,k}$, $b \in A_{k,k}$ and $\{a,b\} \in A_{j,k}^{(2)}$, for distinct $i,j,k \in \{1,2,3\}$ (see Part (a) of Fig. 6). This section presents some properties of edges a and b. These results are used in the next sections.

Lemma 3.6. *Suppose that $A_{i,j} \cap A(1,2,3) = \emptyset$, for each $i,j \in \{1,2,3\}$ with $i \neq j$. Let a and b be two edges of E such that $a \in A_{i,i} \cap A_{j,j} \cap A_{i,k}, b \in A_{k,k}$ and $\{a,b\} \in A_{j,k}^{(2)}$ for distinct $i,j,k \in \{1,2,3\}$, then*

(1) $a \notin A(1,2,3)$.

(2) $a \notin A_{k,k}$.

(3) $a \notin R^*$.

(4) Let R_1 be the region in D^{**} such that $a \in R_1$,then $R^* \nrightarrow R_1$ and $R_1 \rightarrow R^*$.

(5) $b \notin R^*$.

(6) $S_k \nrightarrow R_1$.

(7) Let R_2 be the region in D^{**} such that $b \in R_2$, then $R_2 \nrightarrow R^*$ and $R^* \rightarrow R_2$.

(8) $R_1 \neq R_2$.

(9) $R_1 \rightarrow R_2$ and $R_2 \nrightarrow R_1$.

Proof:

(1) For the sake of contradiction, suppose that $a \in A(1,2,3)$. By $a \in A_{i,i} \cap A_{j,j} \cap A_{i,k}$, we get $A(1,2,3) \cap A_{i,k} \neq \emptyset$, which is contradiction.

(2) By $a \in A_{i,i} \cap A_{j,j} \cap A_{i,k}$, if $a \in A_{k,k}$, then $a \in A(1,2,3)$, which is a contradiction with Part (1).

(3) For the sake of contradiction, suppose that $a \in R^*$. By $a \notin A_{k,k}$, there is an $S_k - T_k$ path Q that does not pass through edge a. By $lead(R^*) \in A(1,2,3)$, we get that path Q passes through $lead(R^*)$. Let Q_k be the subpath of Q from $lead(R^*)$ to T_k. By the definition of R^*, there exists a path Q_i' from S_i to $lead(R^*)$. By $a \in R^*$, path Q_i' does not pass through a. Thus, $Q_i' \cup Q_k$ is an $S_i - T_k$ path, which does not pass through a, contradicting $a \in A_{i,k}$.

(4) For the sake of contradiction, suppose that $R^* \rightarrow R_1$. Let Q be an $S_i - T_i$ path and Q_i be the subpath of Q from S_i to R^*. The subpath Q_i does not pass through R_1 (otherwise, the graph would contain a cycle). By Lemma 3.6(2) and a $\notin A_{k,k}$, there exists an $S_k - T_k$ path Q' which passes through R^*, but does not pass through edge a. Let Q_k' be the subpath of Q' from R^* to T_k. Hence, $Q_i \cup Q_k'$ is an $S_i - T_k$ path that does not pass through edge a, contradicting $a \in A_{i,k}$. Thus $R^* \nrightarrow R_1$, which means, by the definition of R^*, $R_1 \rightarrow R^*$.

(5) By $\{a,b\} \in A^{(2)}_{j,k}$, there exists an $S_j - T_k$ path Q that passes through the edge b, but does not pass through a. For the sake of contradiction, suppose that $b \in R^*$, so, path Q passes through R^*. Let Q_j be the subpath of Q from S_j to R^* and Q'_j be a path from R^* to T_j . By Part (4) of this lemma, path Q'_j does not pass through edge a. Therefore, $Q_j \cup Q'_j$ is an $S_j - T_j$ path that does not pass through a, contradicting $a \in A_{j,j}$. .

(6) By the assumption of this lemma, there exists an $S_i - T_k$ path Q that does not pass through $A(1,2,3)$. By $lead(R^*) \in A(1,2,3)$, path Q does not pass through R^*. But, by $a \in A_{i,k}$ and $a \in R_1$, path Q passes through R_1. Let Q_k be the subpath of Q from R_1 to T_k. For the sake of contradiction, suppose that $S_k \to R_1$ and let Q'_k be an $S_k - R_1$ path. Path Q'_k does not pass through R^*, because $R^* \nrightarrow R_1$. Consequently, $Q'_k \cup Q_k$ is an $S_k - T_k$ path that does not pass through R^*, which is a contradiction.

(7) By $b \in A_{k,k}$ there exists an $S_k - R_2$ path Q_k. For the sake of contradiction, suppose that $R_2 \to R^*$ and let Q be a $R_2 - R^*$ path. By the definition of R^*, there exist a $R^* - T_j$ path Q_j (see Figure 7). By $S_k \nrightarrow R_1$, we get that paths Q and Q_j do not pass through $a \in R1$. On the other hand, by $\{a,b\} \in A^{(2)}_{j,k}$, there exists an $S_j - T_k$ path Q' that does not pass through a, but it passes through b. Let Q'_j be the subpath of Q' from S_j to R_2. Thus, $Q'_j \cup Q \cup Q_j$ is an $S_j - T_j$ path that does not pass through a, which is a contradiction with $a \in A_{j,j}$. By $b \in A_{k,k}$, we have $R_2 \to R^*$ or $R^* \to R_2$, because the graph is acyclic. Consequently, $R^* \to R_2$.

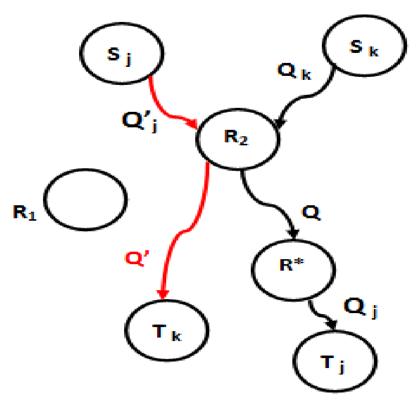

Figure 7. An illustration of the proof of Part (7) of Lemma 3.6.

(8) For the sake of contradiction, suppose that $R_1 = R_2$, which means $a, b \in R_1$. Thus, by $a \in A_{i,i} \cap A_{j,j}$ and $b \in A_{k,k}$, we get $lead(R_1) \in A(1,2,3)$. Hence, by Lemma 3.3(1) and $lead(R_1) \in R^*$, we have $a, b \in R^*$, contradicting Parts (3) and (5) of this lemma.

(9) By Parts (4) and (7) of this lemma, we get $R_1 \to R_2$, which means $R_2 \nrightarrow R_1$.

Example 3.3. Consider the networks in Fig. 6. In Part (a) of Fig. 6, there exist two edges a and b of E such that $a \in A_{1,1} \cap A_{2,2} \cap A_{1,3}$, $b \in A_{3,3}$ and $\{a,b\} \in A^{(2)}_{2,3}$. In Part (b) of Fig. 6, there are two different regions R_1 and R_2 such that $a = (v_1, v_2) \in R_1$ and $b = (v_6, v_7) \in R_2$. Moreover, we have $(v_3, v_5) \in A(1,2,3)$ and $a, b \notin A(1,2,3) \subseteq R^*$. Also, we have $S_3 \nrightarrow R_1$, $R_1 \to R_2$ *and* $R_1 \to R^* \to R_2$.

Lemma 3.7. Suppose that $A_{i,j} \cap A(1,2,3) = \emptyset$, for each $i,j \in \{1,2,3\}$ with $i \neq j$. Let $a \in R_1$ and $b \in R_2$ be two edges of E such that $a \in A_{i,i} \cap A_{j,j} \cap A_{i,k}$, $b \in A_{k,k}$ and $\{a,b\} \in A^{(2)}_{j,k}$, for distinct $i,j,k \in \{1,2,3\}$. Then

(1) $R_1 \in reg^0(S_i, S_j)$.

(2) Let R and R' be two regions in D^{**} such that $R \notin reg(S_i, R')$, for some $i \in \{1,2,3\}$. Then, there exists a $j \in \{1,2,3\} \backslash \{i\}$ and an $S_j - R$ path Q such that Q does not pass through R'.

(3) $R^* \in reg^0(R_1, S_k)$.

Proof:

(1) By $a \in R_1$ and $a \in A_{i,i} \cap A_{j,j} \cap A_{i,k}$, we have $S_i \to R_1$ and $S_j \to R_1$. Thus, Lemma 3.6(6) concludes $R_1 \in reg^0(S_i, S_j)$.

(2) By $R \notin reg(S_i, R')$, region R has a parent R_1 such that $R_1 \notin reg(S_i, R')$, which means $R_1 \neq R'$. Also, region R_1 has a parent R_2 such that $R_2 \notin reg(S_i, R')$ and $R_2 \neq R'$. Graph $RG(D^{**})$ is finite, so, we finally find a region $R_l = S_j$, for $j \in \{1,2,3\} \backslash \{i\}$, such that $R_l \notin reg(S_i, R')$ and $R_l \neq R'$. Hence $Q: R_l - R_{l-1} - \cdots - R_2 - R_1 - R$ is an $S_j - R$ path, which does not pass through R'.

(3) For the sake of contradiction, suppose that $R^* \notin reg^0(R_1, S_k)$, so, by Part (2) of this lemma, there exists at least an $S_i - R^*$ path Q_i or an $S_j - R^*$ path Q_j that does not pass through R_1. By the definition of R^*, there exist a $R^* - T_i$ path Q_i' and a $R^* - T_j$ path Q_j' . By Lemma 3.6(4), paths Q_i' and Q_j' do not pass through R_1. Therefore, $Q_i \cup Q_i'$ and $Q_j \cup Q_j'$ are $S_i - T_i$ and $S_j - T_j$ paths, respectively. These paths do not pass through R_1, contradicting $a \in A_{i,i} \cap A_{j,j}$.

Example 3.4. Consider the networks in Fig. 6. In Part (a) of Fig. 6, there exist two edges a and b of E such that $a \in A_{1,1} \cap A_{2,2} \cap A_{1,3}$, $b \in A_{3,3}$ and $\{a,b\} \in A^{(2)}_{2,3}$. In Part (b) of Fig.6, there is a region R_1 such that $a \in R_1$ and $R_1 \in reg^0(S_1, S_2)$. Moreover, there is a bottleneck region R^* such that $R^* \in reg^0(R_1, S_3)$. So, the claims of Lemma 3.7 are concluded.

Lemma 3.8. Suppose that $A_{i,j} \cap A(1,2,3) = \emptyset$, for each $i,j \in \{1,2,3\}$ with $i \neq j$. Let a $\in R_1$ and $b \in R_2$ be two edges of E such that $a \in A_{i,i} \cap A_{j,j} \cap A_{i,k}$, $b \in A_{k,k}$ and $\{a,b\} \in A^{(2)}_{j,k}$, for distinct $i,j,k \in \{1,2,3\}$. Then $R_1 \in \Lambda_k$.

Proof : By the assumption, there exists an $S_i - T_k$ path Q which does not pass through R^*. By $a \in A_{i,k}$ and $a \in R_1$, path Q must pass through R_1. Let Q_1 (resp. Q') be the subpath of Q from S_i to R_1 (resp. from R_1 to T_k). By $a \in A_{j,j}$, there exists an $S_j - R_1$ path Q_2, which, by Part (4) of Lemma 3.6, does not pass through R^*. For the sake of contradiction, suppose that $R_1 \notin \Lambda_k$, so, for each $R_1 - T_k$ path Q', there exists a region $R_3 \in Q'$, where $R_3 \in \Lambda_k$ (See Figure 8). Let Q_3 (resp. Q_4) be the subpath of Q' from R_1 to R_3 (resp. from R_3 to T_k). By $R_3 \in \Lambda_k$, region R_3 is connected to more than one terminal, so, by $R_3 \to T_k$, there exists at least a $R_3 - T_i$ path Q_i' or a $R_3 - T_j$ path Q_j' . Paths Q_i' and Q_j' pass through R^* (because, in otherwise, path $Q_1 \cup Q_3 \cup Q_i'$ (resp. $Q_2 \cup Q_3 \cup Q_j'$) is an $S_i - T_i$ path (resp. $S_j - T_j$ path) that does not pass through R^*), so $R_3 \to R^*$. Thus, by Lemma 3.7(3), we get $R_3 \in reg^0(S_k, R_1)$. Consequently, there exists an $S_k - R_3$ path Q'' , which does not pass through R^* (because paths Q_i' and Q_j' pass through R^*).Therefore, $Q'' \cup Q_4$ is an $S_k - T_k$ path that does not path through R^*, which is a contradiction.

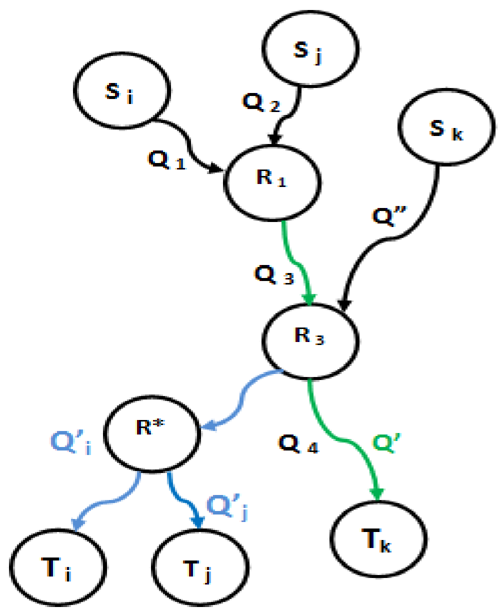

Figure 8. An illustration of the proof of Lemma 3.8.

Example 3.5. Consider the depicted networks in Fig. 6. In Part (a) of Fig. 6, there exist two edges a and b of E such that $a \in A_{1,1} \cap A_{2,2} \cap A_{1,3}$, $b \in A_{3,3}$ and $\{a, b\} \in A_{2,3}^{(2)}$. By Part (3) of Definition 3.1 and Part (b) of Fig. 6, we have $a \in R_1$ and $R_1 \in \Lambda_3 = \{R_1, R_2\}$, which means Lemma 3.8 is held.

Lemma 3.9. Suppose that $A_{i,j} \cap A(1,2,3) = \emptyset$, for each $i, j \in \{1,2,3\}$ with $i \neq j$. Let $a \in R_1$ and $b \in R_2$ be two edges of E such that $a \in A_{i,i} \cap A_{j,j} \cap A_{i,k}$, $b \in A_{k,k}$ and $\{a, b\} \in A_{j,k}^{(2)}$, for distinct $i, j, k \in \{1,2,3\}$. If $\{a, b\} \in A_{j,i}^{(2)}$ or $b \in A_{j,i}$, then, there is no region R_3 such that $R_2 \to R_3$, $R_3 \to T_i$, $R_3 \to T_k$, and $R_3 \in reg(R^*, S_j)$.

Proof : For the sake of contradiction, suppose that there is a region R_3 such that $R_2 \to R_3$, $R_3 \to T_i$, $R_3 \to T_k$, and $R_3 \in reg(R^*, S_j)$. Let R' be the first region (after R_2) in path $R_2 \to R_3$, (see Figure 9). Also, let Q'_i, Q'_k and Q''_k be $R_3 \to T_i$, $R_3 \to T_k$ and $S_k - R^*$ paths, respectively. By Part (7) of Lemma 3.6, paths Q'_i and Q'_k do not pass through R^* and Q''_k does not pass through R_2. By $R_3 \in reg(R^*, S_j)$, we get $R' \in reg(R^*, S_j)$. First, we need the following claim:

Claim 1. Each $R^* - R'$ and $S_j - R'$ path passes through R_2.

Proof of Claim 1: For the sake of contradiction, suppose that there exist a $R^* - R'$ path Q_1 and an $S_j - R'$ path Q_j which do not pass through R_2. Hence, path $Q''_k \cup Q_1 \cup \{R' \to R_3\} \cup Q'_k$ is an $S_k - T_k$ path that does not pass through R_2, contradicting $b \in A_{k,k}$. Path $Q = Q_j \cup \{R' \to R_3\} \cup Q'_i$ is an $S_j - T_i$ path, which does not pass through $b \in R_2$.

Case 1: $\{a, b\} \in A^{(2)}_{j,i}$.

Path Q passes through $a \in R_1$, so, by Part (9) of Lemma 3.6, region R_1 belongs to Q_j . Let Q_i be an $S_i - R_1$ path and Q''_j be the subpath of Q_j from R_1 to R'. Path Q''_j passes through R^*, because, otherwise, path $Q_i \cup Q''_j \cup \{R' \to R_3\} \cup Q'_i$ is an $S_i - T_i$ path which does not pass through R^*. Region R^* connects to S_k, so, we have an $S_k - T_k$ path that does not pass through R_2, which is a contradiction.

Case 2: $b \in A_{j,i}$.

In this case, the $S_j - T_i$ path Q does not pass through $b \in R_2$, which is a contradiction. Therefore, claim 1 is correct and each $R^* - R'$ and $S_j - R'$ path passes through R_2, so, by $R' \in reg(R^*, S_j)$, region R_2 is the only parent of region R', which is contradiction (since each region has at least two parents).

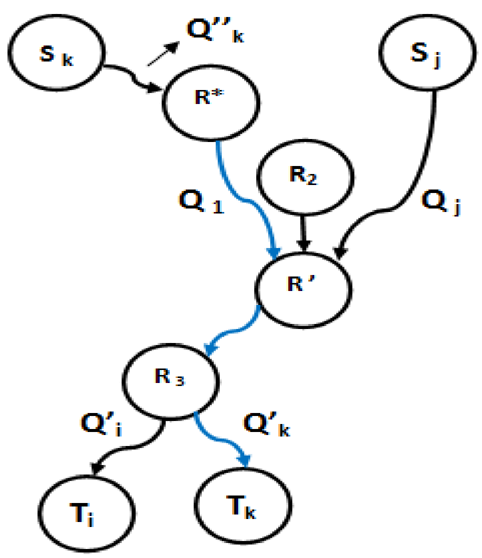

Figure 9. An illustration of the proof of Lemma 3.9.

3.2. Properties of regions R_1, R_2 and R^* such that $R_1 \in reg^0(S_i, S_j)$, $R^* \in reg^0(S_k, R_1)$ and $R_2 \in reg^0(S_j, R^*)$, for distinct i, j, k ∈ {1, 2, 3}

In this section, we present some properties of regions R_1, R_2 and R^* such that $R_1 \in reg^0(S_i, S_j)$, $R^* \in reg^0(S_k, R_1)$, and $R_2 \in reg^0(S_j, R^*)$, for distinct $i, j, k \in \{1,2,3\}$. These results are used in the next section.

Lemma 3.10. *For $i \in \{1,2,3\}$, if $R \in reg^0(S_i, R^*)$, then $R \nrightarrow T_i$.*

Proof : Consider $i \in \{1,2,3\}$. For the sake of contradiction, suppose that $R \to T_i$ and Q be a $R \to T_i$ path. By $R^* \to R$, path Q does not pass through R^*. Since $R \in reg^0(S_i, R^*)$, there exists an $S_i \to R$ path Q' which does not pass through R^*. Path $Q' \cup Q$ is an $S_i - T_i$ path that does not pass through R^*, which is contradiction.

Lemma 3.11. *Suppose that $R^* \in reg^0(S_k, R_1)$. Then for distinct $i, j, k \in \{1,2,3\}$*

(1) $lead(R_1) \in A_{i,i}$.

(2) $lead(R_1) \in A_{j,j}$.

Proof:

(1) For the sake of contradiction, suppose that $lead(R_1) \notin A_{i,i}$, thus, there exists an $S_i - T_i$ path Q that does not pass through R_1. By the definition of R^*, path Q passes through R^*, so $R^* \notin reg^0(S_k, R_1)$, which is a contradiction.

(2) The claim is concluded in the similar way to the proof of the last part.

Lemma 3.12. *Let $R_1 \in reg^0(S_i, S_j)$, $R^* \in reg^0(S_k, R_1)$ and $R_2 \in reg^0(S_j, R^*)$, for distinct $i, j, k \in \{1,2,3\}$. If $\Lambda_k = \{R_1, R_2, R_1'', R_2'', \ldots, R_r'' \mid R_l'' \in reg^0(R_1, R_2), 1 \le l \le r\}$, then $lead(R_1) \in A_{i,k}$.*

Proof : For the sake of contradiction, suppose that $\mathrm{lead}(R_1) \notin A_{i,k}$, which means there exists an $S_i - T_k$ path Q such that it does not pass through R_1. By Lemma 3.4(1), path Q does not pass through R^*. By $\Lambda_k = \{R_1, R_2, R_1'', R_2'', \ldots, R_r'' \mid R_l'' \in reg^0(R_1, R_2), 1 \le l \le r\}$ and Lemma 3.4(4), path Q passes through R_2. Hence, the subpath of Q from S_i to R_2 is an $S_i - R_2$ path that does not pass through R^*, contradicting $R_2 \in reg^0(S_j, R^*)$.

Lemma 3.13. *Let $R_1 \in reg^0(S_i, S_j)$, $R^* \in reg^0(S_k, R_1)$ and $R_2 \in reg^0(S_j, R^*)$, for distinct $i, j, k \in \{1,2,3\}$. If $\Lambda_k = \{R_1, R_2, R_1'', R_2'', \ldots, R_r'' \mid R_l'' \in reg^0(R_1, R_2), 1 \le l \le r\}$, then $lead(R_1) \in A_{k,k}$.*

Proof : For the sake of contradiction, suppose that $\mathrm{lead}(R_2) \notin A_{k,k}$. There exists an $S_k - T_k$path Q that does not pass through R_2. Path Q passes through R^*. Let Q_k be the subpath of Q from R^* to T_k. By definitions, there exists a region R on Q_k such that $R \in \Lambda_k$. Path Q does not pass through R_2, so $R \neq R_2$. Also, by $R^* \in reg^0(S_k, R_1)$, we get $R \neq R_1$. On the other hand, by $R^* \notin reg^0(R_1, R_2)$ and

$\mathrm{R}^* \to \mathrm{R}$, we get $\mathrm{R} \notin \mathrm{reg}^0(\mathrm{R}_1, \mathrm{R}_2)$, which is a contradiction with $\mathrm{R} \in \Lambda_\mathrm{k} = \{\mathrm{R}_1, \mathrm{R}_2, \mathrm{R}_1'', \mathrm{R}_2'', \ldots, \mathrm{R}_\mathrm{r}'' | \, \mathrm{R}_\mathrm{l}'' \in \mathrm{reg}^0(\mathrm{R}_1, \mathrm{R}_2), 1 \le \mathrm{l} \le \mathrm{r}\}$.

Example 3.6. Consider the depicted networks in Fig. 6. In Part (b) of Fig. 6, there are three different regions R_1 and R_2 and R^* such that $R_1 \in reg^0(S_1, S_2)$, $R_2 \in reg^0(S_2, R^*)$ and $R^* \in reg^0(S_3, R_1)$. We can check $R_2 \to T_2$, which concludes Lemma 3.10. Moreover, we have $lead(R_1) \in A_{1,1} \cap A_{2,2}$. So, Lemma 3.11 is concluded. On the other hand, by Part (3) of Definition 3.1, we have $\Lambda_3 = \{R_1, R_2\}$. We can check $lead(R_1) \in A_{1,3}$ and $lead(R_2) \in A_{3,3}$. So, Lemmas 3.12 and 3.13 are concluded.

Lemma 3.14. *Let $R_1 \in reg^0(S_i, S_j)$, $R^* \in reg^0(S_k, R_1)$ and $R_2 \in reg^0(S_j, R^*)$, for distinct $i, j, k \in \{1, 2, 3\}$. If $\Lambda_k = \{R_1, R_2, R_1'', R_2'', \ldots, R_r'' | \, R_l'' \in reg^0(R_1, R_2), 1 \le l \le r\}$, then.*

(1) $lead(R_1) \in A_{j,k}$.

(2) $lead(R_2) \in A_{j,k}$.

(3) $\{lead(R_1), lead(R_2)\}$ is an $S_j \to T_k$ cut.

(4) $\{lead(R_1), lead(R_2)\} \in A_{j,k}^{(2)}$.

Proof:

(1) We show that there is an $S_j \to T_k$ path that does not pass through R_1. By $R_2 \in reg^0(S_j, R^*)$, there exists an $S_j \to R_2$ path Q_1, which, by $R_1 \in reg^0(S_i, S_j)$, it does not pass through R_1. On the other hand, by $\{R_1, R_2\} \in \Lambda_k$, there exists an $R_2 \to T_k$ path Q_2, which, by Lemma 3.4(2), it does not pass through R_1. Hence, $Q_1 \cup Q_2$ is an $S_j \to T_k$ path that does not pass through R_1.

(2) We find an $S_j \to T_k$ path that does not through R_2. By $R_1 \in reg^0(S_i, S_j)$, there exists an $S_j \to R_1$ path Q_1, which, by $R_2 \in reg^0(S_j, R^*)$, it does not pass through R_2. Also, by $\{R_1, R_2\} \in \Lambda_k$, there exists a $R_1 \to T_k$ path Q_2, which, by Lemma 3.4(2), it does not pass through R_2. Consequently, $Q_1 \cup Q_2$ is an $S_j \to T_k$ path that does not pass through R_2.

(3) By Lemma 3.4(4), each $S_j \to T_k$ path passes through Λ_k. Thus, by $\Lambda_k = \{R_1, R_2, R_1'', R_2'', \ldots, R_r'' | \, R_l'' \in reg^0(R_1, R_2), 1 \le l \le r\}$ and Lemma 3.1, set $\{lead(R_1), lead(R_2)\}$ is an $s_j \to t_k$ cut.

(4) By Parts (1), (2) and (3), the claim is concluded.

3.3. Inverse Network

This section defines the inverse network G' and describes some relationships between G and G', which are used in the next section.

Definition 3.2. *(Inverse Network) [4]. The inverse of a network G is a network G' satisfying the following conditions.*

(1) G' has the same nodes as G.

(2) The edges of G' are the same as in G but with the reversed direction.

(3) Each node that emits messages in G demands the same messages in G'.

(4) Each node which demands messages in G, instead emits the same messages in G'.

Lemma 3.15. *Let G be the inverse of G. Then, for each*

$i, j \in \{1, 2, 3\}$ and for each $a, b \in E$,

(1) $a \in A_{i,j}$ in G if and only if $a \in A_{j,i}$ in G'.

(2) $\{a, b\} \in A_{i,j}^{(2)}$ in G if and only if $\{a, b\} \in A_{j,i}^{(2)}$ in G'.

Proof:

(1) Suppose that, in G, we have $a \in A_{i,j}$ and in G', for the sake of contradiction, suppose that $a \notin A_{j,i}$, so, there exists an $S_j \to T_i$ path Q such that $a \notin Q$. Network G' is the inverse of G, so, in the network G, path Q is an $S_i \to T_j$ path such that $a \notin Q$, which is a contradiction with $a \in A_{i,j}$. Also, in a similar way, we can conclude that if, in G', $a \in A_{j,i}$, then $a \in A_{i,j}$ in G.

(2) In the network G', for the sake of contradiction, suppose that $\{a, b\} \notin A_{j,i}^{(2)}$, so $a \in A_{j,i}$ or $b \in A_{j,i}$ or there exists an $S_j \to T_i$ path such that it does not pass through a and b. Network G' is the inverse of G, so, by Part (1) of this lemma, in network G, we have either $a \in A_{i,j}$ or $b \in A_{i,j}$ or there exists an $S_i \to T_j$ path such that it does not pass through a and b, which is a contradiction with $\{a, b\} \in A_{i,j}^{(2)}$. In a similar way, we can conclude that if $\{a, b\} \in A_{j,i}^{(2)}$ in G', then $\{a, b\} \in A_{i,j}^{(2)}$ in G.

Lemma 3.16. *Let G' be the inverse network of G, then*

(1) Condition (2) of Theorem 2.1 holds in G if and only if Condition (3) of Theorem 2.1 holds in G'.

(2) Condition (4) of Theorem 2.1 holds in G if and only if Condition (5) of Theorem 2.1 holds in G'.

(3) Condition (6) of Theorem 2.1 holds in G if and only if Condition (7) of Theorem 2.1 holds in G'.

Proof:

(1) Suppose that the Condition (2) of Theorem 2.1 holds in G, so, there exist two edges $a, b \in E$ such that $a \in A_{i,i} \cap A_{j,j} \cap A_{i,k}$, $b \in A_{k,k}$ and $\{a, b\} \in A_{j,i}^{(2)} \cap A_{j,k}^{(2)}$. By Lemma 3.15, in G', we have

$$a \in A_{i,i} \cap A_{j,j} \cap A_{k,i},\ b \in A_{k,k} \text{ and } \{a, b\} \in A_{i,j}^{(2)} \cap A_{k,j}^{(2)}.$$

Hence, if i, j and k replaced by k, i and j respectively, then

$$a \in A_{k,k} \cap A_{i,i} \cap A_{j,k},\ b \in A_{j,j} \text{ and } \{a, b\} \in A_{k,i}^{(2)} \cap A_{j,i}^{(2)}.$$

which means, Condition (3) of Theorem 2.1 holds in G'' . In a similarly way, we conclude that if Condition (3) of Theorem 2.1 holds in G', then Condition (2) of Theorem 2.1 holds in G.

(2) The claim can be proven in a similar way to Part (1) of this lemma.

(3) The claim can be proven in a similar way to Part (1) of this lemma.

4. Describing the conditions of Theorem 2.1 in the basic region graph

In this section, the conditions of Theorem 2.1 are described in the basic region graph.

4.1. Describing Condition 1 of Theorem 2.1 in the basic region graph

The next lemma is essential for describing Condition 1 of Theorem 2.1 in the basic region graph.

Lemma 4.1. *For distinct* $i, j, k \in \{1,2,3\}$,

$A_{i,j} \cap A(1,2,3) \neq \emptyset$ if and only if $\Lambda_j \subseteq reg(S_k, R^*)$.

Proof : Suppose that $\Lambda_j \subseteq \mathrm{reg}(S_k, R^*)$. Each source region has no incoming link, so, each $S_i - T_j$ path passes through R^*, which means, each $S_i - T_j$ path passes through the leader of R^*. Hence, by Lemma 3.3(2), we get $A_{i,j} \cap A(1,2,3) \neq \emptyset$. Now, suppose that $A_{i,j} \cap A(1,2,3) \neq \emptyset$. We have the following cases for terminal region T_j :

Case 1: *$T_j \rightarrow T_i$, for $i \in \{1,2,3\} \setminus \{j\}$.*

In this case, we have $\Omega_j = \emptyset$, which means $\Lambda_j = \emptyset$, and the claim follows.

Case 2: $T_j \nrightarrow T_i$, for each $i \in \{1,2,3\} \setminus \{j\}$.

In this case, we get $T_j \in \Omega_j \neq \emptyset$, thus $\Lambda_j \neq \emptyset$. Each $S_i - T_j$ path passes through R^*, so, by $R^* \notin \Omega j$ and Lemma 3.4(2), we have $S_i \notin \Lambda_j$. Also, since each $S_j - T_j$ path passes through R^*, by Lemma3.4(2), we conclude $S_j \notin \Lambda_j$, which means $\Lambda_j \subseteq reg(S_k, R^*)$.

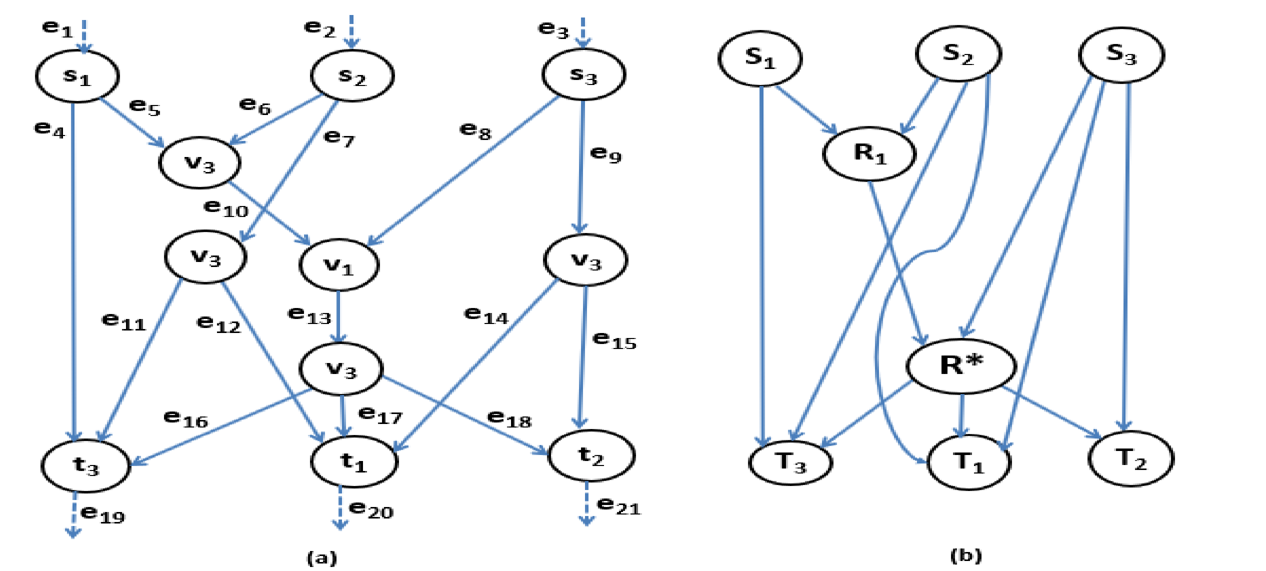

Figure 10. An illustration of Lemma 4.1: (a) is a three-pair network G_1 where $A(1,2,3) \neq \emptyset$ and $A_{1,2} \cap A(1,2,3) = \{e_{13}\} \neq \emptyset$. Thus, G_1 satisfies Condition (1) of Theorem 2.1 and is unsolvable. (b) is the basic region graph $RG(D^{**})$ with bottleneck region R^*. Here, $\Lambda_2 = \{R^*, S_3\} \subseteq reg(R^*, S_3)$ and $RG(D^{**})$ is unsolvable.

By Lemma 4.1, the next corollary describes Condition 1 of Theorem 2.1 in the basic region graph.

Corollary 4.1. *Condition* (1) *of Theorem 2.1 holds if and only if the following condition holds:*

C-IR1: There exist distinct $j, k \in \{1,2,3\}$ *such that* $\Lambda_j \subseteq reg(S_k, R^*)$.

4.2. Describing Conditions 2 and 3 of Theorem 2.1 in the basic region graph

In this section, assume Condition (1) of Theorem 2.1 does not hold, which means $A_{i,j} \cap A(1,2,3) = \emptyset$, for each $i, j \in \{1,2,3\}$ with $i \neq j$. The next theorem describes Condition (2) of Theorem 2.1 in the basic region graph.

Theorem 4.1. *Suppose that* $A_{i,j} \cap A(1,2,3) = \emptyset$*, for each* $i, j \in \{1,2,3\}$ *with* $i \neq j$*. Condition* (2) *of Theorem 2.1 holds if and only if the following condition (C-IR2) holds:*

C-IR2: There exist $R_1 \in reg^0(S_i, S_j)$, $R^* \in reg^0(S_k, R_1)$, *and* $R_2 \in reg^0(S_j, R^*)$ *such that* $\Lambda_k = \{R_1, R_2, R_1'', R_2'', \dots, R_r'' \mid R_l'' \in reg^0(R_1, R_2), 1 \leq l \leq r\}$ *and* $\Lambda_i = \{R_2, R_1', R_2', \dots, R_r' \mid R_l' \in reg(R^*, S_k), 1 \leq l \leq r\}$*. Moreover, if* $|\Lambda_i| = 2$*, then* $R_1' \neq S_k$*, for distinct* $i, j, k \in \{1,2,3\}$.

We need some lemmas and corollaries in order to prove Theorem 4.1.

Lemma 4.2. *Suppose that* $A_{i,j} \cap A(1,2,3) = \emptyset$*, for each* $i, j \in \{1,2,3\}$ *with* $i \neq j$*. Let* a *and* b *be two edges of* E *such that* a *and* b *satisfy Condition (2) of Theorem 2.1, then* $R_2 \in reg^0(S_j, R^*)$.

Proof : By Parts (7) and (9) of Lemma 3.6, there exists a region $R_2 \neq R^*$ such that $b \in R_2$. By $b \in R_2$ and $\{a, b\} \in A_{j,i}^{(2)}$, there exists an $S_j - T_i$ path Q_i, which passes through R_2. Also, by $\{a, b\} \in A_{j,k}^{(2)}$, there exists an $S_j - T_k$ path Q_k, which passes through R_2. Let Q_i' (resp. Q_k') be the subpath of Q_i (resp. Q_k) from R_2 to T_i (resp. T_k). Lemma 3.6(9) concludes that subpaths Q_i' and Q_k' do not pass through R^*. For the sake of contradiction, suppose that $R_2 \notin reg^0(S_j, R^*)$, so, by Part (2) of Lemma 3.7, there exists at least an $S_i - R_2$ path l_i or an $S_k - R_2$ path l_k such that l_i (resp. l_k) does not pass through R^*. Hence, $l_i \cup Q_i'$ and $l_k \cup Q_k'$ are $S_i - T_i$ and $S_k - T_k$ paths respectively, which do not pass through R^*. This is a contradiction with the definition of R^*.

Lemma 4.3. *Suppose that* $A_{i,j} \cap A(1,2,3) = \emptyset$*, for each* $i, j \in \{1,2,3\}$ *with* $i \neq j$*. Let* a *and* b *be two edges of* E *such that* a *and* b *satisfy Condition (2) of Theorem 2.1. Then*

(1) $R_2 \nrightarrow T_j$.

(2) $R_2 \in \Lambda_k$.

(3) $R_2 \in \Lambda_i$.

(4) $R_1 \in \Lambda_k$.

Proof:

(1) By Lemmas 4.2 and 3.10, the claim is concluded.

(2) For the sake of contradiction, suppose that $R_2 \notin \Lambda_k$. Thus, there exists a region R_3 such that $R_3 \in \Lambda_k$, $R_2 \to R_3$ and $R_3 \to T_k$. By $R_2 \to R_3$ and Lemma 3.10, we get $R_3 \nrightarrow T_j$. Hence, by $R_3 \in \Lambda_k$, we conclude $R_3 \to T_i$. Let Q_i and Q_k be a $R_3 \to T_i$ and $R_3 \to T_k$ path, respectively. Lemma 3.9 concludes $R_3 \notin reg^0(S_j, R^*)$, so, by Lemma 3.7(2), there exists at least an $S_i \to R_3$ path Q'_i or an $S_k \to R_3$ path Q'_k which does not pass through R^*. Consequently, $Q'_i \cup Q_i$ (resp. $Q'_k \cup Q_k$) is an $S_i - T_i$ path (resp. $S_k - T_k$ path) that does not pass through R^*, which is a contradiction.

(3) By Parts (1) and (2) of this lemma, we get $R_2 \to T_i$. For the sake of contradiction, suppose that $R_2 \notin \Lambda_i$. In a similar way to the last part of this lemma, there exists at least an $S_i - T_i$ path or an $S_k - T_k$ path that does not pass through R^*, which is a contradiction.

(4) By Lemma 3.8, the claim is concluded.

Lemma 4.4. *Suppose that $A_{i,j} \cap A(1,2,3) = \emptyset$, for each $i, j \in \{1,2,3\}$ with $i \neq j$. Let a and b be two edges of E such that a and b satisfy Condition (2) of Theorem 2.1. Then*

(1) There exists a non-source region $R' \neq R_2$ such that $R' \in \Lambda_i$.
(2) $R' \in reg(S_k, R^*)$.
(3) If R'' be a non-source region of D^{**} such that $R'' \in \Lambda_k$ and $R'' \neq R_1, R_2$, then $R'' \in reg^0(R_1, R_2)$.

Proof:

(1) By Lemmas 3.5 and 4.3(3), there exists a region R' such that $R' \neq R_2$. Now, we show that R' is a non-source region. By $\{a, b\} \in A^{(2)}_{j,i}$, we get $a \notin A_{j,i}$, so, there exists an $S_j - T_i$ path Q such that $a \notin Q$ and $b \in Q$. By Lemma 3.6(9), path Q passes through R_2. Let $e = lead(R_2)$, we have $e \neq a$ (because, by Parts (4) and (8) of Lemma 3.6, $a \in R_1 \neq R_2$). Let Q_j be the subpath of Q from S_j to tail(e). By $b \in R_2$, we get $b \notin Q_j$. Also, by $a \notin Q$, we have $a \notin Q_j$. By $b \notin A_{j,i}$, there exists an $S_j - T_i$ path Q' such that $b \notin Q'$ and $a \in Q'$. We prove that Q' does not pass through R_2. For the sake of contradiction, suppose that Q' passes through R_2, so, $e \in Q'$. Let Q'_i be the subpath of Q' from head(e) to T_i. Also, $a \notin Q'_i$ because $a \notin R_2$ and $R_2 \nrightarrow R_1$. By $e \notin \{a, b\}$, path $Q_j \cup \{e\} \cup Q'_i$ is an $S_j - T_i$ path that does not pass through a and b, which is a contradiction with $\{a, b\} \in A^{(2)}_{j,i}$. By Lemma 3.4(4), $Q' \cap \Lambda_i \neq \emptyset$, so, there exists a region R' on Q' that $R' \in \Lambda_i$ and $R' \neq R_2$. By $a \in Q'$, path Q' passes through R_1. Also, by Lemma 3.8 and Remark 3.1(d), we get $R_1 \notin \Omega_i$, thus $R_1 \to R'$, which means, R' is a non-source region.

(2) Let $\theta = \{S_k, R^*\}$ and R' be a non-source region of D^{**}, where $R' \neq R_2$ and $R' \in \Lambda_i$. By Lemma 2.2(1), it is sufficient to show that each $S_i - R'$ path Q_i, $S_j - R'$ path Q_j and $S_k - R'$ path Q_k passes through a region of θ. By definitions, for each $S_k - R'$ path, the claim is satisfied. Also, by Lemma 3.4(3), each $S_i - R'$ path passes through R^*. Let Q_j be an $S_j - R'$

path. By $R' \in \Lambda_i$, there exists a $R' - T_i$ path Q'_i, which does not passes through R^*. Consequently, $Q = Q_j \cup Q'_i$ is an $S_j - T_i$ path, which, by $\{a, b\} \in A^{(2)}_{j,i}$, it passes through R_1 or R_2 (or both of them). By Lemma 4.3(3), we have $R_2 \in \Lambda_i$, so, by $R' \in \Lambda_i$ and $R' \neq R_2$, we conclude that path Q does not pass through R_2, which means, it passes through R_1. By Lemma 3.8, we have $R_1 \in \Lambda_k$, which means that path Q'_i does not pass through R_1. Hence, by $R' \neq R_1$, path Q_j passes through R_1. Let Q' be the subpath of Q_j from R_1 to R'. By $R_1 \in reg^0(S_i, S_j)$, there exists an $S_i - R_1$ path Q_i, which, by Lemma 3.6(4), does not pass through R^*. By definitions, each $S_i - T_i$ path passes through R^* and $Q_i \cup Q' \cup Q'_i$ is an $S_i - T_i$ path, which means that path Q' passes through R^*.

(3) Define $\theta = \{R_1, R_2\}$. Let Q_i be an $S_i - R''$ path, Q_j be an $S_j - R''$ path and Q_k be an $S_k - R''$ path. By $R'' \in \Lambda_k$ and $R_1, R_2 \in \Lambda_k$, we get that $R'' - T_k$ path Q'_k does not pass through R_1 and R_2. Also, we have $Q_i \cap \theta \neq \emptyset$, $Q_j \cap \theta \neq \emptyset$ and $Q_k \cap \theta \neq \emptyset$ (in otherwise, $Q = Q_i \cup Q'_k$, $Q' = Q_k \cup Q'_k$ and $Q'' = Q_j \cup Q'_k$ are $S_i - T_k$, $S_k - T_k$ and $S_j - T_k$ paths respectively, such that Q, Q' and Q'' do not pass through $R_1$and R_2, which are contradiction with $a \in A_{i,k}$, $b \in A_{k,k}$ and $\{a, b\} \in A^{(2)}_{j,k}$). Hence, by Lemma 2.2(1), we conclude $R'' \in reg^0(R_1, R_2)$.

The following corollary is a result of Lemma 4.4.

Corollary 4.2. *If $\Lambda_i = \{R_2, R'\}$, then R' is a non-source region and $R' \in reg(S_k, R^*)$.*

Example 4.1. Figure 6 is an illustration of a three-pair network that satisfies Condition (2) of Theorem 2.1 and its basic region graph. In Part (a) of Fig. 6, there are two edges a and b such that $a \in A_{1,1} \cap A_{2,2} \cap A_{1,3}$, $b \in A_{3,3}$ and $\{a, b\} \in A^{(2)}_{2,1} \cap A^{(2)}_{2,3}$. In Part (b) of Fig. 6, the basic region graph of the given network in Part(a) is depicted. We see that $R_2 \in reg^0(S_2, R^*)$, so, Lemma 4.2 is hold. Also, by Part (3) of Definition 3.1, we conclude that $\Lambda_3 = \{R_1, R_2\}$ and $\Lambda_1 = \{R_2, R^*, S_3\}$. So, $R_2 \in \Lambda_3 \cap \Lambda_1$ which concludes Lemma 4.3. Moreover, by considering the members of set Λ_1, the claims in Lemma 4.4 are concluded.

Lemma 4.5. *If Condition C-IR2 holds, then*

(1) $R^* \to R_2$ and $R_2 \nrightarrow R^*$.

(2) $R^* \nrightarrow R_1$ and $R_1 \to R^*$.

(3) $R_1 \to R_2$ and $R_2 \nrightarrow R_1$.

(4) $lead(R_1) \in A_{i,i} \cap A_{j,j}$ and $lead(R_1) \in A_{i,k}$.

(5) $lead(R_2) \in A_{k,k}$ and $\{ lead(R_1), lead(R_2)\} \in A^{(2)}_{j,k}$.

Proof : By Condition C-IR2, we have $R_2 \in reg^0(S_j, R^*)$, so $R^* \to R_2$, which means, $R_2 \nrightarrow R^*$.

The proof of Parts (2) and (3) are similar to Part (1). By Lemmas 3.11 and 3.12, Part (4) is resulted. Also, by Lemmas 3.13 and 3.14, Part (5) is concluded.

The next lemma presents some other properties of Condition C-IR2.

Lemma 4.6. *If Condition C-IR2 holds, then*

(1) $lead(R_1) \notin A_{j,i}$.

(2) $lead(R_2) \notin A_{j,i}$.

(3) $\{ lead(R_1), lead(R_2)\}$ *is an* $s_j - t_i$ *cut.*

(4) $\{lead(R_1), lead(R_2)\} \in A_{j,i}^{(2)}$.

Proof:

(1) By Lemma 3.14, there exists an $S_j - T_k$ path Q that does not pass through R_1, but it passes through R_2. Let Q_j be the subpath of path Q from S_j to R_2. By $R_2 \in \Lambda_k$, there exits a $R_2 - T_i$ path Q_i, which, by Lemma 4.5(3), it does not pass through R_1. Thus, $Q_j \cup Q_i$ is an $S_j - T_i$ path which does not pass through R_1.

(2) By $R_1 \in reg^0(S_i, S_j)$, there exists an $S_j - R_1$ path Q_1, which, by Lemma 4.5(3), it does not pass through R_2. Moreover, since $R^* \in reg^0(R_1, S_k)$, there exists a $R_1 - R^*$ path Q_2, which by Lemma 4.5(1), it does not pass through R_2 . On the other hand, by $R' \in \Lambda_i$ and $R' \in reg(S_k, R^*)$, there exists a $R^* - R'$ path Q_3, which, by $R_2 \in reg^0(S_j, R^*)$, it does not pass through R_2 . Also, by $\{R_2 , R'\} \in \Lambda_i$ and Lemma 3.4(2), there exists a $R' - T_i$ path Q_4 that does not pass through R_2 . Thus $Q = Q_1 \cup Q_2 \cup Q_3 \cup Q_4$ is an $S_j - T_i$ path that does not pass through R_2 , which means $lead(R_2) \notin A_{j,i}$.

(3) By Lemma 3.4(4), each $S_j - T_i$ path passes through Λ_i. Since $\Lambda_i = \{R_2, R'_1, R'_2, \ldots, R'_r | R'_l \in reg(R^*, S_k), 1 \leq l \leq r\}$, set $\{ lead(R'_l), lead(R_2), 1 \leq l \leq r\}$ *is an* $s_j - t_i$ *cut.* Consequently, by $R'_l \in reg(R^*, S_k)$, $R^* \in reg(R_1, S_k)$, Lemma 3.1 and Lemma 3.2(3), set) $\{ lead(R_1), lead(R_2)\}$ is an $s_j - t_i$ cut.

(4) By Parts (1), (2) and (3) of this lemma, the claim is concluded.

Proof of Theorem 4.1: If Condition (2) of Theorem 2.1 holds, then, by Parts (1) and (3) of Lemma 3.7, there are two regions $R_1 \in reg^0(S_i, S_j)$ and $R^* \in reg^0(R_1, S_k)$. Moreover, by Lemma 3.8, $R_1 \in \Lambda_k$. Also, by Lemma 4.2, there is a region $R_2 \in reg^0(R^*, S_j)$. On the other hand, by Parts (2),(3) and (4) of Lemma 4.3, $R_1 \in \Lambda_k$ and $R_2 \in \Lambda_i \cap \Lambda_k$. So, by Lemma 4.4 and Corollary 4.2, we conclude that Condition C-IR2 holds. Also, if Condition C-IR2 holds, then, by Lemmas 4.5, and 4.6, $lead(R_1)$ and $lead(R_2)$ are two edges that satisfy Condition (2) of Theorem 2.1. So, the claim is concluded.

By Lemma 3.16 and Theorem 4.1, the next theorem is concluded.

Theorem 4.2. *Suppose that* G *is a three-pair network such that* $A_{i,j} \cap A(1,2,3) = \emptyset$*, for each* $i, j \in \{1,2,3\}$ *with* $i \neq j$ *and* G' *is its inverse network.* G *satisfies Condition* (3) *of Theorem 2.1*

if and only if the basic region graph of G' *satisfies Condition (C-IR2).*

4.3. Describing Conditions 4,5,6 and 7 of Theorem 2.1 in the basic region graph

In this section, we suppose that Condition (1) of Theorem 2.1 is not satisfied, which means $A_{i,j} \cap A(1,2,3) = \emptyset$, for each $i,j \in \{1,2,3\}$ *with* $i \neq j$. The next theorem describes Condition 4 of Theorem 2.1 in the basic region graph.

Theorem 4.3. *Suppose that* $A_{i,j} \cap A(1,2,3) = \emptyset$, *for each* $i,j \in \{1,2,3\}$ *with* $i \neq j$. *Condition* (4) *of Theorem 2.1 holds if and only if the following condition (C-IR4) holds:*

C-IR4: There exist $R_1 \in reg^0(S_i, S_j)$, $R_2 \in reg^0(R^*, S_j)$ *and* $R^* \in reg^0(R_1, S_k)$ *such that* $\Lambda_k = \{R_1, R_2, R_1'', R_2'', \dots, R_r'' \mid R_l'' \in reg^0(R_1, R_2), 1 \leq l \leq r\}$ *and* $\Lambda_i = \{R_2, R_1', R_2', \dots, R_r' | R_l' \in reg(R_2, S_k), R_l' \neq R_2, 1 \leq l \leq r\}$, *for distinct* $i,j,k \in \{1,2,3\}$.

We need some lemmas and corollaries in order to prove Theorem 4.3. Figure 11 is an illustration of a three-pair network that satisfies Condition (4) of Theorem 2.1 and its basic region graph. In Part(a) of Fig. 11, there are two edges a and b such that $a \in A_{1,1} \cap A_{2,2} \cap A_{1,3}$, $b \in A_{3,3} \cap A_{2,1}$ and $\{a,b\} \in A^{(2)}_{2,3}$. In Part (b) of Fig. 11, the basic region graph of the given network in Part(a) is depicted. By Theorem 4.3, in this network condition C-IR4 is held because there are $R_1 \in reg^0(S_1, S_2)$, $R_2 \in reg^0(R^*, S_2)$ and $R^* \in reg^0(R_1, S_3)$ such that $\Lambda_3 = \{R_1, R_2\}$ and $\Lambda_1 = \{R_2, S_3\}$.

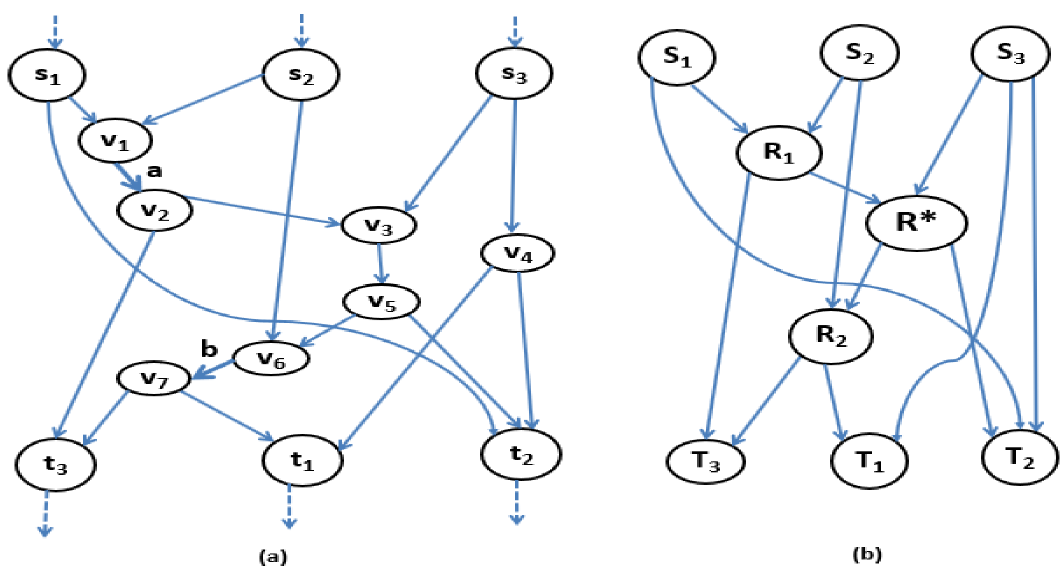

Figure 11. An illustration of Theorem 4.3: (a) is a three-pair network that satisfies Condition (4) of Theorem 2.1. (b) is the basic region graph of the given network in part (a).

Lemma 4.7. *Suppose that* $A_{i,j} \cap A(1,2,3) = \emptyset$, *for each* $i,j \in \{1,2,3\}$ *with* $i \neq j$. *If* a, b *are two edges of E such that* a *and* b *satisfy Condition (4) of Theorem 2.1, then*

(1) $R_1 \in reg^0(S_i, S_j)$.

(2) $R^* \in reg^0(R_1, S_k)$.

Proof : Edges a and b satisfy in Condition (4), so $a \in A_{i,i} \cap A_{j,j} \cap A_{i,k}$ and $\{a, b\} \in A^{(2)}_{j,k}$, which means, by Part (1) and (3) of Lemma 3.7, the claims are true.

Lemma 4.8. *Suppose that* $A_{i,j} \cap A(1,2,3) = \emptyset$, *for each* $i, j \in \{1,2,3\}$ *with* $i \neq j$. *If* a, b *are two edges of E such that* a *and* b *satisfy Condition (4) of Theorem 2.1. Then* $R_2 \in reg^0(R^*, S_j)$.

Proof : For the sake of contradiction, suppose that $R_2 \notin reg^0(R^*, S_j)$. Then, by Lemma 3.7(2), there exists at least an $S_i - R_2$ path Q_i or an $S_k - R_2$ path Q_k that does not pass through R^*. On the other hand, by $b \in A_{k,k} \cap A_{j,i}$ and $b \in R_2$, there exists a $R_2 - T_k$ path Q'_k and a $R_2 - T_i$ path Q'_i. By Lemma 3.6(7), Q'_k and Q'_i do not pass through R^*. Thus $Q_i \cup Q'_i$ and $Q_k \cup Q'_k$ are $S_i - T_i$ and $S_k - T_k$ paths, respectively. These paths do not pass through R^*, which is a contradiction.

Lemma 4.9. *Suppose that* $A_{i,j} \cap A(1,2,3) = \emptyset$, *for each* $i, j \in \{1,2,3\}$ *with* $i \neq j$. *If* a, b *are two edges of E such that* a *and* b *satisfy Condition (4) of Theorem 2.1, then* $R_2 \in \Lambda_k$.

Proof : For the sake of contradiction, suppose that $R_2 \notin \Lambda_k$. Thus, there exists a region R_3 such that $R_3 \in \Lambda_k$, $R_2 \to R_3$ and $R_3 \to T_k$. By $R_2 \to R_3$ and Lemma 3.10, we get $R_3 \nrightarrow T_j$. Hence, by $R_3 \in \Lambda_k$, we conclude $R_3 \to T_i$. Let Q_i and Q_k be $R_3 - T_i$ and $R_3 - T_k$ paths, respectively. Lemma 3.9 concludes $R_3 \notin reg(R^*, S_j)$, so, by Lemma 3.7(2), there exists at least an $S_i - R_3$ path Q'_i or an $S_k - R_3$ path Q'_k which does not pass through R^*. Consequently, $Q'_i \cup Q_i$ and $Q'_k \cup Q_k$ are $S_i - T_i$ and $S_k - T_k$ paths, respectively. These paths do not pass through R^*, which is a contradiction.

Lemma 4.10. *Suppose that* $A_{i,j} \cap A(1,2,3) = \emptyset$, *for each* $i, j \in \{1,2,3\}$ *with* $i \neq j$. *If* a, b *are two edges of E such that a and b satisfy Condition (4) of Theorem 2.1, then*

(1) $R_1 \in \Lambda_k$.

(2) $R_2 \in \Lambda_i$.

(3) If R' *is a non-source region of* D^{**} *such that* $R' \in \Lambda_i$ *and* $R' \neq R_2$, *then* $R' \in reg(R_2, S_k)$.

(4) If R'' *is a non-source region of* D^{**} *such that* $R'' \in \Lambda_k$ *and* $R'' \neq R_1, R_2$, *then* $R'' \in reg^0(R_1, R_2)$.

Proof:

(1) Edges a, b satisfy Condition (4) of Theorem 2.1, so, we have $a \in A_{i,i} \cap A_{j,j} \cap A_{i,k}$, $b \in A_{k,k}$ and $\{a, b\} \in A^{(2)}_{j,k}$, which means, by Lemma 3.8, $R_1 \in \Lambda_k$.

(2) By Lemma 4.9, we have $R_2 \in \Lambda_k$. Thus, the proof is similar to Lemma 4.3(3).

(3) Since $R' \in \Lambda_i$, there exists a $R' - T_i$ path Q'_i. By $\{R', R_2\} \in \Lambda_i$ and Lemma 3.4(2), we get that Q'_i does not pass through R_2. For the sake of contradiction, suppose that $R' \notin reg(R_2, S_k)$. By Lemma 3.7(2), we have the following two cases.

Case 1: There exists an $S_i - R'$ path Q_i such that Q_i does not pass through R_2. In this case, $l_i = Q_i \cup Q'_i$ is an $S_i - T_i$ path such that Q_i does not pass through R_2. Let l'_i be a subpath of Q_i from R^* to T_i. Suppose that l_j is an $S_j - R^*$ path. By Lemma 3.6(7), path l_j does not pass through R_2. Therefore, $l_j \cup l'_i$ is an $S_j - T_i$ path that does not pass through R_2, which is a contradiction with $b \in R_2$ and $b \in A_{j,i}$.

Case 2: There exists an $S_j - R'$ path l'_j such that it does not pass through R_2. In this case, , $l'_j \cup Q'_i$ is an $S_j - T_i$ path that does not pass through R_2, which is a contradiction with $b \in R_2$ and $b \in A_{j,i}$.

(4) The claim can be proven in a similar way to Lemma 4.4(3).

Lemma 4.11 *If Condition C-IR4 holds, then*

(1) $lead(R_1) \in A_{i,i} \cap A_{j,j}$ *and* $lead(R_1) \in A_{i,k}$.

(2) $lead(R_1) \in A_{i,i} \cap A_{j,j} \cap A_{i,k}$ *and* $lead(R_2) \in A_{k,k}$.

(3) $lead(R_2) \in A_{j,i}$.

(4) $lead(R_2) \in A_{k,k} \cap A_{j,i}$ *and* $\{lead(R_1), lead(R_2)\} \in A^{(2)}_{j,k}$.

Proof : By $R^* \in reg^0(R_1, S_k)$ and Lemma 3.11, Part (1) is concluded. By Part (1) of this lemma, and Lemma 3.13 we get Part (2).

3. By $\Lambda_i = \{R_2, R'_1, R'_2, \ldots, R'_r | R'_l \in reg(R_2, S_k),\ R'_l \neq R_2, 1 \leq l \leq r\}$ and Lemma 3.4(4), each $S_j - T_i$ path passes through R_2 or R'_l, for $1 \leq l \leq r$. On the other hand, by $R'_l \in reg(R_2, S_k)$ and Lemma 3.2(2), each $S_j - R'_l$ path passes through R_2. Thus, by Lemma 3.1, $lead(R_2) \in A_{j,i}$.

4. By Lemma 3.14 and Parts (2) and (3) of this lemma, the claim is concluded.

Proof of Theorem 4.3. If Condition (4) of Theorem 2.1 holds, then, by Lemmas 4.7 and 4.8, there are regions $R_1 \in reg^0(S_i, S_j)$, $R_2 \in reg^0(R^*, S_j)$ *and* $R^* \in reg^0(R_1, S_k)$. Moreover, by Lemmas 4.9 and 4.10, we conclude that Condition C-IR4 holds. Also, if Condition C-IR4 holds, by Lemma 4.11, *lead*(*R*1) and *lead*(*R*2) satisfy Condition (4) of Theorem 2.1. So, the claim is concluded.

Condition (5) of Theorem 2.1 in the basic region graph is described as follows.

Theorem 4.4. Suppose that G is a three-pair network such that $A_{i,j} \cap A(1,2,3) = \emptyset$, for each $i, j \in \{1,2,3\}$ with $i \neq j$ and G' is its inverse network. G satisfies Condition (5) of Theorem 2.1 if and only if the basic region graph of G' satisfies Condition (C-IR4).

Proof: By Lemma 3.16 and Theorem 4.3, the claim is concluded.

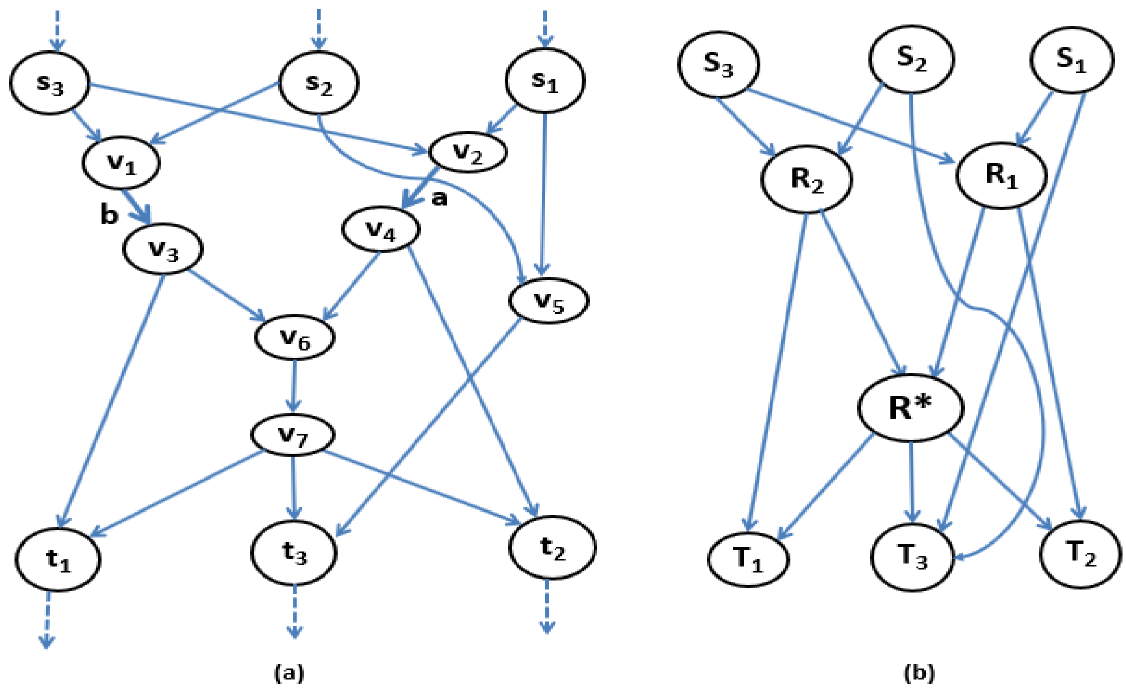

Figure 12. An illustration of Theorem 4.5: (a) is a three-pair network that satisfies Condition (6) of Theorem 2.1. (b) is the basic region graph of the given network in part (a).

The next theorem describes Conditions (6) and (7) of Theorem 2.1 in the basic region graph. This theorem can be proven in similar way to the last theorem.

Theorem 4.5. Suppose that G is a three-pair network such that $A_{i,j} \cap A(1,2,3) = \emptyset$, for each $i, j \in \{1,2,3\}$ with $i \neq j$ and G' is its inverse network. Condition (6) of Theorem 2.1 holds if and only if *the following condition (C-IR6) holds:*

C-IR6: There exist $R_1 \in reg^0(S_i, S_k)$, $R_2 \in reg^0(S_j, S_k)$ and $R^* \in reg^0(R_1, R_2)$ such that $\Lambda_i = \{R_2, R_1'', R_2'', \dots, R_r'' \mid R_l'' \in reg^0(R_1, R_2), 1 \leq l \leq r\}$ and $\Lambda_j = \{R_1, R_1', R_2', \dots, R_r' \mid R_l' \in reg(R_1, R_2), 1 \leq l \leq r\}$, for distinct $i, j, k \in \{1,2,3\}$. Also, G satisfies Condition (7) of Theorem2.1 if and only if the basic region graph of G' satisfies Condition (C-IR6).

For example, Figure 12 is an illustration of a three-pair network that satisfies Condition (6) of Theorem 2.1 and its basic region graph. In Part (a) of Fig. 12, there are two edges a and b such that $a \in A_{1,1} \cap A_{1,2}$, $b \in A_{2,2}$ and) $\{a, b\} \in A_{3,1}^{(2)} \cap A_{3,2}^{(2)} \cap A_{3,3}^{(2)}$. In Part (b) of Fig. 12, the basic region

graph of the given network in Part(a) is depicted. By Theorem 4.5, in this network condition C-IR6 is held because there are $R_1 \in reg^0(S_1, S_3)$, $R_2 \in reg^0(S_2, S_3)$ and $R^* \in reg^0(R_1, R_2)$ such that $\Lambda_1 = \{R_2, R^*\}$ and $\Lambda_2 = \{R_1, R^*\}$.

4.4. Complexity of diagnosing the solvability or unsolvability of three-pair problem

This section presents our algorithm to diagnose the solvability or unsolvability of three-pair problem. By Theorem 2.1, Corollary 4.1 and Theorems 4.1,...,4.5, region R^* should be found. By Lemma 3.3, region R^* is found by having a bottleneck edge. Algorithm 1 presents a method to find all bottleneck edges.

Algorithm 1. Computing bottleneck edges of $G = (V, E)$;

Begin

(1) For each $e \in E$ do

Begin

Delete edge e from E;

For each $i \in \{1, 2, 3\}$ do

Begin

(2) Search an $S_i - T_i$ path;

If there does not exist an $S_i - T_i$ path then goto (1);

End

Write "Edge *e* is bottleneck";

End

End.

The next lemma concludes that region R^* is found in $O(|E|^2)$ time.

Lemma 4.12.

(1) Algorithm 1 computes all bottleneck edges in $O(|E|^2)$ time.

(2) Region R^* is found in $O(|E|^2)$ time.

Proof:

(1) Step (1) of Algorithm 1 examines each link of the network. By *search algorithm*[3], Step (2) is done in $O(|E|)$ time. Therefore, the time complexity of computing a bottleneck edge is $O(|E|^2)$.

(2) By [16], all regions are found in $O(|D^{**}|)$ time. Also, Lemma 3.3 concludes that, by having a bottleneck edge, the region R^* is found. Hence, by Part (1), the claim is concluded.

Algorithm 2 presents our method to diagnose the solvability or unsolvability of three-pair networks using Theorem 2.1, Corollary 4.1 and Theorems 4.1,...,4.5.

Algorithm 2. Solvability of $G = (V, E)$;

Begin

(i) For each distinct $i, j, k \in \{1, 2, 3\}$ do

Begin

Procedure-preprocess;

1-1 Procedure-C-IR1;

1-2 If Condition C-IR1 is satisfied, then goto 1-15;

1-3 Procedure-C-IR2;

1-4 If Condition C-IR2 is satisfied, then goto 1-15;

1-5 Apply procedure-C-IR2 for the inverse graph G';

1-6 If Condition C-IR2 is satisfied in graph G', then goto 1-15;

1-7 Procedure-C-IR4;

1-8 If Condition C-IR4 is satisfied, then goto 1-15;

1-9 Apply procedure-C-IR4 for the inverse graph G';

1-10 If Condition C-IR4 is satisfied in graph G', then goto 1-15;

1-11 Procedure-C-IR6;

1-12 If Condition C-IR6 is satisfied, then goto 1-15;

1-13 Apply procedure-C-IR6 for the inverse graph G';

1-14 If Condition C-IR6 is satisfied in graph G', then goto 1-15, otherwise goto (i);

1-15 Write "The three-pair problem is not solvable" and Break;

End,

Write" The three-pair problem is solvable" and Break;

End.

Procedure-preprocess;

Begin

1-1 Construct basic region graph $RG(D^{**})$;

1-2 Find region R^*;

1-3 Find Λ_i, Λ_j and Λ_k;

1-4 Let $\theta_1 = \mathrm{reg}^0(\mathrm{S_i}, \mathrm{S_j})$;

1-5 Let $\theta_2 = \mathrm{reg}^0(\mathrm{S_j}, \mathrm{S_k})$;

1-6 Let $\theta_3 = \mathrm{reg}^0(R^*, \mathrm{S_j})$;

1-7 Let $\theta_4 = \mathrm{reg}^0(R^*, \mathrm{S_k})$;

1-8 Let $\theta_5 = \mathrm{reg}^0(\mathrm{S_i}, \mathrm{S_k})$;

End.

Procedure-C-IR1;

Begin

If $\Lambda_\mathrm{j} \subseteq \theta_4$, then write "Condition C-IR1 is satisfied";

End.

Procedure-C-IR2;

Begin

1-1 If there exists $R \in \Lambda_i$ such that $R \in \theta_3$, then let $R_2 = R$, otherwise Break;

1-2 If $|\Lambda_i| = 2$ and $\{S_k\} = \Lambda_i - \{R_2\}$, then Break;

1-3 If there exists region $R' \in \Lambda_i$ such that $R' \neq R_2$ and $R' \notin \theta 4$, then Break;

1-4 If $R_2 \notin \Lambda_k$, then Break;

1-5 If there exists $R_l \in \Lambda_k$ such that $R_l \neq R_2$ and $R_l \in \theta_1$, then let $R_1 = R_l$, otherwise Break;

1-6 If $R^* \notin \mathrm{reg}^0(\mathrm{R_1}, \mathrm{S_k})$, then Break;

1-7 If there exists region $R_l \in \Lambda_k$ such that $R_l \notin \{R_1, R_2\}$ and $R_l \notin \mathrm{reg}^0(\mathrm{R_1}, \mathrm{R_2})$, then Break;

1-8 Write "Condition C-IR2 is satisfied";

End.

Procedure-C-IR4;

Begin

1-1 If there exists $R \in \Lambda_i$ such that $R \in \theta_3$, then let $R_2 = R$, otherwise Break;

1-2 If there exists $R' \in \Lambda_i$ such that $R' \neq R_2$, and $R' \notin \mathrm{reg}(\mathrm{R_2}, \mathrm{S_k})$, then Break;

1-3 If $R_2 \notin \Lambda_k$, then Break;

1-4 If there exists $R_l \in \Lambda_k$ such that $R_l \neq R_2$ and $R_l \in \theta_1$, then let $R_1 = R_l$, otherwise Break;

1-5 If $R^* \notin \mathrm{reg}^0(\mathrm{R}_1, \mathrm{S}_\mathrm{k})$, then Break;

1-6 If there exists $R_l \in \Lambda_k$ such that $R_l \notin \{R_1, R_2\}$ and $R_l \notin \mathrm{reg}^0(\mathrm{R}_1, \mathrm{R}_2)$, then Break;

1-7 Write "Condition C-IR4 is satisfied";

End.

Procedure-C-IR6;

Begin

1-1 If there exists $R \in \Lambda_i$ such that $R \in \theta_2$, then let $R_2 = R$, otherwise Break;

1-2 If there exists $R' \in \Lambda_j$ such that $R' \in \theta_5$, then let $R_1 = R'$, otherwise Break;

1-3 If $R^* \notin \mathrm{reg}^0(\mathrm{R}_1, \mathrm{R}_2)$, then Break;

1-4 If there exists $R_l \in \Lambda_i$ such that $R_l \neq R_2$ and $R_l \notin \mathrm{reg}^0(\mathrm{R}_1, \mathrm{R}_2)$, then Break;

1-5 If there exists $R_l \in \Lambda_j$ such that $R_l \neq R_1$ and $R_l \notin \mathrm{reg}^0(\mathrm{R}_1, \mathrm{R}_2)$, then Break;

1-6 Write "Condition C-IR6 is satisfied";

End.

The next theorem concludes that the running time of Algorithm 2 is $O(|E|^2)$.

Theorem 4.6. *Algorithm 2 diagnoses the solvability of three-pair networks in $O(|E|^2)$-time.*

Proof: By Theorem 2.1, Corollary 4.1 and Theorems 4.1,...,4.5, we conclude that Algorithm 2 diagnoses the solvability of three-pair networks. By Lemma 4.12, Step 1-2 of

Procedure-preprocess runs in $O(|E|^2)$-time (note, by [10], Steps 1-1, 1-3, $\cdots$, 1-8 of Procedure preprocess are done in $O(|E|)$ time). Procedure-CIR1 is done in $O(|E|)$-time. Also, the running time of Procedure-CIR2, Procedure-CIR4 and Procedure-CIR6 is $O(|E|^2)$. On the other hand, Steps 1-5, 1-9 and 1-13 of Algorithm 2 are done in $O(|E|^2)$ $-$time. Therefore, Algorithm 2 runs in $O(|E|^2)$ $-$time.

5. Conclusion

This paper presents an $O(|E|^2)$-time algorithm for determining the solvability of three-pair networks with common bottleneck links using the region decomposition method and the basic region graph, where ($|E|$) denotes the number of links in the network. Compared to the previous $O(|E|^3)$-time algorithm in [4], the proposed approach achieves a significant reduction in computational complexity while preserving the correctness of the solvability diagnosis. This improvement enhances the efficiency of analyzing large communication networks. From an algorithmic perspective, the proposed method follows a principle that is widely adopted in combinatorial optimization and graph algorithms [1,11,13]. Although determining network coding solvability is fundamentally a decision problem rather than a classical optimization problem, both areas pursue the common objective of reducing computational complexity through structural decomposition and efficient graph analysis. In this work, the region decomposition framework provides an effective mechanism for identifying and eliminating unnecessary computations, thereby reducing the overall running time of the solvability test.

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