Iranian Journal of Operations Research
Vol. 16, No. 1, 2025, pp.159 -170

A New Approach for Solving Fuzzy Linear Programming Problems Using a K-Scale Ordering Method

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Abstract: *In this paper, we deal with a fully fuzzy linear programming (FFLP) when the constraints are described as equality and inequality. With respect to Hadi method which is a new and a comfortable ranking method for ordering the trapezoidal fuzzy numbers, we introduce a new ranking function. We show that this function has some smooth properties when we use it for new classes of the trapezoidal fuzzy numbers which we called them k-scale trapezoidal fuzzy numbers. The k- scale trapezoidal fuzzy numbers are in fact a generalization of symmetric trapezoidal fuzzy numbers. Based on this ranking function, a new method is proposed to find the fuzzy solution for solving k-scale FFLP. Numerical examples are providing to illustrate the method.*

Keywords: *Fully fuzzy linear programming, Trapezoidal fuzzy numbers, Hadi ranking method.*

1. Introduction

In many real-world optimization and decision-making problems, uncertainty and imprecision are unavoidable. Parameters such as costs, processing times, and resource availability are often not known exactly and may vary due to incomplete information or subjective estimation. Classical linear programming assumes deterministic data; however, this assumption is often unrealistic in practical applications. To overcome this limitation, fuzzy set theory introduced by Zadeh provides a mathematical framework for modeling vagueness and uncertainty in complex systems.

Fuzzy Linear Programming (FLP) extends classical linear programming by allowing some or all parameters to be represented as fuzzy numbers. This extension enables more flexible and realistic modeling of decision-making problems under uncertainty. FLP has been widely applied in various fields such as project management, production planning, supply chain optimization, and engineering design, where precise data is not always available.

One of the main challenges in solving fuzzy linear programming problems is the ranking and comparison of fuzzy numbers. Since fuzzy numbers do not have a natural total ordering, different ranking methods have been developed, including score functions, distance-based approaches, and lexicographic ordering techniques. The choice of ranking method significantly affects the final solution, as it determines how fuzzy alternatives are compared and prioritized.

In recent years, improved ranking techniques have been proposed to enhance consistency and discrimination among fuzzy numbers. Among these approaches, ordering-based methods aim to

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transform fuzzy quantities into comparable structures that preserve more information about uncertainty. However, existing methods may still suffer from limitations such as loss of information or sensitivity to parameter scaling, motivating the development of more robust approaches.

In this context, advanced ordering techniques such as the k-scale ordering method have been proposed to improve the ranking process in fuzzy optimization problems. These methods aim to provide more reliable comparisons between fuzzy numbers and enhance the solution quality of fuzzy linear programming models. As a result, they contribute to more accurate and stable decision-making in environments characterized by uncertainty.

In this paper, we consider the following fuzzy linear programming

Maximize/Minimize $\sum_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j$

Subject to (2.1)

$$\sum_{j=1}^{n} \tilde{a}_{ij} \otimes \tilde{x}_j \preccurlyeq, \approx, \succcurlyeq \tilde{b}_i, \quad \forall i = 1, 2, \dots, m,$$

where $\tilde{x}_j$ is a non-negative trapezoidal fuzzy number, for each $j = 1, \dots, n$.

If the right-hand side, constraint coefficients, the objective function coefficients, and the variables are represented by fuzzy numbers, then model (2.1) is known as Fully Fuzzy Linear Programming (FFLP) problem. This problem has been considered by many researchers and different methods are proposed for solving it; see e.g. [1, 2, 11] and the references therein. In this paper, we introduce a new method for a class of FFLP problems. We convert this fuzzy model to a crisp linear optimization problem using a proposed ranking function which is corresponding to Hadi ranking method for trapezoidal fuzzy numbers [10]. Hadi method is developed in [10] for ranking trapezoidal fuzzy numbers to overcome shortcomings which are found in ranking fuzzy numbers with some convenient methods such as Asady, Chen method, Cheng distance and Chu-Tsao methods; for more information see [10].

Ghoushchi et al [8] presented a fully fuzzy linear programming (FFLP) problem with triangular fuzzy decision parameters and variables based on alpha-cut theory and modified triangular fuzzy numbers.

Hasan and Al-kanani [6], for fully fuzzy linear programming (FFLP) problems, where all variables and parameters are of the same type of decagonal fuzzy numbers, presents a new ranking function technique and a new membership function of decagonal fuzzy numbers.

Hasankhani et.al [7] introduced the Z-complete linear programming (FZLP) problem with all parameters, including the coefficients of variables in the objective functions, the coefficients of variables in the constraints, the right-hand side of the constraints, as well as the decision variables, as Z numbers and used the concept of Z numbers to solve the problem.

The present study, proposes a novel ranking framework for trapezoidal fuzzy numbers and applies it to the solution of fully fuzzy linear programming (FFLP) problems. Specifically, a new ranking function is developed as an extension of the Hadi ranking method, leading to the introduction of k-scale trapezoidal fuzzy numbers (KSTFNs). This paper is organized into 6 sections. In the next section, some preliminaries of fuzzy numbers and Hadi method reviewed. In Section 3, we introduce a new ranking function and investigate some properties of it. In Section 4, a new method is proposed and, in this sense, we will illustrate this method by a numerical example in Section 5. The conclusion and some suggestions are given in Section 6.

2. Preliminaries

In this section, we provide some preliminaries. The notations and results in this section are taken from [10,11].

Definition 2.1. A fuzzy number $\tilde{a} = \left(a^L, a^U, a^\alpha, a^\beta\right)$ is said to be a trapezoidal fuzzy number, if $a^L \leq a^U$ and $a^\alpha, a^\beta > 0$ and

$$\tilde{a}(x) = \begin{cases} \dfrac{x}{a^\alpha} + \dfrac{a^\alpha - a^L}{a^\alpha}, & x \in [a^L - a^\alpha, a^L] \\ 1, & x \in [a^L, a^U] \\ \dfrac{-x}{a^\beta} + \dfrac{a^U + a^\beta}{a^\beta}, & x \in [a^U, a^U + a^\beta] \end{cases} \quad (2.2)$$

The set of all trapezoidal fuzzy numbers denote by F(ℝ). A trapezoidal fuzzy number $\tilde{a} = \left(a^L, a^U, a^\alpha, a^\beta\right)$ is said to be non-negative if and only if $a^L - a^\alpha \geq 0$. Also, $\tilde{a} = \left(a^L, a^U, a^\alpha, a^\beta\right)$ is said to be symmetric if $a^\alpha = a^\beta$.

Assume that $\tilde{a} = \left(a^L, a^U, a^\alpha, a^\beta\right)$ and $\tilde{b} = \left(b^L, b^U, b^\alpha, b^\beta\right)$ are two trapezoidal fuzzy numbers, then

i) $\tilde{a} \oplus \tilde{b} = (a^L + b^L, a^U + b^U, a^\alpha + b^\alpha, a^\beta + b^\beta)$, (2.3)

ii) $\tilde{a} \ominus \tilde{b} = (a^L - b^U, a^U - b^L, a^\alpha + b^\beta, a^\beta + b^\alpha)$, (2.4)

ii) $\tilde{a} \otimes \tilde{b} = (c^L, c^U, c^\alpha, c^\beta)$, (2.5)

where

$$c^L = \min\{a^L b^L, a^L b^U, a^U b^L, a^U b^U\},$$

$$c^U = \max\{a^L b^L, a^L b^U, a^U b^L, a^U b^U\},$$

$$c^\alpha = c^L - \min\left\{(a^L - a^\alpha)(b^L - b^\alpha), (a^L - a^\alpha)\left(b^U + b^\beta\right), \left(a^U + a^\beta\right)(b^L - b^\alpha), \left(a^U + a^\beta\right)\left(b^U + b^\beta\right)\right\},$$

$$c^\beta = \max\{(a^L - a^\alpha)(b^L - b^\alpha), (a^L - a^\alpha)\left(b^U + b^\beta\right), \left(a^U + a^\beta\right)(b^L - b^\alpha), \left(a^U + a^\beta\right)\left(b^U + b^\beta\right)\} - c^U.$$

Note that if $\tilde{a} = \left(a^L, a^U, a^\alpha, a^\beta\right)$ and $\tilde{b} = \left(b^L, b^U, b^\alpha, b^\beta\right)$ are two non-negative trapezoidal fuzzy numbers, then

$$\tilde{a} \otimes \tilde{b} = \left(a^L b^L, a^U b^U, a^L b^L - (a^L - a^\alpha)\left(b^L - b^\beta\right), \left(a^U + a^\beta\right)\left(b^U + b^\beta\right) - a^U b^U\right). \quad (2.6)$$

Assume that $\tilde{a} = \left(a^L, a^U, a^\alpha, a^\beta\right)$ and $\tilde{b} = \left(b^L, b^U, b^\alpha, b^\beta\right)$ are two trapezoidal fuzzy numbers, then

$$\tilde{a} \approx \tilde{b} \Leftrightarrow a^L = b^L, a^U = b^U, a^\alpha = b^\alpha, a^\beta = b^\beta \quad (2.7)$$

Now, we review some definitions and results which are established by Nasseri (2010). Let $\tilde{a} = (a^L, a^U, a^\alpha, a^\beta)$, be an arbitrary trapezoidal fuzzy number. Define

$$\underline{a} = a^m - \frac{1}{2}h_{\underline{a}}, \quad \overline{a} = a^m + \frac{1}{2}h_{\overline{a}}, \tag{2.8}$$

where $a^m = \frac{a^L + a^U}{2}$, and $h_{\underline{a}} = \frac{a^\alpha}{a^\alpha + a^\beta}$, $h_{\overline{a}} = \frac{a^\beta}{a^\alpha + a^\beta}$.

Now assume that $\tilde{a} = (a^L, a^U, a^\alpha, a^\beta)$ and $\tilde{b} = (b^L, b^U, b^\alpha, b^\beta)$ are two trapezoidal fuzzy numbers. Let

$$\overline{R}(\tilde{a}, \tilde{b}) = \overline{a} - \overline{b}, \tag{2.9}$$

$$\underline{R}(\tilde{a}, \tilde{b}) = \underline{a} - \underline{b}, \tag{2.10}$$

then, we have

$$\overline{R}(\tilde{a}, \tilde{b}) = -\overline{R}(\tilde{b}, \tilde{a}) = \overline{R}(-\tilde{b}, -\tilde{a})$$
$$\underline{R}(\tilde{a}, \tilde{b}) = -\underline{R}(\tilde{b}, \tilde{a}) = \underline{R}(-\tilde{b}, -\tilde{a})$$

Definition 2.2. Assume that $\tilde{a} = (a^L, a^U, a^\alpha, a^\beta)$ and $\tilde{b} = (b^L, b^U, b^\alpha, b^\beta)$ are two trapezoidal fuzzy numbers and $\underline{R}(\tilde{b}, \tilde{a}) \geq 0$. Define the relations $\prec$ and $\approx$ on F(ℝ) as given below:

i) $\tilde{a} \approx \tilde{b}$ if and only if $\underline{R}(\tilde{b}, \tilde{a}) = \overline{R}(\tilde{a}, \tilde{b})$,
ii) $\tilde{a} \prec \tilde{b}$ if and only if $\underline{R}(\tilde{b}, \tilde{a}) > \overline{R}(\tilde{a}, \tilde{b})$.

We denote $\tilde{a} \preccurlyeq \tilde{b}$ if and only if $\tilde{a} \approx \tilde{b}$ or $\tilde{a} \prec \tilde{b}$. Therefore, $\tilde{a} \preccurlyeq \tilde{b}$ if and only if $\underline{R}(\tilde{b}, \tilde{a}) \geq \overline{R}(\tilde{a}, \tilde{b})$. Also $\tilde{b} \succ \tilde{a}$ if and only if $\tilde{a} \prec \tilde{b}$. We set $\tilde{0} \coloneqq (0,0,0,0)$ as zero trapezoidal fuzzy numbers.

Lemma 2.1. Assume $\tilde{a} \prec \tilde{b}$, then $-\tilde{a} \succ -\tilde{b}$.

Proof. The proof is straightforward by Definition 2.2.

Nasseri (2010) showed that $\approx$ is an equivalence relation (reflexive, symmetric, and transitive) and $\preccurlyeq$ is a partial order on F(ℝ). Note that the relation $\preccurlyeq$ is a linear order on F(ℝ) too, because any two elements in F(ℝ) are comparable by this relation.

3. Proposed ranking function and its properties

In this section, we are going to define a ranking function $\mathcal{R}: F(\mathbb{R}) \to \mathbb{R}$ corresponding to Hadi ranking, which maps each trapezoidal fuzzy number into the real line. Then, we also define k-scale trapezoidal fuzzy number as a generalization of symmetric trapezoidal fuzzy number.

Definition 3.1. Assume that $\tilde{a} = (a^L, a^U, a^\alpha, a^\beta)$ is an arbitrary trapezoidal fuzzy number. Define:

$$\mathcal{R}(\tilde{a}) = a^L + a^U + \frac{1}{2}\left(\frac{a^\beta - a^\alpha}{a^\alpha + a^\beta}\right). \tag{2.11}$$

Note that for the symmetric trapezoidal fuzzy number, the above formula can be defined as $\mathcal{R}(\tilde{a}) = a^L + a^U$.

Theorem 3.1. Assume that $\tilde{a} = (a^L, a^U, a^\alpha, a^\beta)$ and $\tilde{b} = (b^L, b^U, b^\alpha, b^\beta)$ are two trapezoidal fuzzy numbers and $\underline{R}(\tilde{b}, \tilde{a}) \geq 0$.

i) $\tilde{a} \approx \tilde{b}$ if and only if $\mathcal{R}(\tilde{a}) = \mathcal{R}(\tilde{b})$,

ii) $\tilde{a} \prec \tilde{b}$ if and only if $\mathcal{R}(\tilde{a}) < \mathcal{R}(\tilde{b})$.

Proof: i) Let $\tilde{a} \approx \tilde{b}$. Therefore, $\underline{R}(\tilde{b}, \tilde{a}) = \overline{R}(\tilde{a}, \tilde{b})$. Thus, $\underline{b} - \underline{a} = \overline{a} - \overline{b}$. This implies that

$$b^m - \frac{1}{2}h_{\underline{b}} - a^m + \frac{1}{2}h_{\underline{a}} = a^m + \frac{1}{2}h_{\overline{a}} - b^m - \frac{1}{2}h_{\overline{b}},$$

so,

$$\frac{b^L + b^U}{2} - \frac{1}{2}\frac{b^\alpha}{b^\alpha + b^\beta} - \frac{a^L + a^U}{2} + \frac{1}{2}\frac{a^\alpha}{a^\alpha + a^\beta} = \frac{a^L + a^U}{2} + \frac{1}{2}\frac{a^\beta}{a^\alpha + a^\beta} - \frac{b^L + b^U}{2} - \frac{1}{2}\frac{b^\beta}{b^\alpha + b^\beta}.$$

Now, we have,

$$b^L + b^U + \frac{1}{2}\left(\frac{b^\beta}{b^\alpha + b^\beta} - \frac{b^\alpha}{b^\alpha + b^\beta}\right) = a^L + a^U + \frac{1}{2}\left(\frac{a^\beta}{a^\alpha + a^\beta} - \frac{a^\alpha}{a^\alpha + a^\beta}\right).$$

Therefore, $\mathcal{R}(\tilde{a}) = \mathcal{R}(\tilde{b})$. The converse part can be obtained same as the above relations.

ii) Let $\tilde{a} \prec \tilde{b}$. Therefore $\underline{R}(\tilde{b}, \tilde{a}) > \overline{R}(\tilde{a}, \tilde{b})$. This implies that $\overline{a} - \overline{b} < \underline{b} - \underline{a}$. Thus, $\overline{a} + \underline{a} < \overline{b} + \underline{b}$. So,

$$a^m + \frac{1}{2}h_{\overline{a}} + a^m - \frac{1}{2}h_{\underline{a}} < b^m + \frac{1}{2}h_{\overline{b}} + b^m - \frac{1}{2}h_{\underline{b}}.$$

Hence,

$$a^L + a^U + \frac{1}{2}\left(\frac{a^\beta - a^\alpha}{a^\alpha + a^\beta}\right) < b^L + b^U + \frac{1}{2}\left(\frac{b^\beta - b^\alpha}{b^\alpha + b^\beta}\right).$$

Therefore, $\mathcal{R}(\tilde{a}) < \mathcal{R}(\tilde{b})$. The converse part is clear and we omit it here. ■

Similar to the results of Theorem 3.1, we have

$$\tilde{a} \preccurlyeq \tilde{b} \text{ if and only if } \mathcal{R}(\tilde{a}) \leq \mathcal{R}(\tilde{b}). \quad (2.12)$$

However, the function $\mathcal{R}$ is not linear, i.e. $\mathcal{R}(\tilde{a} + \tilde{b})$ is not necessarily equal to $\mathcal{R}(\tilde{a}) + \mathcal{R}(\tilde{b})$. For example, consider $\tilde{a} = (10,20,10,10)$ and $\tilde{b} = (20,30,10,20)$. We have $\mathcal{R}(\tilde{a} + \tilde{b}) = 80 + \frac{1}{5}$ and $\mathcal{R}(\tilde{a}) + \mathcal{R}(\tilde{b}) = 80 + \frac{1}{3}$. Also, if $\tilde{a} = (a^L, a^U, a^\alpha, a^\beta)$ is a non-negative trapezoidal fuzzy number and $\tilde{x} = (x^L, x^U, x^\alpha, x^\beta)$ is a non-negative trapezoidal fuzzy variable, then $\mathcal{R}(\tilde{a} \otimes \tilde{x})$ is a fractional term. Therefore, if we use $\mathcal{R}$ for transform a fuzzy linear objective function to a crisp one, the result might be complex to solve. In this paper, we deal to a special class of trapezoidal fuzzy number to overcome this shortage.

Definition 3.2. A trapezoidal fuzzy number $\tilde{a} = (a^L, a^U, a^\alpha, a^\beta)$ is said to be an k-scale trapezoidal fuzzy number if $a^\beta = ka^\alpha$ where $k \in \mathbb{N}$. The class of k-scale trapezoidal fuzzy numbers is denoted by $F_k(\mathbb{R})$.

It is clear that $F_1(\mathbb{R})$ is the class of symmetric trapezoidal fuzzy numbers. $F_1(\mathbb{R})$ is considered in many researches; see e.g. [Ganesan and Veeramani (2006), Ebrahimnejad (2011), Ebrahimnejad and Verdegay (2016)] and the references therein. Therefore, our results covered a wider class of fuzzy optimization problems.

Theorem 3.2. If $\left\{\tilde{a}_i = \left(a_i^L, a_i^U, a_i^\alpha, a_i^\beta\right)\right\}_{i=1}^m \subseteq F_k(\mathbb{R})$, then

$$\sum_{i=1}^{m} \mathcal{R}(\tilde{a}_i) = \mathcal{R}\left(\sum_{i=1}^{m} \tilde{a}_i\right) + \frac{m-1}{2}\left(\frac{k-1}{k+1}\right)$$

Proof: By definition of addition of two trapezoidal fuzzy numbers which is given in (2.3), we have

$$\mathcal{R}\left(\sum_{i=1}^{m} \tilde{a}_i\right) = \mathcal{R}\left(\sum_{i=1}^{m} a_i^L, \sum_{i=1}^{m} a_i^U, \sum_{i=1}^{m} a_i^\alpha, \sum_{i=1}^{m} a_i^\beta\right)$$

$$= \sum_{i=1}^{m} a_i^L + \sum_{i=1}^{m} a_i^U + \frac{1}{2}\left(\frac{\sum_{i=1}^{m} a_i^\beta - \sum_{i=1}^{m} a_i^\alpha}{\sum_{i=1}^{m} a_i^\beta + \sum_{i=1}^{m} a_i^\alpha}\right)$$

$$= \sum_{i=1}^{m} a_i^L + \sum_{i=1}^{m} a_i^U + \frac{1}{2}\left(\frac{\sum_{i=1}^{m} k a_i^\alpha - \sum_{i=1}^{m} a_i^\alpha}{\sum_{i=1}^{m} k a_i^\alpha + \sum_{i=1}^{m} a_i^\alpha}\right)$$

$$= \sum_{i=1}^{m} a_i^L + \sum_{i=1}^{m} a_i^U + \frac{1}{2}\left(\frac{(k-1)\sum_{i=1}^{m} a_i^\alpha}{(k+1)\sum_{i=1}^{m} a_i^\alpha}\right)$$

$$= \sum_{i=1}^{m} a_i^L + \sum_{i=1}^{m} a_i^U + \frac{1}{2}\left(\frac{k-1}{k+1}\right).$$

On the other hand,

$$\sum_{i=1}^{m} \mathcal{R}(\tilde{a}_i) = \sum_{i=1}^{m} a_i^L + \sum_{i=1}^{m} a_i^U + \sum_{i=1}^{m} \frac{1}{2}\left(\frac{a_i^\beta - a_i^\alpha}{a_i^\beta + a_i^\alpha}\right)$$

$$= \sum_{i=1}^{m} a_i^L + \sum_{i=1}^{m} a_i^U + \sum_{i=1}^{m} \frac{1}{2}\left(\frac{k-1}{k+1}\right)$$

$$= \sum_{i=1}^{m} a_i^L + \sum_{i=1}^{m} a_i^U + \frac{m}{2}\left(\frac{k-1}{k+1}\right)$$

$$= \sum_{i=1}^{m} a_i^L + \sum_{i=1}^{m} a_i^U + \frac{1}{2}\left(\frac{k-1}{k+1}\right) + \frac{m-1}{2}\left(\frac{k-1}{k+1}\right)$$

$$= \mathcal{R}\left(\sum_{i=1}^{m} \tilde{a}_i\right) + \frac{m-1}{2}\left(\frac{k-1}{k+1}\right)$$

■

Note that for $\tilde{a}, \tilde{b} \in F_k(\mathbb{R})$, in generally $\tilde{a} \otimes \tilde{b}$ does not belong to $F_k(\mathbb{R})$, even for k=1. To overcome this limitation for k=1, Ganesan and Veeramani (2006) have proposed a new product $\odot_{GV}$ for symmetric trapezoidal fuzzy numbers. In the following definition, we generalize their definition for k-scale trapezoidal fuzzy numbers.

Definition 3.3. Assume that $\tilde{a} = \left(a^L, a^U, a^\alpha, a^\beta\right)$ and $\tilde{b} = \left(b^L, b^U, b^\alpha, b^\beta\right)$ are two non-negative trapezoidal fuzzy numbers, we define

$$\tilde{a}\odot_{NK}\tilde{b}=\left[\left(\frac{a^L+a^U}{2}\right)\left(\frac{b^L+b^U}{2}\right)-\left(\frac{a^Ub^U-a^Lb^L}{2}\right),\left(\frac{a^L+a^U}{2}\right)\left(\frac{b^L+b^U}{2}\right)+\left(\frac{a^Ub^U-a^Lb^L}{2}\right),a^Ub^\alpha+b^Ua^\alpha,a^Ub^\beta+b^Ua^\beta\right]$$

From this definition, if $\tilde{a},\tilde{b}\in F_k(\mathbb{R})$, then $\tilde{a}\odot_{NK}\tilde{b}\in F_k(\mathbb{R})$. It is clear that for $\lambda\geq 0$, $\lambda\tilde{a}=(\lambda a^L,\lambda a^U,\lambda a^\alpha,\lambda a^\beta)$.

4. Proposed method

Let $I=\{1,\dots,m\}, J=\{1,\dots,n\}$. In our model, there are some equalities and inequalities, and we will see, using proposed ranking function; we can convert it to a crisp linear programming. We assume that $\tilde{c}_j$, $\tilde{a}_{ij}$, $\tilde{x}_j$, and $\tilde{b}_i$ are non-negative k-scale trapezoidal fuzzy numbers. Consider the following model which is named as P_{NK} model.

Maximize/Minimize $\sum_{j=1}^{n}\tilde{c}_j\odot_{NK}\tilde{x}_j$

Subject to (2.13)

$$\sum_{j=1}^{n}\tilde{a}_{ij}\odot_{NK}\tilde{x}_j\preccurlyeq,\approx,\succcurlyeq\tilde{b}_i,\qquad\forall i=1,2,\dots,m,$$
$$\tilde{x}_j\succcurlyeq\tilde{0},\qquad j=1,\dots,n.$$

Step 1: Assume that $\tilde{c}_j=\left(c_j^L,\ c_j^U,c_j^\alpha,c_j^\beta\right)$, $\tilde{x}_j=\left(x_j^L,x_j^U,x_j^\alpha,x_j^\beta\right)$, $\tilde{a}_{ij}=\left(a_{ij}^L,a_{ij}^U,a_{ij}^\alpha,a_{ij}^\beta\right)$, $\tilde{b}_i=\left(b_i^L,b_i^U,b_i^\alpha,b_i^\beta\right)$ .Therefore Problem (2.13) can be written as:

Maximize/Minimize $\sum_{j\in J}\left(c_j^L,c_j^U,c_j^\alpha,c_j^\beta\right)\odot_{NK}\left(x_j^L,x_j^U,x_j^\alpha,x_j^\beta\right)$

Subject to (2.14)

$$\sum_{j\in J}\left(a_{ij}^L,a_{ij}^U,a_{ij}^\alpha,a_{ij}^\beta\right)\odot_{NK}\left(x_j^L,x_j^U,x_j^\alpha,x_j^\beta\right)\preccurlyeq,\approx,\succcurlyeq\left(b_i^L,b_i^U,b_i^\alpha,b_i^\beta\right)\qquad i\in I$$
$$\left(x_j^L,x_j^U,x_j^\alpha,x_j^\beta\right)\succcurlyeq(0,0,0,0)\qquad j\in J$$

Step 2. Assume that

$$\left(c_j^L,c_j^U,c_j^\alpha,c_j^\beta\right)\odot_{NK}\left(x_j^L,x_j^U,x_j^\alpha,x_j^\beta\right)=\left(d_j^L,d_j^U,d_j^\alpha,d_j^\alpha\right),$$
$$\left(a_{ij}^L,a_{ij}^U,a_{ij}^\alpha,a_{ij}^\beta\right)\odot_{NK}\left(x_j^L,x_j^U,x_j^\alpha,x_j^\beta\right)=\left(e_j^L,e_j^U,e_j^\alpha,e_j^\alpha\right)$$

With these notations using Definition 3.3, Problem (2.14) can be written as

$$\text{Maximize/Minimize} \quad \sum_{j\in J}\left(d_j^L, d_j^U, d_j^\alpha, d_j^\alpha\right)$$

Subject to (2.15)

$$\sum_{j\in J}\left(e_j^L, e_j^U, e_j^\alpha, e_j^\alpha\right) \preccurlyeq, \approx, \succcurlyeq \left(b_i^L, b_i^U, b_i^\alpha, b_i^\beta\right) \qquad i \in I$$
$$\left(x_j^L, x_j^U, x_j^\alpha, x_j^\beta\right) \succcurlyeq (0,0,0,0) \qquad j \in J$$

Step 3. Using proposed ranking function which is given in Definition 3.1, we solve the following linear programming:

$$\text{Maximize/Minimize} \quad \mathcal{R}\left(\sum_{j\in J}\left(d_j^L, d_j^U, d_j^\alpha, d_j^\alpha\right)\right)$$

Subject to (2.16)

$$\mathcal{R}\left(\sum_{j\in J}\left(e_j^L, e_j^U, e_j^\alpha, e_j^\alpha\right)\right) \leq, =, \geq \mathcal{R}\left(b_i^L, b_i^U, b_i^\alpha, b_i^\beta\right) \quad i \in I$$
$$x_j^L - x_j^\alpha \geq 0 \qquad j \in J.$$

Step 4. Now, from Theorem 3.2, we convert (2.16) to the following model:

$$\text{Maximize/Minimize} \quad \sum_{i=1}^{n}\mathcal{R}\left(d_j^L, d_j^U, d_j^\alpha, d_j^\alpha\right) - \frac{n-1}{2}\left(\frac{k-1}{k+1}\right)$$

Subject to (2.17)

$$\sum_{i=1}^{n}\mathcal{R}\left(e_j^L, e_j^U, e_j^\alpha, e_j^\alpha\right) - \frac{n-1}{2}\left(\frac{k-1}{k+1}\right) \leq, =, \geq \mathcal{R}\left(b_i^L, b_i^U, b_i^\alpha, b_i^\beta\right), \quad i \in I,$$
$$x_j^L - x_j^\alpha \geq 0 \qquad j \in J.$$

Step 5. By solving the crisp programming problem (2.17), find the fuzzy optimal solution as: $x_j^* = \left(x_j^{L*}, x_j^{U*}, x_j^{\alpha*}, x_j^{\beta*}\right)$.

Step 6. Find the fuzzy optimal value of objective function by putting the optimal values of $x_j^* = \left(x_j^{L*}, x_j^{U*}, x_j^{\alpha*}, x_j^{\beta*}\right)$ in $\sum_{j\in J}\left(c_j^L, c_j^U, c_j^\alpha, c_j^\beta\right) \odot_{NK} \left(x_j^L, x_j^U, x_j^\alpha, x_j^\beta\right)$.

5. Illustrative example

In this section, we illustrate our method with an example. This example is a symmetric form of the illustrated model (P_{GV} model), which is given by Ganesan and Veeramani (2006) to may compare with our results.

Example 5.1. Consider the following problem:

$$\max[13,15,2,2]\odot_{NK}\tilde{x}_1 + [12,14,3,3]\odot_{NK}\tilde{x}_2 + [15,17,2,2]\odot_{NK}\tilde{x}_3$$

Subject to

$$12\tilde{x}_1+13\,\tilde{x}_2+12\tilde{x}_3 \preccurlyeq [475, 505, 6,6],$$
$$14\tilde{x}_1+0\ \ \tilde{x}_2+13\tilde{x}_3 \preccurlyeq [460, 480, 8,8],$$
$$12\tilde{x}_1+15\,\tilde{x}_2+0\ \ \tilde{x}_3 \preccurlyeq [465, 495, 5,5],$$
$$\tilde{x}_1 \succcurlyeq \tilde{0}, \tilde{x}_2 \succcurlyeq \tilde{0}, \tilde{x}_3 \succcurlyeq \tilde{0}$$

Step 1: Assume that $\tilde{x}_j = \left(x_j^L, x_j^U, x_j^\alpha, x_j^\beta\right)$, then we have

$$\max[13,15,2,2]\odot_{NK}\left(x_1^L, x_1^U, x_1^\alpha, x_1^\beta\right) + [12,14,3,3]\odot_{NK}\left(x_2^L, x_2^U, x_2^\alpha, x_2^\beta\right) + [15,17,2,2]\odot_{NK}\left(x_3^L, x_3^U, x_3^\alpha, x_3^\beta\right)$$

Subject to

$$\left(12x_1^L, 12\,x_1^U, 12\,x_1^\alpha, 12x_1^\beta\right)+\left(13x_2^L, 13\,x_2^U, 13\,x_2^\alpha, 13x_2^\beta\right)+\left(12x_3^L, 12x_3^U, 12\,x_3^\alpha, 12x_3^\beta\right) \preccurlyeq [475, 505, 6,6],$$
$$\left(14x_1^L, 14\,x_1^U, 14\,x_1^\alpha, 14x_1^\beta\right)+\left(13x_3^L, 13\,x_3^U, 13\,x_3^\alpha, 13x_3^\beta\right) \preccurlyeq [460, 480, 8,8],$$
$$\left(12x_1^L, 12\,x_1^U, 12\,x_1^\alpha, 12x_1^\beta\right)+\left(15x_2^L, 15\,x_2^U, 15\,x_2^\alpha, 15x_2^\beta\right) \preccurlyeq [465, 495, 5,5],$$
$$x_1^L - x_1^\alpha \geq 0, x_2^L - x_2^\alpha \geq 0, x_3^L - x_3^\alpha \geq 0$$

Step 2: Obtain the fuzzy equivalent problem as:

$$\max\left(\frac{27x_1^L}{2} - \frac{x_1^U}{2}, \frac{x_1^L}{2} + \frac{29x_1^U}{2}, 2x_1^U + 15x_1^\alpha, 2x_1^U + 15x_1^\beta\right) + \left(\frac{25x_2^L}{2} - \frac{x_2^U}{2}, \frac{x_2^L}{2} + \frac{27x_2^U}{2}, 3x_2^U + 14x_2^\alpha, 3x_2^U + 14x_2^\beta\right) + \left(\frac{31x_3^L}{2} - \frac{x_3^U}{2}, \frac{x_3^L}{2} + \frac{33x_3^U}{2}, 2x_3^U + 17x_3^\alpha, 2x_3^U + 17x_3^\beta\right)$$

Subject to

$$\left(12x_1^L, 12\,x_1^U, 12\,x_1^\alpha, 12x_1^\beta\right)+\left(13x_2^L, 13\,x_2^U, 13\,x_2^\alpha, 13x_2^\beta\right)+\left(12x_3^L, 12x_3^U, 12\,x_3^\alpha, 12x_3^\beta\right) \preccurlyeq [475, 505, 6,6],$$
$$\left(14x_1^L, 14\,x_1^U, 14\,x_1^\alpha, 14x_1^\beta\right)+\left(13x_3^L, 13\,x_3^U, 13\,x_3^\alpha, 13x_3^\beta\right) \preccurlyeq [460, 480, 8,8],$$
$$\left(12x_1^L, 12\,x_1^U, 12\,x_1^\alpha, 12x_1^\beta\right)+\left(15x_2^L, 15\,x_2^U, 15\,x_2^\alpha, 15x_2^\beta\right) \preccurlyeq [465, 495, 5,5],$$
$$x_1^L - x_1^\alpha \geq 0, x_2^L - x_2^\alpha \geq 0, x_3^L - x_3^\alpha \geq 0$$

Step 3. We must compute the crisp equivalent problems as:

$$\max \mathcal{R}\Bigg(\left(\frac{27x_1^L}{2}-\frac{x_1^U}{2},\frac{x_1^L}{2}+\frac{29x_1^U}{2},2x_1^U+15x_1^\alpha,2x_1^U+15x_1^\beta\right)+\left(\frac{25x_2^L}{2}-\frac{x_2^U}{2},\frac{x_2^L}{2}+\frac{27x_2^U}{2},3x_2^U+14x_2^\alpha,3x_2^U+14x_2^\beta\right)+\left(\frac{31x_3^L}{2}-\frac{x_3^U}{2},\frac{x_3^L}{2}+\frac{33x_3^U}{2},2x_3^U+17x_3^\alpha,2x_3^U+17x_3^\beta\right)\Bigg)$$

Subject to

$$\mathcal{R}\left(\left(12x_1^L,12\,x_1^U,12\,x_1^\alpha,12x_1^\beta\right)+\left(13x_2^L,13\,x_2^U,13\,x_2^\alpha,13x_2^\beta\right)+\left(12x_3^L,12x_3^U,12\,x_3^\alpha,12x_3^\beta\right)\right)\le\mathcal{R}(475,505,6,6),$$

$$\mathcal{R}\left(\left(14x_1^L,14\,x_1^U,14\,x_1^\alpha,14x_1^\beta\right)+\left(13x_3^L,13\,x_3^U,13\,x_3^\alpha,13x_3^\beta\right)\right)\le\mathcal{R}(460,480,8,8),$$

$$\mathcal{R}\left(\left(12x_1^L,12\,x_1^U,12\,x_1^\alpha,12x_1^\beta\right)+\left(15x_2^L,15\,x_2^U,15\,x_2^\alpha,15x_2^\beta\right)\right)\le\mathcal{R}(465,495,5,5),$$

$$x_1^L-x_1^\alpha\ge0,x_2^L-x_2^\alpha\ge0,x_3^L-x_3^\alpha\ge0$$

Step 4. In this example $k=1$, therefore $\frac{n-1}{2}\left(\frac{k-1}{k+1}\right)=0$. So we compute

$$\max \mathcal{R}\left(\frac{27x_1^L}{2}-\frac{x_1^U}{2},\frac{x_1^L}{2}+\frac{29x_1^U}{2},2x_1^U+15x_1^\alpha,2x_1^U+15x_1^\beta\right)+\mathcal{R}\left(\frac{25x_2^L}{2}-\frac{x_2^U}{2},\frac{x_2^L}{2}+\frac{27x_2^U}{2},3x_2^U+14x_2^\alpha,3x_2^U+14x_2^\beta\right)+\mathcal{R}\left(\frac{31x_3^L}{2}-\frac{x_3^U}{2},\frac{x_3^L}{2}+\frac{33x_3^U}{2},2x_3^U+17x_3^\alpha,2x_3^U+17x_3^\beta\right)$$

Subject to

$$\mathcal{R}\left(12x_1^L,12\,x_1^U,12\,x_1^\alpha,12x_1^\beta\right)+\mathcal{R}\left(13x_2^L,13\,x_2^U,13\,x_2^\alpha,13x_2^\beta\right)+\mathcal{R}\left(12x_3^L,12x_3^U,12\,x_3^\alpha,12x_3^\beta\right)\le\mathcal{R}(475,505,6,6),$$

$$\mathcal{R}\left(14x_1^L,14\,x_1^U,14\,x_1^\alpha,14x_1^\beta\right)+\mathcal{R}\left(13x_3^L,13\,x_3^U,13\,x_3^\alpha,13x_3^\beta\right)\le\mathcal{R}(460,480,8,8),$$

$$\mathcal{R}\left(12x_1^L,12\,x_1^U,12\,x_1^\alpha,12x_1^\beta\right)+\mathcal{R}\left(15x_2^L,15\,x_2^U,15\,x_2^\alpha,15x_2^\beta\right)\le\mathcal{R}(465,495,5,5),$$

$$x_1^L-x_1^\alpha\ge0,x_2^L-x_2^\alpha\ge0,x_3^L-x_3^\alpha\ge0$$

Therefore, thanks to definition

$$\mathcal{R}(\tilde{a})=a^L+a^U+\frac{1}{2}\left(\frac{a^\beta-a^\alpha}{a^\alpha+a^\beta}\right).$$

we must solve the following linear programming:

$$\max\frac{27x_1^L}{2}-\frac{x_1^U}{2}+\frac{x_1^L}{2}+\frac{29x_1^U}{2}+\frac{25x_2^L}{2}-\frac{x_2^U}{2}+\frac{x_2^L}{2}+\frac{27x_2^U}{2}+\frac{31x_3^L}{2}-\frac{x_3^U}{2}+\frac{x_3^L}{2}+\frac{33x_3^U}{2}$$

Subject to

$$12x_1^L+12\,x_1^U+13x_2^L+13\,x_2^U+12x_3^L+12x_3^U\le475+505,$$

$$14x_1^L+14\,x_1^U+13x_3^L+13\,x_3^U\le460+480,$$

$$12x_1^L+12\,x_1^U+15x_2^L+15\,x_2^U\le465+495,$$

$x_1^L - x_1^\alpha \geq 0, x_2^L - x_2^\alpha \geq 0, x_3^L - x_3^\alpha \geq 0$

with some simplifications, we have

$$\max 14x_1^L + 14x_1^U + 13x_2^L + 13x_2^U + 16x_3^L + 16x_3^U$$

Subject to

$$12x_1^L + 12\,x_1^U + 13x_2^L + 13\,x_2^U + 12x_3^L + 12x_3^U \leq 980,$$
$$14x_1^L + 14\,x_1^U + 13x_3^L + 13\,x_3^U \leq 940,$$
$$12x_1^L + 12\,x_1^U + 15x_2^L + 15\,x_2^U \leq 960,$$
$$x_1^L - x_1^\alpha \geq 0, x_2^L - x_2^\alpha \geq 0, x_3^L - x_3^\alpha \geq 0$$

In the following table, the results of proposed method and Ganesan and Veeramani (2006) method are compared.

	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{z}$	$\mathcal{R}(\tilde{z})$
Ganesan and Veeramani (2006)	$\tilde{0}$	$\left(\frac{415}{169},\frac{1045}{169},\frac{174}{169},\frac{174}{169}\right)$	$\left(\frac{460}{13},\frac{480}{13},\frac{8}{13},\frac{8}{13}\right)$	$\left(\frac{94235}{169},\frac{120265}{169},\frac{19819}{169},\frac{19819}{169}\right)$	1269.2307
Proposed method	$\tilde{0}$	(8.639,0,0,0)	(72.308,0,0,0)	(1228.7615, 40.4735,0,0)	1269.2350

Note that in Ganesan and Veeramani (2006) method we need to add some slack to our model to make it a standard model and it might complex the problem while in proposed method we do not need to do it.

6. Conclusion

In this paper, we focused on a convenient kind of fuzzy mathematical models where there are some shortcomings on the essential tools such as arithmetic operations, fuzzy ordering and in particular an efficient solving process. The main results in our work are listed as follows:

- Hadi method as a suitable tool for fuzzy ordering is extended for the generalized/symmetric trapezoidal fuzzy numbers entitled k- scale trapezoidal fuzzy numbers.
- A new role for multiplication of two k- scale trapezoidal fuzzy numbers are established.
- Some new results in order to proposing a new ranking function are proved.
- To overcoming the limitation of the proposed solving method by Ganesan and Veeramani (2006), in particular the shortcoming which is there in their proposed product for symmetric trapezoidal fuzzy numbers, we extended the multiplication of fuzzy numbers to the general kind, in fact to the k- scale type.
- Study of a more general model entitled P_{NK} fully fuzzy linear programming problem, where the parameters are not essentially in symmetric type. In particular, the proposed method for solving the P_{NK} model as a general kind of P_{GV} model can solve both kinds of these problems with a simpler method. The illustrative example verifies our claim.

Finally, we emphasize that this approach can be extended and use for the other convenient models such as Transportation problem, Assignment problem and etc, in the future studies.

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