Iranian Journal of Operations Research
Vol. 9, No. 2, 2018, pp. 49-80

A Scenario-Based Nonlinear Programming Model for a Two-Level Inventory Control: A Case Study in Dairy Product Industry

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Here, a novel scenario-based two-level inventory control model with a limited budget is formulated. The demand during the selling period is considered to follow a uniform probability distribution. It is assumed that there will be some customers who are willing to wait for their demands to be satisfied; thus, a service level is considered for these customers. The aim is to find the optimal order quantities of the products and the required raw materials at the beginning of the selling period such that the relevant expected total profit obtained during the period is maximized. A penalty function along with a barrier method is proposed to solve the developed nonlinear stochastic programming problem. The problem is solved under different scenarios including good, fair, and low demands. Finally, a case study in a dairy manufacturing company is provided to illustrate the application of the proposed methodology in real-world inventory control systems.

***Keywords**: Newsvendor model; Two-level inventory control; Scenario-based modeling; Penalty and barrier functions.*

Manuscript was received on 12/07/2018 revised on 16/10/2018 and accepted for publication on 27/10/2018.

1. Introduction

On time delivery of quality products in proper quantities to the customers is a fundamental aim of any manufacturing company working in today's global market. However, the uncertainty involved in demand is a major problem the companies face with to increase service level and to decrease cost. One of the most popular approaches used by companies to control the demand uncertainty during the selling period is the so-called newsvendor modeling devised for the single-period inventory control problems. In this approach, a prior knowledge of demand variations is required to prepare products before the selling period. The newsvendor model was firstly introduced by Hadley and Whitin [1] with the extensions provided by Silver et al. [2] and Khouja [3] later on. As the selling process became more complex, further extensions of the model involving service level, lost sales, salvage value, emergency order, risk and many other interesting parameters were proposed by various researchers in the literature.

The present study considers the single-period inventory control problem as a two-level model in which the vendor procures raw materials and converts a certain amount of the raw materials into final products before the selling season starts. As the cost of the raw materials left at the end of the period

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is lower than that of the products, it is more appropriate for the vendor to have some raw materials remained at the end of the selling period, although it is ideal for the vendor to sell out all the purchased raw materials and products with the highest profit and least lost sales. When the demand during the selling period exceeds the stocked products, it is assumed that the vendor turns a part of the raw materials into the final products. As this transformation is time-consuming, some customers cannot wait and at least some impatient customers' demands will be lost. Therefore, the vendor should decide how much raw material and products to be stocked at the beginning of the period in order to maximize the profit. Similar to all newsvendor models, in our work here, the vendor predicts the sale quantity before the beginning of the selling season using the historical demand data and prepares raw materials and products according to this prediction. More specifically, a scenario-based two-level single-period inventory problem under a limited budget is investigated in an attempt to examine the influence of the demand prediction on the newsvendor's profit. The modeling and the solution approach proposed here are applicable in different real-world systems such as inventory systems of multi-echelon supply chains, spare part distribution systems, food industries and in general in any inventory control system involving products and raw materials. In short, our main contributions are:

1. A new model is developed to enable the newsvendor to compare her decision under different scenarios.
2. When the demand during the selling period exceeds the stored products at the beginning of the period, a re-production during the period is allowed for those customers who are willing to wait to receive their products.
3. A stochastic optimization approach is taken to deal with uncertain demand in order to investigate the effect of demand prediction on the total expected profit. In other words, different cases (demand scenarios) are considered to investigate their effects on the newsvendor expected profit.

2. Literature Review

Several extensions of the single-period (newsvendor) problem have been proposed in the literature, since the pioneering work of Hadley and Whitin [1] was introduced for a single product. Vairaktarakis [4] investigated a multi-item newsboy problem with a budget constraint and made a single-period stocking decision prior to realizing the demand for all items. Abdel-Malek et al. [5] developed a multi-product newsvendor model with budget limitation and proposed an exact solution approach when the demand follows a uniform distribution. Later, their work was extended by Abdel-Malek and Montanari [6] for a multi-product newsboy problem with the budget constraint assumed in three ranges of large, medium and very tight. Some numerical examples were solved to illustrate the application of the proposed procedures. Mostard and Teunter [7] analyzed a newsvendor problem with resalable returns, assuming that all unsatisfied demands were lost. They derived a simple closed-form equation to determine the optimal order quantity given the demand distribution, the probability that a sold product was returned and all relevant revenues and costs. Abdel-Malek and Areeratchakul [8] devised a quadratic programming solution approach for a multi-product newsvendor problem with side constraints. Their model considers a lower bound on the demand and is suitable to conduct sensitivity analysis to allow for adjusting the available resources when necessary. Niederhoff [9] proposed an approximating programming technique to solve a multi-product newsvendor problem with independent products demands. He found the optimal solution for any demand distribution using a convex separable program. Panda et al. [10] developed a mathematical model for a single-period multi-product manufacturing system consisting of stochastically imperfect items with a continuous stochastic demand under budget and shortage constraints. They solved their problem using a nonlinear optimization technique called the generalized reduced gradient method. Zhang and Hua

[11] established the structural properties for the optimal decisions of their proposed profit-maximization model, in which the procurement strategy for newsboy product was designed as a portfolio contract. They proposed an efficient algorithm to solve the problem. Huang et al. [12] studied a multi-product competitive newsvendor problem with shortage penalty cost and partial product substitution. They devised an iterative algorithm on the basis of approximating the effective demand as well as the expected profit function for each product. Performing sensitivity analysis, Khanra et al. [13] identified conditions for symmetry and skewness of the cost deviation and established a lower bound on the cost deviation for a symmetric unimodal demand distribution. Recently, Kim et al. [14] introduced a multi-period newsvendor problem and formulated it as a multi-stage stochastic programming model with integer recourse decisions.

To name a few works on the two-level inventory control problem, Axsäter [15] extended a new model in two ways: first he provided a complete probability distribution for the retailer inventory levels, and second he generalized the case from having Poisson demands to compound Poisson demands. He determined the distribution of the inventory levels at the retailers in the steady state. Tee and Rossetti [16] developed a robust version of a standard model for a two-echelon inventory system proposed by Axsäter [15]. They concluded that their model performed well at the low demand and large retailer order batch size situation. For a single-item, two-echelon, continuous-review inventory model, Hill et al. [17] assumed a number of retailers to have stock replenished from a central warehouse and the demand processes on the retailers to be independent Poisson. They presented a means of studying the steady-state behavior of a multi-level batch ordering model which, on the basis of simulation results, provided very accurate answers. Pasandideh et al. [18] proposed a new model for a two-echelon inventory control system for a non-repairable item where the system consisted of one warehouse and m identical retailers. They solved their nonlinear integer-programming problem using a parameter-tuned genetic algorithm. Tsai and Zheng [19] presented a simulation optimization algorithm to solve a two-echelon constrained inventory control problem. They determined the optimal setting of stocking levels to minimize the total inventory investment costs while satisfying the expected response time targets for each field depot. They showed that their proposed algorithm required less simulation effort to guarantee to achieve a better solution than the ones obtained by other existing approaches. Alvarez and van der Heijden [20] considered a two-echelon supply chain system under Poisson demand with a one-for-one replenishment policy. They assumed that the demand was lost if no items were available at the local warehouse, central depot or in the pipeline in between. In order to approximate the service level, they proposed a simple and fast approach. Recently, Priyan and Uthayakumar [21] proposed a two-echelon multi-product multi-constraint product returns inventory model with a permissible delay in payments. A distributor and a warehouse consisting of a serviceable part and a recoverable part were considered in their work. They showed that the model of their problem was a constrained nonlinear programming one and solved it using the Lagrangian relaxation method. Keramatpour et al. [22] developed a new model for the two-level newsvendor problem with budget constraint. They considered a service level for customers and solved their problem using meta-heuristic algorithms.

The rest of our work is organized as follows. The problem statement, the assumptions involved, and the notation are presented in Section 3. In Section 4, the problem is modeled. Under a uniform distribution of the demand a real case-study is investigated in Section 5 to demonstrate the application of the proposed approach. The convexity proof and the obtained results are provided in Section 6. Finally, the conclusion and future research directions are presented in Section 7. Some necessary mathematical derivations are given in Appendices A and B.

3. Problem Statement

A single-item single-period two-level inventory control problem with a limited budget is investigated. Similar to the classical newsvendor problem, the demand is a random variable and shortage is allowed in the form of lost sales. It is assumed that when the demand during the selling period exceeds the stocked products at the beginning of the period, the newsvendor turns a part of the raw materials into the final products. However, as turning the raw materials into the final products is time-consuming, all customers cannot wait. As a result, some of the impatient customers' demands will be lost. Therefore, the newsvendor should decide how much raw materials and products to prepare at the beginning of the selling period in order to maximize her profit. Although some researchers such as Serel [23] and Karimi et al. [24] considered this special case as the so-called "emergency order," the approach taken here is different. Here, the demand is a random variable and shortage is permitted. As such, the newsvendor profit is directly affected by the actual demand happening during the period. It is obvious that in the newsvendor problem, the final product inventory is prepared at the beginning of the period based on the predicted demand. So, the demand pattern of the final product should be analyzed in depth to maximize the newsvendor's expected profit. To this aim, three scenarios including good, fair, and low are assumed for the product demand. The newsvendor decides how much products and raw materials should be prepared at the beginning of the period under different scenarios. The first major contribution of our work is to develop a new modeling for a two-level inventory control problem. A real-world case is provided to show the efficiency of the developed model. Our second major contribution is the proposed solution method. An exact solution method rarely used by researchers is presented.

3.1. Assumptions and Notations

As stated before, it is assumed that the newsvendor prepares raw materials and final products just before the beginning of the selling period. If the customers are willing to wait for their products in case a shortage happens during the period, i.e. the demand during the period exceeds the prepared products, then the vendor produces products using raw materials, each with a different usage coefficient.

In general, the following assumptions are made:

1. There is only a single type of product for sale during the period.
2. There is a limited budget available to purchase raw materials and to transform them into final products just before the beginning of the selling period.
3. A product is produced using raw materials with different usage coefficients.
4. Raw materials can be stocked just once before the beginning of the selling period.
5. Products are produced and stocked before the selling period.
6. If the demand during the period exceeds the already stocked products at the start of the selling period, then some customers wait for their orders to get ready for delivery.
7. In addition to the products that are prepared prior to the selling period, some products can be prepared and delivered during the period for customers who are willing to wait for their orders to be met.
8. Shortage is considered as lost sale.
9. Excess products and raw materials left at the end of the period are sold at a lower price.
10. As revenue is obtained by selling products within the period, no budget constraint is considered for the transformation of raw materials into final products during the period.

11. The demand for the product is assumed to follow a discrete set of three scenarios: good, fair, and low.
12. The demand quantity does not affect the price of the products.

3.2. Notations

The following indices, parameters, and decision variables are used throughout the paper.

3.2.1. Indices

$i =$ index used for a raw material ($i = 1, 2, 3, \dots, I$)
$s =$ index used for a demand scenario ($s = 1, 2, 3$)

3.2.2. Parameters

$T =$ Unit cost of transforming raw materials into products
$\gamma_i =$ Consumption rate of ith raw material in products
$C_i =$ Unit purchasing cost of raw material i needed to produce one unit of the finished product
$D_i =$ Unit discounted sale price of ith raw material left at the end of the selling period
$D' =$ Unit discounted sale price of the product remained at the end of the selling period
$H_i =$ Holding cost per unit time of ith raw material
$H' =$ Holding cost per unit time of the product
$P =$ Selling price per unit of the product
$\mu =$ Service level or a percentage of the customers willing to wait (if the product is sold out during the selling period)
$\pi =$ shortage cost per unit of the products
$x_S =$ Customers' demand for the products in scenario s (a random variable)
$f_X(x_S) =$ Probability density function of the demand in scenario s
$F_X(x_S) =$ Cumulative probability distribution function of the demand in scenario s
$B =$ available budget to purchase raw materials and to transfer them into the final product
$p_S =$ probability of scenario s to happen ($0 < p_S < 1$)

3.2.3. Decision Variables

$Q_{R,S} =$ Quantity of the raw materials purchased before the beginning of the selling period for scenario-s demand
$Q_{F,S} =$ Quantity of the final product stocked just before the beginning of the selling period for scenario-s demand.

Note that the maximum number of products one can manufacture from $Q_{R,S}$ with the raw material consumption rates γ_i is $Q^+_{F,S}$, i.e., $Q^+_{F,S} = \min_{i}\left(\frac{Q_{R,S}}{\gamma_i}\right)$. After solving the problem and obtaining the optimal solution for $Q_{R,S}$, the quantity of the i-th raw material is easily obtained by $\gamma_i Q_{R,S}$.

4. Mathematical Formulation

Given the assumptions and notations, the product demand during the selling period in each scenario happens to fall within the following three states.

***State* 1:** Product demand is not bigger than the quantity of the products prepared before the period, i.e. $x_S \leq Q_{F,S}$. In this case, products are partially sold and fractions of products alongside the raw materials that are not used in production are sold at lower prices at the end of the period.

***State* 2:** All products prepared before the selling period are sold out during the period, i.e., $x_S \geq Q_{F,S}$. In this case, fractions of raw materials are transformed into the products during the selling period and the remaining products are sold at lower prices at the end of the period. Moreover, the demands of some impatient customers who cannot wait are lost. Note that in this case $\left(x_S - Q_{F,S}\right) \leq Q^+_{F,S}$, and hence $Q_{F,S} \leq x_S \leq Q^+_{F,S}/\mu$.

***State* 3:** All products and raw materials prepared before the selling period are sold out, i.e. the product demand is greater than the quantity of all products prepared at the beginning of the selling period and that the number of customers who are willing to wait for their demands be met is so high that all the procured raw materials at the start of the period are transformed into the products during the period. In this case, no products and raw materials remain at the end of the period to be sold at lower prices and the vendor will just have some lost sales. Thus, we have $Q_{F,S} + Q^+_{F,S}/\mu \leq x_S$.

Figure 1 depicts a summary of the above three states.

Using the above assumptions, the costs and the incomes in each state are as follows.

***State* 1:**

Income includes the sale of a fraction of products that is equal to the quantity of the demand plus sale of the remaining raw materials plus the sale of the remaining products at the end of the period at lower prices. In other words,

$$Income_1 = P(x_s) + \sum_{i=1}^{I} D_i\left(Q_{R,S}\right) + D'\left(Q_{F,S} - x_S\right). \tag{1}$$

Costs include the purchasing cost of raw materials plus the transformation cost of raw materials into products plus the holding costs of raw materials along with the holding cost of a fraction of products until the end of the period, that is,

$$Cost_1 = \sum_{i=1}^{I} C_i\left(Q_{R,S}\right) + \left(T + \sum_{i=1}^{I} C_i\right) Q_{F,S} + \sum_{i=1}^{I} H_i\left(Q_{R,S}\right) + H'\left(Q_{F,S} - x_S\right). \tag{2}$$

Note that $Q_{R,S}$ is the number of raw materials purchased at the beginning of the period that is used to produce the products during the selling period if required. The cost of raw materials used in the product ($Q_{,S}$) at the beginning of the period is considered separately. Then, the newsvendor's expected profit is obtained using (3) below, given the demand scenario:

$$Profit_1 = \emptyset_1 = \int_0^{Q_{F,S}} (Income_1 - Cost_1) f_X(x_S)\, dx_S. \tag{3}$$

***State* 2:**

Income includes the sale of all products plus sale of a fraction of the products produced during the period plus the sale of raw materials left at the end of the period at a lower price. In other words,

$$Income_2 = P(Q_{F,S}) + P(\mu)(x_S - Q_{F,S}) + \sum_{i=1}^{I} D_i[Q_{R,S} - \gamma_i(\mu)(x_S - Q_{F,S})], \tag{4}$$

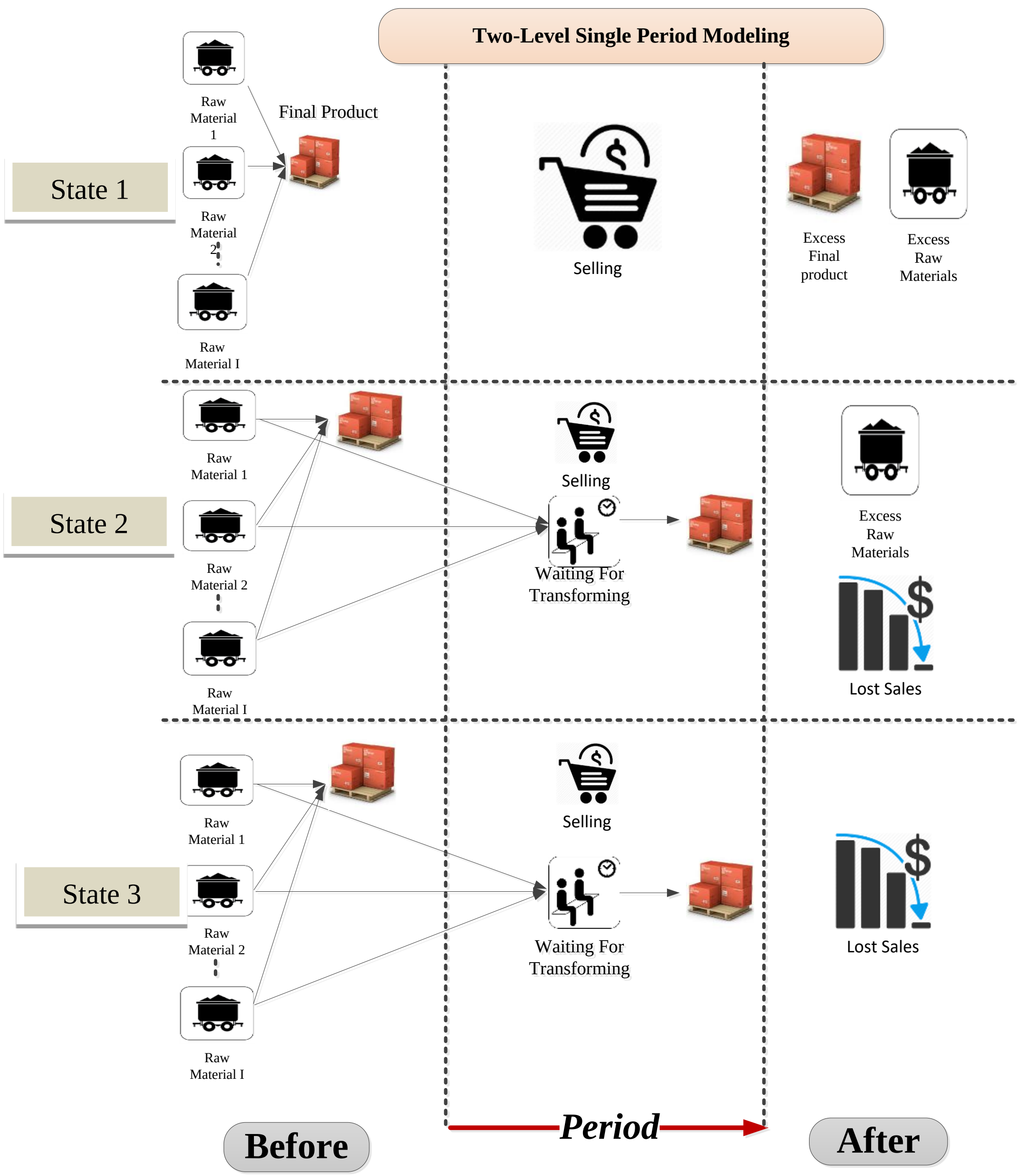

Figure 1. Different states of the two-level single-period problem

where the last term on the right-hand-side of (4) is constrained to be non-negative as a constraint of the problem (see Equation (11) as given later).

Costs include the purchasing cost of raw materials plus the transformation cost of raw materials into the products at the start of the period plus the transformation cost of raw materials during the period plus the lost cost of impatient customers who cannot wait plus the holding cost of raw materials until the end of the period, that is,

$$Cost_2 = \sum_{i=1}^{I} C_i(Q_{R,S}) + \left(T + \sum_{i=1}^{I} C_i\right) Q_{F,S} + T(\mu)(x_S - Q_{F,S}) + \pi(1-\mu)(x_S - Q_{F,S}) \\ + \sum_{i=1}^{I} H_i[Q_{R,S} - \gamma_i(\mu)(x_S - Q_{F,S})]. \tag{5}$$

Then, $Profit_2$, added to $Profit_1$, that may be obtained in $(Q_{F,S}, Q_{F,S} + Q_{F,S}^{+}/\mu)$, becomes

$$Profit_2 = \emptyset_2 = \int_{Q_{F,S}}^{Q_{F,S}+\frac{Q_{F,S}^{+}}{\mu}} (Income_2 - Cost_2) f_X(x_S) dx_S. \tag{6}$$

Note that in (6), if all the customers can wait, i.e. $\mu = 1$, and if the consumption rates of raw materials are all equal to 1, i.e. , $\gamma_i = 1$, then all the products and raw materials are sold out and the interval of the integration will change from $Q_{F,S}$ to $Q_{F,S} + Q_{R,S}$. This means that the demand that exceeds the quantity of the prepared products is at most as high as all prepared the products at the beginning of the period plus the quantity produced during the period using the raw materials provided at the start of the period.

***State* 3:**

Income includes sales of all the prepared products at the beginning of the period plus the sale of all the products produced from raw materials during the period. In other words

$$Income_3 = P(Q_{F,S} + Q_{F,S}^{+}). \tag{7}$$

Costs include the purchasing cost of raw materials plus the transformation cost of raw material into the products plus the transformation of the raw material into the products during the selling period given the consumption factor plus the loss of impatient customers who cannot wait. In other words,

$$Cost_3 = \sum_{i=1}^{I} C_i(Q_{R,S}) + \left(T + \sum_{i=1}^{I} C_i\right) Q_{F,S} + T(Q_{F,S}^{+}) + \pi(x_S - Q_{F,S} - Q_{F,S}^{+}). \tag{8}$$

In this state, the newsvendor's expected profit is obtained by (9) below based on the demand interval:

$$Profit_3 = \emptyset_3 = \int_{Q_{F,S}+\frac{Q_{F,S}^{+}}{\mu}}^{\infty} (Income_3 - Cost_3) f_X(x_S) dx_S. \tag{9}$$

Based on the expected profits obtained for states 1-3 in (3), (6), and (9) respectively, the newsvendor's expected profit is derived by $Z = (\emptyset_1 + \emptyset_2 + \emptyset_3)$. This profit is aimed to be maximized within the budget constraint as

$$\sum_{i=1}^{I} C_i(Q_{R,S}) + (T + Q_{F,S}) \leq B. \tag{10}$$

The inequality (10) shows that the purchasing and the transformation costs of raw materials into the products cannot exceed the available budget before the selling period. Note that as the revenue is obtained by selling the products within the period, no budget constraint is assumed to transform raw materials into the products during the period. Moreover, the total quantity of the products to be produced either at the start or within the selling period cannot exceed the potential number of products that can be produced using the raw materials based on their consumption factor. In other words,

$$[Q_{F.S} + \mu(x_S - Q_{F,S})] \leq \frac{Q_{R,S}}{\gamma_i}. \tag{11}$$

In short, the mathematical formulation of the problem at hand is

$$\begin{aligned}
&\max Z = (\emptyset_1 + \emptyset_2 + \emptyset_3)\\
&\text{s.t.}\\
&(1)\ \emptyset_1 = \int_{0}^{Q_{F,S}} (Income_1 - Cost_1) f_X(x_S) dx_S\\
&(2)\ \emptyset_2 = \int_{Q_{F,S}}^{Q_{F,S}+\frac{Q_{F,S}^{+}}{\mu}} (Income_2 - Cost_2) f_X(x_S) dx_S\\
&(3)\ \emptyset_3 = \int_{Q_{F,S}+\frac{Q_{F,S}^{+}}{\mu}}^{\infty} (Income_3 - Cost_3) f_X(x_S) dx_S\\
&(4) \sum_{i=1}^{I} C_i(Q_{R,S}) + (T + Q_{F,S}) \leq B\\
&(5)\ [Q_{F,S} + \mu(x_S - Q_{F,S})] \leq \frac{Q_{R,S}}{\gamma_i}\\
&Q_{F,S}.\, Q_{R,S} > 0, \qquad \forall\, S = 1, 2, 3.
\end{aligned} \tag{12}$$

where the *Incomes* and the Costs in States 1-3 have been obtained previously. In the above model, the last constraint shows that the quantities of raw materials and products must be positive. It should be mentioned that the above model can be easily extended to a multi-product case by adding a new index for each product in all the necessary parameters.

4.1. Case of Uniform Distribution of the Demand

As used in previous studies, a uniform probability distribution in the interval $[a.b]$is a proper choice for probable values of the random demand between a and b (Wanke [25]). This is also suitable for new products whose demand is unknown. The properties of a uniform distribution allow one to obtain a maximum and a minimum value as simple estimations of the newsvendor's expected profit.

Stochastic programming deals with situations where some or all of the parameters of the optimization problem are considered to be stochastic rather than deterministic quantities. In this section, the demand for the product is assumed to be a random variable X that follows a uniform distribution in the interval $[a.b]$. As the problem is examined using a scenario-based approach, the probability distribution of the demand under scenario s , i.e. x_S , is:

$$f(x_S) = \begin{cases} \dfrac{1}{b_S - a_S}, & \text{if } a_S \leq x_S \leq b_S \\ 0, & \text{otherwirse.} \end{cases} \quad \text{with } 0 \leq a_S \leq b_S \tag{13}$$

Given the objective function presented in the previous section and the probability density function of the demand given by (13), the mathematical formulation of the expected profit for each state follows. Note that the expected profit in each state is used in the integration first. Then, the start and the endpoints of the integral are applied to evaluate the expected profit.

***State* 1:**

$$\begin{aligned} \emptyset_1 &= \left(\begin{matrix} \left\{ \frac{1}{2} P(x_S^2) + x_S \sum_{i=1}^{I} D_i(Q_{R.S}) + x_S(D') \left(Q_{F.S} - \frac{1}{2} x_S \right) \right\} \\ - \left\{ \begin{matrix} x_S \sum_{i=1}^{I} C_i(Q_{R.S}) + x_S \left(T + \sum_{i=1}^{I} C_i \right) Q_{F.S} + \ x_S \sum_{i=1}^{I} H_i(Q_{R.S}) + \\ x_S(H') \left(Q_{F.S} - \frac{1}{2} x_S \right) \end{matrix} \right\} \end{matrix} \right) \Bigg|_0^{Q_{F.S}} \frac{1}{b_S - a_S} \\ &= \left(\begin{matrix} \left\{ \frac{1}{2} P(Q_{F.S}^2) + Q_{F.S} \sum_{i=1}^{I} D_i(Q_{R.S}) + \frac{1}{2}(D') Q_{F.S}^2 \right\} \\ - \left\{ \begin{matrix} Q_{F.S} \sum_{i=1}^{I} C_i(Q_{R.S}) + Q_{F.S}^2 \left(T + \sum_{i=1}^{I} C_i \right) + \ Q_{F.S}^2 \sum_{i=1}^{I} H_i(Q_{R.S}) + \\ \frac{1}{2}(H')(Q_{F.S}^2) \end{matrix} \right\} \end{matrix} \right) \frac{1}{b_S - a_S} \end{aligned} \tag{14}$$

***State* 2:**

$$\emptyset_2 = x_S\left(\begin{array}{c}\left\{\begin{array}{c}(P)(Q_{F.S}) + (P)(\mu)\left(\frac{1}{2}x_S - Q_{F.S}\right) + \\ \sum_{i=1}^{I} D_i\left[Q_{R.S} - \gamma_i(\mu)\left(\frac{1}{2}x_S - Q_{F.S}\right)\right]\end{array}\right\} \\ -\left\{\begin{array}{c}\sum_{i=1}^{I} C_i(Q_{R.S}) + \left(T + \sum_{i=1}^{I} C_i\right)Q_{F.S} + (T)(\mu)\left(\frac{1}{2}x_S - Q_{F.S}\right) \\ +\pi(1-\mu)\left(\frac{1}{2}x_S - Q_{F.S}\right) + \sum_{i=1}^{I} H_i\left[Q_{R.S} - \gamma_i(\mu)\left(\frac{1}{2}x_S - Q_{F.S}\right)\right]\end{array}\right\}\end{array}\right)_{Q_{F.S}}^{Q_{F.S}+\frac{Q_{F.S}^{+}}{\mu}} \frac{1}{b_S - a_S}$$

$$= \left(\begin{array}{c}\left\{\begin{array}{c}P(Q_{F.S}) + P(\mu)\left(\frac{1}{2}\left(Q_{F.S} + \frac{Q_{F.S}^{+}}{\mu}\right) - Q_{F.S}\right) \\ +\sum_{i=1}^{I} D_i\left[Q_{R.S} - \gamma_i(\mu)\left(\frac{1}{2}\left(Q_{F.S} + \frac{Q_{F.S}^{+}}{\mu}\right) - Q_{F.S}\right)\right]\end{array}\right\} \\ -\left\{\begin{array}{c}\sum_{i=1}^{I} C_i(Q_{R.S}) + \left(T + \sum_{i=1}^{I} C_i\right)Q_{F.S} + (T)(\mu)\left(\frac{1}{2}\left(Q_{F.S} + \frac{Q_{F.S}^{+}}{\mu}\right) - Q_{F.S}\right) \\ +\pi(1-\mu)\left(\frac{1}{2}\left(Q_{F.S} + \frac{Q_{F.S}^{+}}{\mu}\right) - Q_{F.S}\right) \\ +\sum_{i=1}^{I} H_i\left[Q_{R.S} - \gamma_i(\mu)\left(\frac{1}{2}\left(Q_{F.S} + \frac{Q_{F.S}^{+}}{\mu}\right) - Q_{F.S}\right)\right]\end{array}\right\}\end{array}\right) \left(Q_{F.S} + \frac{Q_{F.S}^{+}}{\mu}\right)\frac{1}{b_S - a_S} \quad (15)$$

$$-\left(\begin{array}{c}\left\{P(Q_{F.S}) + P(\mu)\left(\frac{1}{2}(Q_{F.S}) - Q_{F.S}\right) + \sum_{i=1}^{I} D_i\left[Q_{R.S} - \gamma_i(\mu)\left(\frac{1}{2}(Q_{F.S}) - Q_{F.S}\right)\right]\right\} \\ -\left\{\begin{array}{c}\sum_{i=1}^{I} C_i(Q_{R.S}) + \left(T + \sum_{i=1}^{I} C_i\right)Q_{F.S} + (T)(\mu)\left(\frac{1}{2}(Q_{F.S}) - Q_{F.S}\right) + \\ \pi(1-\mu)\left(\frac{1}{2}(Q_{F.S}) - Q_{F.S}\right) + \sum_{i=1}^{I} H_i\left[Q_{R.S} - \gamma_i(\mu)\left(\frac{1}{2}(Q_{F.S}) - Q_{F.S}\right)\right]\end{array}\right\}\end{array}\right)(Q_{F.S})\frac{1}{b_S - a_S}$$

***State* 3:**

$$\emptyset_3 = x_S\left(\begin{array}{c}\{(P)(Q_{F.S}) + (P)(Q^+_{F.S})\} \\ -\left\{\begin{array}{c}\sum_{i=1}^{I} C_i(Q_{R.S}) + \left(T + \sum_{i=1}^{I} C_i\right)Q_{F.S} + (T)(Q^+_{F.S}) \\ +\pi\left(\frac{1}{2}x_S - Q_{F.S} - Q^+_{F.S}\right)\end{array}\right\}\end{array}\right)_{Q_{F.S}+\frac{Q^+_{F.S}}{\mu}}^{b_S} \frac{1}{b_S - a_S}$$

$$= \left(P(Q_{F.S} + Q^+_{F.S}) - \left\{\begin{array}{c}\sum_{i=1}^{I} C_i(Q_{R.S}) + \left(T + \sum_{i=1}^{I} C_i\right)Q_{F.S} + (T)(Q^+_{F.S}) \\ +\pi\left(\frac{1}{2}\left(Q_{F.S} + \frac{Q^+_{F.S}}{\mu}\right) - Q_{F.S} - Q^+_{F.S}\right)\end{array}\right\}\right) \left(Q_{F.S} + \frac{Q^+_{F.S}}{\mu}\right)\frac{1}{b_S - a_S} \quad (16)$$

4.2. Solution Approach

Here, the steps involved in the solution approach are given below:

Step 1. Given the parameters, check for the convexity of the maximization problem modeled in (12) using the first partial derivative, the second partial derivative, and the Hessian of the second partial derivative of the objective function with respect to the decision variables.

Step 2. If the convexity does not hold, then a meta-heuristic solution algorithm is needed to solve the problem in order to find a near optimal solution. Otherwise, go to Step 3.

Step 3. Find an initial feasible point using a penalty function method, i.e., a point that satisfies the first two constraints. The penalty function method ignores the objective function entirely by replacing the constraints for the objective function.

Step 4. Transform the constrained optimization problem in (12) to an unconstrained nonlinear optimization problem using a barrier method. Here, the barrier method is employed to avoid approaching an infeasible point (Bazaraa et al. [26]).

Step 5. Use the sequential unconstrained maximization technique (SUMT) to solve the transformed problem (Bazaraa et al. [26]). In this method, the initial feasible point iteratively converges to the optimal solution using the Newton-Raphson method.

Step 6. The solution approach discussed above applies to a deterministic constrained non-linear optimization problem. However, as the problem modeled in (17) is stochastic, three common stochastic optimization procedures, namely “wait and see (WS),” “expected value (EV),” and “here and now” or “resource problem (RP)” are used (Birge and Louveaux [27]).

5. A Case Study

A case study for a two-level single-period dairy product problem is presented in this section in order to demonstrate the application of the proposed methodology in real-world environments. The commodity under investigation is a special dessert produced by a relatively large dairy firm. This product has only a single-period 17-day opportunity to be sold. The firm needs to prepare the products along with some raw materials before the beginning of the selling period in order to be able to immediately produce the products during the period in case of excessive demands. There are 13 types of raw materials with various consumption rates required to produce the final product. The firm has a limited available budget to purchase raw materials and to transform them into the product. In short, similar to the problem modeled in previous sections, a two-level single-period problem is involved in which the final product is made by combining different raw materials with different consumption rates within a certain budget. The objective is to find the number of raw materials and final product needed to be purchased and stocked before the selling period so as to maximize the profit. As demand prediction plays a major role in single-period problems, the problem is solved under different demand scenarios of good, fair, and low.

5.1. Data Sources

Based on historical data available for the past 48 periods, uniform distributions have shown table good fits for the probability distributions of the demands in different scenarios. Table 1 presents the intervals of these scenarios where $\boldsymbol{a_s}$ and $\boldsymbol{b_s}$ shows the start and the end of the intervals respectively for each scenario. The raw materials, consumption units, consumption rates, purchase price, discounted sale price, and holding costs of all raw materials are shown in Table 2. The discounted price of the product is assumed to be 60% of the selling price, while holding costs of the raw materials are 2% of their purchase prices. The selling price, shortage cost, holding cost of products, discounted sale price, and production costs are summarized in Table 3. The production cost includes direct labor cost, machinery overhead, and operation cost. Shortage cost is considered as lost sale. Since the profit margin of the product is 10% in this firm, the same is assumed for the shortage cost. Historical data in this firm shows that only 50% of the customers will wait for the preparation of the products in case of shortage and the rest will be lost. Thus, μ = 50%. Finally, the total available budget for the procurement and transformation of the raw material before the selling period is 150,000,000 currency units.

Table 1. Intervals of demand scenarios for products

Scenario	Demand Interval	
	$\boldsymbol{a_s}$	$\boldsymbol{b_s}$
Scenario 1 (Good)	38,000	55,000
Scenario 2 (Fair)	32,000	53,000
Scenario 3 (Low)	29,000	50,000

Table 2. Raw material data

i	*Raw Material*	*Unit*	*consumption rate*	*purchasing cost*	*Di*	*Hi*
1	T Phosphate	Kg	0.00027	10	6	0.2
2	40% bulk cream	Kg	0.036594	256	154	5.1
3	PBSS starch	Kg	0.0045	469	281	9.4
4	Industrial zero fat milk	Kg	0.22572	1,129	677	22.6
5	Gelatin powder	Kg	0.0027	674	404	13.5
6	Pure salt	Kg	0.00243	6	4	0.1
7	C-Polar text 6748	Kg	0.0018	134	80	2.7
8	5 layer	Peace	0.00185185	22	13	0.4
9	Punched aluminum foil 116 mm	Peace	1	334	200	6.7
10	Printed Poly-styrene 116 mm cup	Peace	1	1,511	907	30.2
11	3.2% fat milk	Kg	0.808209	1,010	606	20.2
12	Termtex Latex 5625 starch	Kg	0.0018	99	59	2.0
13	C-polar Tex 06716 additive	Kg	0.0018	108	65	2.2

Table 3. Products data

P	π	H'	D'	T
28,500	2,850	410	1,300	4,494

5.2. Demand Scenarios

Assuming good, fair, and low demand scenarios, the problem defined in the case study is solved under the same selling price for each scenario. Then, the scenario-based model is as follows:

$$\begin{aligned}
&\max Z_S = \left(\emptyset_{1,S} + \emptyset_{2,S} + \emptyset_{3,S}\right)\\
&\text{s.t.}\\
&(1)\ \emptyset_1 = \int_0^{Q_{F,S}} (Income_1 - Cost_1) f_X(x_S)\, dx_S\\
&(2)\ \emptyset_2 = \int_{Q_{F,S}}^{Q_{F,S}+\frac{Q_{F,S}^{+}}{\mu}} (Income_2 - Cost_2) f_X(x_S)\, dx_S\\
&(3)\ \emptyset_3 = \int_{Q_{F,S}+\frac{Q_{F,S}^{+}}{\mu}}^{\infty} (Income_3 - Cost_3) f_X(x_S)\, dx_S\\
&(4) \sum_{i=1}^{I} C_i\left(Q_{R,S}\right) + \left(T + Q_{F,S}\right) \le B\\
&(5)\ \left[Q_{F,S} + \mu\left(x_S - Q_{F,S}\right)\right] \le \frac{Q_{R,S}}{\gamma_i}\\
&Q_{F,S}, Q_{R,S} \ge 0. \qquad \forall\, S = 1, 2, 3,
\end{aligned} \tag{17}$$

where $\emptyset_{1,S}$, $\emptyset_{2,S}$, and $\emptyset_{3,S}$ are the first, the second, and the third state of profit under the sth scenario, respectively. As the three scenarios cannot occur at the same time, these scenarios are presented to show the demand variations and its effect on newsboy optimal solutions.

6. Solution

In order to solve the maximization problem shown in (17) using an exact method, it is necessary to first prove the convexity using the first and the second partial derivatives. Having the first partial derivatives of the objective function with respect to the decision variables in Appendix A, the second partial derivatives with respect to the decision variables are

$$\begin{aligned}
&\frac{\partial^2 Z}{\partial^2 Q_{F,S}} = (-P - H' + \pi + D')\frac{1}{b_s - a_s} \\
&\frac{\partial^2 Z}{\partial^2 Q_{R,S}} = 2 * \left[\left(\frac{1}{2}(P+T) + \sum_{i=1}^{I}(D_i - H_i)\left(1 - \frac{\gamma_i}{2}\right)\right) - P\right]\left(\frac{1}{\mu}\right)\frac{1}{b_s - a_s} \\
&\frac{\partial^2 Z}{\partial Q_{F,S}\partial Q_{R,S}} = \left(\sum_{i=1}^{I} D_i - \sum_{i=1}^{I} H_i - (T - \pi) - P\right)\frac{1}{b_s - a_s}
\end{aligned} \tag{18}$$

Then, the Hessian matrix of the second derivatives that is a 2×2 matrix with two decision variables including $Q_{F,S}$ and $Q_{R,S}$ is obtained to be

$$\begin{aligned}
&H_s = \begin{bmatrix} \dfrac{\partial^2 Z}{\partial^2 Q_{F,S}} & \dfrac{\partial^2 Z}{\partial Q_{F,S}\partial Q_{R,S}} \\ \dfrac{\partial^2 Z}{\partial Q_{F,S}\partial Q_{R,S}} & \dfrac{\partial^2 Z}{\partial^2 Q_{R,S}} \end{bmatrix} \\
&H_s = \frac{1}{b_s - a_s} \\
&\begin{bmatrix} (-P - H' + \pi + D') & \left(\sum_{i=1}^{I} D_i - \sum_{i=1}^{I} H_i - (T - \pi) - P\right) \\ \left(\sum_{i=1}^{I} D_i - \sum_{i=1}^{I} H_i - (T - \pi) - P\right) & 2 * \left[\left(\frac{1}{2}(P+T) + \sum_{i=1}^{I}(D_i - H_i)\left(1 - \frac{\gamma_i}{2}\right)\right) - P\right]\left(\frac{1}{\mu}\right) \end{bmatrix}
\end{aligned} \tag{19}$$

In the above matrix we assume that $H_s = \begin{bmatrix} a'_s & b'_s \\ c'_s & d'_s \end{bmatrix}$. As shown in Table 4 convexity holds true for all the three scenarios. Hence, an exact method can be used to solve the unconstrained problem in all the demand scenarios.

Table 4. The Hessian matrix under three demand scenarios

Scenario	Hessian Matrix	$a'_s d'_s - b'_s c'_s$
Scenario 1 (Good)	$H_1 = \frac{1}{55000-38000}\begin{bmatrix} -24760 & -26803 \\ -26803 & -38048 \end{bmatrix}$	$a'_1 d'_1 - b'_1 c'_1 = 13156$
Scenario 2 (Fair)	$H_2 = \frac{1}{55000-32000}\begin{bmatrix} -24760 & -26803 \\ -26803 & -38048 \end{bmatrix}$	$a'_2 d'_2 - b'_2 c'_2 = 10650$
Scenario 3 (Low)	$H_3 = \frac{1}{55000-29000}\begin{bmatrix} -24760 & -26803 \\ -26803 & -38048 \end{bmatrix}$	$a'_3 d'_3 - b'_3 c'_3 = 10650$

The constrained optimization problem shown in (17) is first transformed into an unconstrained nonlinear optimization problem for which the penalty function and the barrier method are employed. In this method, the constraints along with their violations are presented in Table 5.

Table 5. Definition of constraint violation

Constraint	*Violation*
$\sum_{i=1}^{I} C_i(Q_{R,S}) + T(Q_{F,S}) \le B$	$\boldsymbol{Max\left(0.\sum_{i=1}^{I} C_i(Q_{,S}) + T(Q_{F,S}) \le B\right)}$
$[Q_{F,S} + \mu(x_S - Q_{F,S})] \le \frac{Q_{R,S}}{\gamma_i}$	$\boldsymbol{Max\left(0.[Q_{F,S} + \mu(x_S - Q_{F,S})] \le \frac{Q_{R,S}}{\gamma_i}\right)}$
$Q_{F,S} \ge 0$	$\boldsymbol{Max(0.-Q_{F,S})}$
$Q_{R,S} \ge 0$	$\boldsymbol{Max(0.-Q_{R,S})}$

For the on-hand constrained optimization problem with two "less than or equal" constraints and two "greater than zero" bounds on the decision variables, the penalty function, $p(x)$, to be minimized is defined as

$$p(x) = \sum_{i=1}^{2} v(g_i(x)) + \sum_{j=1}^{2} v(x_j), \tag{20}$$

where $v(g_i(x))$ is considered for "less than or equal to" constraints and $v(x_j)$ is defined for the bounds on the decision variable x_j (in this case, $Q_{R,S}$ and $Q_{F,S}$). Then, the unconstrained minimization problem in (20) only assumes the penalty function as the objective function, where its least value is zero if a feasible point is obtained; that is if all constraints are satisfied (Bazaraa et al. [26]).

After transforming the constrained optimization problem into an unconstrained optimization problem using the penalty function and the barrier function methods described above, the sequential unconstrained maximization technique (SUMT) is used to solve it (Bazaraa et al. [26]). In short, the following steps are taken to solve the problem:

i. Select an initial value of r, a reduction rate of r, and an acceptable computational error.
ii. Find a feasible point that satisfies all the constraints for which the penalty method can be used.

iii. Build the barrier function and its integration with the main objective function of the problem so that the problem is solved without constraints.
iv. Use the feasible point as the starting point and solve the problem by an unconstrained nonlinear optimization approach such as the Newton-Raphson method.
v. Monitor the computational error; in case it is less than a certain value, the algorithm stops.
vi. Reduce r and monitor the error until the stopping condition is met.

Given the constraints put forth for the model shown in (17), the barrier function becomes

$$B(Q_{F,S}, Q_{R,S}, r) = r\left[\frac{1}{B - \sum_{i=1}^{I} C_i\left(Q_{R,S}\right) - T(Q_{F,S})} + \frac{1}{\frac{Q_{R,S}}{\gamma_i} - Q_{F,S} - \mu(x_S - Q_{F,S})} + \frac{1}{Q_{F,S}} + \frac{1}{Q_{R,S}}\right] \quad (21)$$

Then, by integrating the constraints and the objective function, we will have to maximize the unconstrained non-linear optimization problem in (22) using the Newton-Raphson method, where the required derivatives are presented in Appendix B.

$$\max Z(Q,r) = (\emptyset_{1,S} + \emptyset_{2,S} + \emptyset_{3,S}) - r\left[\frac{1}{B - \sum_{i=1}^{I} C_i(Q_{R,S}) - T(Q_{F,S})} + \frac{1}{\frac{Q_{R,S}}{\gamma_i} - Q_{F,S} - \mu(x_S - Q_{F,S})} + \frac{1}{Q_{F,S}} + \frac{1}{Q_{R,S}}\right] \quad (22)$$

This method starts with an initial reduction rate of $r = 1$ with the computational error of $\varepsilon = 0.001$ The solution approach discussed above applies to a deterministic constrained non-linear optimization problem. However, as the problem modeled in (17) is stochastic, three common stochastic optimization procedures, namely “wait and see (WS),” “expected value (EV),” and “here and now” or “resource problem (RP)” are used in the following subsections [27].

6.1. Wait and See Method

This method assumes that the uncertainty is replaced with certainty and that the actual values of the random parameters are known. Obviously, these assumptions do not hold in the real world because the future cannot be precisely predicted in most cases. In this approach, the problem is solved for each scenario to obtain Z_{WS} as the expected value of the profit using

$$Z_{WS} = \sum_{s \in S} p_S(Z^{*S}), \quad (23)$$

where Z^{*S} is the optimum profit in scenario s obtained by the SUMT algorithm. Tables 6-8 contain the results in consecutive steps of the SUMT algorithm for good, fair, and low demand scenarios, respectively. In these tables, the Z^{*S} value at the bottom of the last column shown in bold, which is very close to $Z(Q,r)$ represents the optimal profit. In addition, the optimal quantities of raw materials and products in each scenario are shown in the last rows of these tables. These figures lead to the quantities of raw materials to be purchased before the selling period in proportion to their consumption rates in Table 9. Moreover, the budget requirements are shown in Table 10, where the budget needed to purchase raw materials and to produce products as well as the total profit obtained under each scenario are shown.

Table 6. Optimal profit obtained by the WS method for the good demand scenario

Iteration	*r*	*Raw Material*	*Products*	$Z(Q,r)$	Z^{*1}
1	1	14,444	21,217	10,836,705	10,844,884
2	0.1	14,813	22,225	10,939,387	10,939,307
3	**0.01**	**14,814**	**22,232**	**10,939,049**	**10,939,041**

Table 7. Optimal profit obtained by the WS method for the fair demand scenario

Iteration	*r*	*Raw Material*	*Products*	$Z(Q,r)$	Z^{*2}
1	1	14,114	21,650	8,229,281	8,237,694
2	0.1	14,126	21,687	8,238,877	8,239,867
3	0.01	14,156	21,753	8,243,422	8,243,582
4	0.001	14,173	21,798	8,246,034	8,246,062
5	0.0001	14,182	21,824	8,247,482	8,247,487
6	0.00001	14,189	21,842	8,248,472	8,248,473
7	0.000001	14,190	21,845	8,248,626	8,248,626
8	**0.0000001**	**14,194**	**21,856**	**8,249,184**	**8,249,184**

Table 8. Optimal profit obtained by the WS method for the low demand scenario

Iteration	*r*	*Raw Material*	*Products*	$Z(Q,r)$	Z^{*3}
1	1	13,299	20,426	7,329,573	7,331,652
2	0.1	13,308	20,435	7,331,920	7,332,128
3	**0.01**	**13,308**	**20,435**	**7,331,920**	**7,332,128**

Table 9. Optimal solution obtained using the WS method

i	Raw Material	Unit	Raw Materials Quantity		
			Scenario 1 (Good)	**Scenario 2 (Fair)**	**Scenario 3 (Low)**
1	T phosphate	Kg	4.00	3.8	3.6
2	40% bulk cream	Kg	542.47	519.4	487.0
3	PBSS starch	Kg	66.71	63.9	59.9
4	Industrial zero fat milk	Kg	3,346.09	3,203.9	3,003.8
5	Gelatin powder	Kg	40.02	38.3	35.9
6	Pure salt	Kg	36.02	34.5	32.3
7	C-polar Tex 6748	Kg	26.68	25.5	24.0
8	5 layer	Peace	27.45	26.3	24.6
9	Punched aluminum foil 116 mm	Peace	14,824.07	14,194.3	13,307.7
10	Printed Poly-styrene 116 mm cup	Peace	14,824.07	14,194.3	13,307.7
11	3.2% fat milk	Kg	11,980.94	11,471.9	10,755.4

Table 9. Optimal solution obtained using the WS method					
i	Raw Material	Unit	Raw Materials Quantity		
			Scenario 1 (Good)	**Scenario 2 (Fair)**	**Scenario 3 (Low)**
12	Termtex Latex 5625 starch	Kg	26.68	25.5	24.0
13	C-polar Tex 06716 additive	Kg	26.68	25.5	24.0

Table 10. Budget consumed using the WS method					
i	**Raw Material**	**Unit**	**Raw Material Cost**		
			Scenario 1 (Good)	Scenario 2 (Fair)	Scenario 3 (Low)
1	T phosphate	Kg	38.40	36.8	34
2	40% bulk cream	Kg	138,958.50	133,054.8	124,745
3	PBSS starch	Kg	31,279.52	29,950.6	28,080
4	Industrial zero fat milk	Kg	3,776,394.99	3,615,953.3	3,390,117
5	Gelatin powder	Kg	26,982.92	25,836.5	24,223
6	Pure salt	Kg	218.15	208.9	196
7	C-Polar Tex 6748	Kg	3,578.64	3,426.6	3,213
8	5 layer	Peace	599.88	574.4	539
9	Punched aluminum foil 116 mm	Peace	4,952,297.80	4,741,897.4	4,445,739
10	Printed poly styrene 116 mm cup	Peace	22,400,875.15	21,449,164.9	20,109,543
11	3.2% fat milk	Kg	12,103,882.60	11,589,644.2	10,865.805
12	Termtex Latex 5625 starch	Kg	2,641.65	2,529.4	2,371
13	C-Polar Tex 06716 additive	Kg	2,881.80	2,759.4	2,587
Consumed Budget for Raw Materials			**43,440,629.98**	**41,595,037.16**	**38,997,192**
Consumed Budget for Products			**99,910,608**	**98,220,864**	**91,834,890**
Total Consumed Budget			**143,351,238**	**139,815,901**	**130,832,082**
Overall Profit			**10,939,041**	**8,249,184**	**7,332,128**

In order to investigate the effect of demand prediction on the total profit obtained, realized actual demands are also considered to follow the good, fair, and low scenarios. In other words, the question "what happens to the profit in each scenario if good, fair, and low scenarios of the actual demand happen?" is answered here. As expected, the results shown in Table 11 indicate that the highest profits in all the scenarios are realized if the good scenario for the actual demand is realized. This investigation enables the newsvendor to estimate the profit reduction in case of fair and low demands so that she would be able to predict the demand using a better approach more precisely. Then, assuming an identical likelihood of the occurrence of the demand scenarios, i.e. $p_s = 1/3$, the long-term (expected) profit obtained using the WS method becomes $Z_{WS} = 8.839.234$, based on (23).

Table 11. The effect of demand prediction on the total profit obtained using the WS method

Scenario	Realized Demand		
	Good	Fair	Low
Scenario 1 (Good)	10,939,041	8,258,900	7,451,800
Scenario 2 (Fair)	10,789,000	8,249,184	7,395,700

6.2. Expected Value Method

In most cases, finding an answer based on the explicit use of random parameters is difficult and time-consuming. Moreover, obtaining a complete information on the parameters being used is costly and sometimes impossible. Thus, it is occasionally preferred to use the expected value of a random parameter to obtain a simpler solution as a certain equivalent. The EV method employed in our work takes the mean of the random demand and inserts it in the optimization problem to find the optimal solution. In this method, the average of the lower $(\bar{a})$ and the upper $(\bar{b})$ bound of the uniform distribution of the demand in the three equally likely scenarios are first obtained using (24):

$$\begin{aligned}\bar{a} &= \frac{1}{3}(29000 + 32000 + 38000) = 33000 \\ \bar{b} &= \frac{1}{3}(50000 + 53000 + 55000) = 52667.\end{aligned} \tag{24}$$

Then, the bounds are used in the SUMT algorithm to find the optimal solution (Z^*) iteratively. Table 12 contains the results for different iterations, where (Z^*) becomes 8,709,068 in the 7th iteration at the bottom of the table. In this case, the optimal solution obtained using the EV method is shown in Table 13.

The budget requirement for the above solution is shown in Table 14, where the budget needed to purchase raw materials and to produce products as well as the total profit obtained are shown. Then, one should determine how much profit the newsvendor will gain if she uses the EV solution when the good, fair or low scenario is realized. To this aim, a similar approach employed for the WS method is taken here. The results are summarized in Table 15, where the long-term profit obtained using the EV method becomes $Z_{EEV} = 8.712.700$, based on (23).

Table 12. Optimal profit obtained by the EV method

Iteration	r	*Raw Material*	*Products*	$Z(Q.r)$	(Z^*)
1	1	14,074	21,720	6,689,445	8,698,326
2	0.1	14,090	21,777	8,700,411	8,701,582
3	0.01	14,112	21,835	8,704,587	8,704,776
4	0.001	14,125	21,872	8,706,711	8,706,743
5	0.0001	14,133	21,893	8,707,891	8,707,897
6	1.00E-05	14,137	21,905	8,708,495	8,708,496
7	**1.00E-06**	**14,141**	**21,916**	**8,709,068**	**8,709,068**

Table 13. Optimal solution of the EV method			
i	**Raw Material**	**Unit**	**Raw Materials Quantity**
1	T phosphate	Kg	3.82
2	40% bulk cream	Kg	517.49
3	PBSS starch	Kg	63.64
4	Industrial zero fat milk	Kg	3,191.99
5	Gelatin powder	Kg	38.18
6	Pure salt	Kg	34.36
7	C-Polar Tex 6748	Kg	25.45
8	5 layer	Peace	26.19
9	Punched aluminum foil 116 mm	Peace	14,141.36
10	Printed poly styrene 116 mm cup	Peace	14,141.36
11	3.2% fat milk	Kg	11,429.17
12	Termtex Latex 5625 starch	Kg	25.45
13	C-Polar Tex 06716 additive	Kg	25.45

Table 14. Budget consumed using the EV method			
i	**Raw Material**	**Unit**	**Raw Material Cost**
1	T phosphate	Kg	36.63
2	40% bulk cream	Kg	132,558.92
3	PBSS starch	Kg	29,838.98
4	Industrial zero fat milk	Kg	3,602,477.23
5	Gelatin powder	Kg	25,740.25
6	Pure salt	Kg	208.10
7	C-Polar Tex 6748	Kg	3,413.83
8	5 layer	Peace	572.25
9	Punched aluminum foil 116 mm	Peace	4,724,225.11
10	Printed poly styrene 116 mm cup	Peace	21,369,227.20
11	3.2% fat milk	Kg	11,546,451.46
12	Termtex Latex 5625 starch	Kg	2,519.99
13	C-Polar Tex 06716 additive	Kg	2,749.08
Consumed Budget for Raw Materials			**41,440,019.02**
Consumed Budget for Products			**98,490,504**
Total Consumed Budget			**139,930,523**
Overall Profit			**8,709,068**

Table 15. Demand effect on the newsvendor profit based on EV	
Scenario	EV
Scenario 1 (Good)	10,909,000
Scenario 2 (Fair)	8,253,000
Scenario 3 (Low)	7,376,100
Z_{EEV}	8,712,700

6.3. The Here and Now Method

The here and now or the resource problem (RP) method offers a practical real-world approach to solve stochastic optimization problems. In this method, the feasible region of the problem is identical in all scenarios and the solution is treated as the real-world answer to the problem. The main drawback of this method is that when the problem becomes relatively large, it can be hardly solved.

The optimization problem to be solved by the RP method for the model shown in (17) is:

$$\begin{aligned}
&\max Z = \frac{1}{3}(\emptyset_{1,1} + \emptyset_{2,1} + \emptyset_{3,1}) + \frac{1}{3}(\emptyset_{1,2} + \emptyset_{2,2} + \emptyset_{3,2}) + \frac{1}{3}(\emptyset_{1,3} + \emptyset_{2,3} + \emptyset_{3,3}) \\
&\text{s.t.} \\
&(1) \sum_{i=1}^{I} C_i(Q_R) + (T + Q_F) \le B \\
&(2)\ [Q_F + \mu(E(x_1) - Q_F)] \le \frac{Q_R}{\gamma_i} \\
&(3)\ [Q_F + \mu(E(x_2) - Q_F)] \le \frac{Q_R}{\gamma_i} \\
&(4)\ [Q_F + \mu(E(x_3) - Q_F)] \le \frac{Q_R}{\gamma_i} \\
&Q_F, Q_R > 0,
\end{aligned} \tag{25}$$

where $E(x_S)$, $S = 1, 2, 3$ is the expected demand in sth scenario. Equation (25) is used to place all the scenarios into the recourse problem, i.e., all the scenarios are considered in one model. Besides, the objective function consists of three parts, each related to a scenario. Moreover, as the demand is a random variable, the constraints containing this variable repeat for each scenario. For instance, the second constraint in (17) involves the demand variable is repeated in (25).

Solving the problem modeled in (25) using the SUMT algorithm results in the optimal profit as shown at the bottom of Table 16.

Table 16. Optimal profit obtained using the RP method					
Iteration	r	*Raw Material*	*Products*	$Z(Q.r)$	(Z^*)
1	1	12,786	19,610	8,729,316	8,729,206
2	**0.1**	**12,796**	**19,610**	**8,730,416**	**8,730,416**

Consequently, the optimal quantities of the raw materials based on their consumption rates are reported in Table 17. In addition, the budget required to purchase raw materials and to produce products as well the profit obtained are presented in Table 18. The results in Table 18 show that if the newsvendor decides to use the plan obtained by the RP method, her long-term profit will be equal to $Z_{RP} = 8.730.416$ units.

Table 17. Optimal solution obtained by the RP method

i	Raw Material	Unit	Raw Materials Quantity
1	T phosphate	Kg	3.45
2	40% bulk cream	Kg	468.26
3	PBSS starch	Kg	57.58
4	Industrial zero fat milk	Kg	2,888.31
5	Gelatin powder	Kg	34.55
6	Pure salt	Kg	31.09
7	C-Polar Tex 6748	Kg	23.03
8	5 layer	Peace	23.70
9	Punched aluminum foil 116 mm	Peace	12,796.00
10	Printed poly styrene 116 mm cup	Peace	12,796.00
11	3.2% fat milk	Kg	10,341.84
12	Termtex Latex 5625 starch	Kg	23.03
13	C-Polar Tex 06716 additive	Kg	23.03

Table 18. Budget requirement using the RP method

i	Raw Material	Unit	Raw Materials Cost
1	T phosphate	Kg	33.14
2	40% bulk cream	Kg	119,947.73
3	PBSS starch	Kg	27,000.20
4	Industrial zero fat milk	Kg	3,259,750.19
5	Gelatin powder	Kg	23,291.41
6	Pure salt	Kg	188.31
7	C-Polar Tex 6748	Kg	3,089.05
8	5 layer	Peace	517.81
9	Punched aluminum foil 116 mm	Peace	4,274,778.91
10	Printed poly styrene 116 mm cup	Peace	19,336,233.94
11	3.2% fat milk	Kg	10,447,962.59
12	Termtex Latex 5625 starch	Kg	2,280.25
13	C-polar Tex 06716 additive	Kg	2,487.54
	Consumed Budget for Raw Materials		**37,497,561.07**
	Consumed Budget for Products		**88,127,340**

Table 18. Budget requirement using the RP method			
i	Raw Material	Unit	Raw Materials Cost
Total Consumed Budget			**125,624,901**
Overall Profit under each Scenario			**8,730,416**

6.4. Significance of the Solutions Obtained

The long-term profits obtained using the above-mentioned methods are summarized in Table 19.

Table 19. Summary of stochastic model	
Method	Profit
Wait and See (Z_{WS})	8,839,234
Expected Value (Z_{EEV})	8,712,700
Recourse Problem (Z_{RP})	8,730,416

The significance of the results obtained using the three solution methods is tested in this section using the expected value of perfect information and the value of the stochastic solution. The expected value of perfect information (*EVPI*) shows to what extent information shortage can be effective. It is calculated by

$$EVPI = Z_{WS} - Z_{RP}. \quad (26)$$

For the two-level single-period inventory control model of the present study, $EVPI$ is

$$EVPI = Z_{WS} - Z_{RP} = 8{,}839{,}234 - 8{,}730{,}416 = 108{,}818 \quad (27)$$

Note that a lower value of *EVPI* shows that the acquisition of information about the future does not entail significant impact, while a higher EVPI shows that ignoring the significance of perfect information will be very costly. In other words, this parameter shows how much can be spent on acquiring perfect information. As a result, if the newsvendor of the investigated problem neglects the uncertainty involved and procures raw materials and final product according to mean demand, she will be imposed a cost of 108,818 units in each period in the long run.

The value of stochastic solution (*VSS*) is another useful measure defined by

$$VSS = Z_{RP} - Z_{EEV}. \quad (28)$$

It shows how much can be saved if the stochastic programming solution is used instead of a certain programming problem solution such as the expected value method. In other words, *VSS* represents the cost of neglecting uncertainty in decision making. A lower value of *VSS* indicates that the solution provided by the expected value method is a good approximation of the real solution of the problem. The value of *VSS* for the problem at hand is obtained to be

$$VSS = Z_{RP} - Z_{EEV} = 8{,}730{,}416 - 8{,}712{,}700 = 17{,}716 \tag{29}$$

This means that 17,716 units can be saved if the newsvendor uses the RP method instead of the EV method.

7. Conclusions and Future Research

A two-level single-period problem with budget constraint under three different demand scenarios was investigated. The newsvendor needs to prepare raw materials and products just before the selling period. Production is allowed during the period for those customers who can wait for their unsatisfied demands by transforming the prepared raw materials into products during the period. It was assumed that the demands of those customers who cannot wait are lost. Inventory control models were first developed for this problem based on three different states of demand to estimate the newsvendor's profit in each state. Then, the budget constraints and the consumption rates of raw materials were taken into account assuming three demand scenarios of good, fair and low. It was next shown that the developed nonlinear optimization problem could be solved using an exact method. Thereafter, an initial feasible point was obtained using a penalty function method and then, the problem was converted to an unconstrained nonlinear optimization problem using a barrier method, based on which the initial feasible point iteratively turns to an optimal solution using the Newton-Raphson method. A real-world case study was presented to illustrate the proposed methodology. Based on some analysis carried out under various conditions, we showed how the newsvendor can learn the effect of demand prediction on her profit.

The mathematical formulation proposed here can be easily extended and used for a multi-product single-period two-level newsvendor problem using a new index for each product in all the necessary parameters. However, the solution approach might be different. This will be considered in our future research. Moreover, while uncertainty was merely assumed for demand, investigating some other uncertainties for other parameters of the problem is left for future. In addition, analyzing the impact of non-conforming product returns on the expected profit obtained as well as price sensitivity analysis are other interesting topics for future studies.

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Appendix A: First derivatives of the objective function with respect to the decision variables

$$\frac{\partial \emptyset_1}{\partial Q_{F,S}} = \left(\begin{array}{c} \left\{ P(Q_{F,S}) + \sum_{i=1}^{I} D_i(Q_{R,S}) + D'(Q_{F,S}) \right\} \\ -\left\{ \sum_{i=1}^{I} C_i(Q_{R,S}) + 2\left(T + \sum_{i=1}^{I} C_i\right) Q_{F,S} + \sum_{i=1}^{I} H_i(Q_{R,S}) + H'(Q_{F,S}) \right\} \end{array} \right) \frac{1}{b_s - a_s}$$

$$\frac{\partial \emptyset_1}{\partial Q_{R,S}} = Q_{F,S} \sum_{i=1}^{I} (D_i - C_i - H_i) \frac{1}{b_s - a_s}$$

$$\frac{\partial \emptyset_2}{\partial Q_{F,S}} = \left(\begin{array}{c} P - \frac{1}{2} P(\mu) + \sum_{i=1}^{I} D_i \left[\gamma_i(\mu) \frac{1}{2} \right] \\ -\left\{ \left(T + \sum_{i=1}^{I} C_i\right) - T(\mu)\frac{1}{2} - \pi(1-\mu)\frac{1}{2} + \sum_{i=1}^{I} H_i \left[\gamma_i(\mu) \frac{1}{2} \right] \right\} \end{array} \right) \left(Q_{F,S} + \frac{Q_{F,S}^{+}}{\mu} \right) \frac{1}{b_S - a_S}$$

$$+ \left\{ \begin{array}{c} \left\{ \begin{array}{c} P(Q_{F,S}) + P(\mu) \left(\frac{1}{2}\left(Q_{F,S} + \frac{Q_{F,S}^{+}}{\mu} \right) - Q_{F,S} \right) + \\ \sum_{i=1}^{I} D_i \left[Q_{R,S} - \gamma_i(\mu) \left(\frac{1}{2}\left(Q_{F,S} + \frac{Q_{F,S}^{+}}{\mu} \right) - Q_{F,S} \right) \right] \end{array} \right\} - \\ \left\{ \begin{array}{c} \sum_{i=1}^{I} C_i(Q_{R,S}) + \left(T + \sum_{i=1}^{I} C_i\right) Q_{F,S} + T(\mu) \left(\frac{1}{2}\left(Q_{F,S} + \frac{Q_{F,S}^{+}}{\mu} \right) - Q_{F,S} \right) \\ +\pi(1-\mu) \left(\frac{1}{2}\left(Q_{F,S} + \frac{Q_{F,S}^{+}}{\mu} \right) - Q_{F,S} \right) \\ + \sum_{i=1}^{I} H_i \left[Q_{R,S} - \gamma_i(\mu) \left(\frac{1}{2}\left(Q_{F,S} + \frac{Q_{F,S}^{+}}{\mu} \right) - Q_{F,S} \right) \right] \end{array} \right\} \end{array} \right\} \frac{1}{b_S - a_S} -$$

$$\left(\begin{array}{c}\left\{P\left(Q_{F,S}\right) + P(\mu)\left(\frac{1}{2}\left(Q_{F,S}\right) - Q_{F,S}\right) + \sum_{i=1}^{I} D_i\left[Q_{R,S} - \gamma_i(\mu)\left(\frac{1}{2}\left(Q_{F,S}\right) - Q_{F,S}\right)\right]\right\} - \\ \left\{\begin{array}{c} \sum_{i=1}^{I} C_i\left(Q_{R,S}\right) + \left(T + \sum_{i=1}^{I} C_i\right)Q_{F,S} + T(\mu)\left(\frac{1}{2}\left(Q_{F,S}\right) - Q_{F,S}\right) \\ +\pi(1-\mu)\left(\frac{1}{2}\left(Q_{F,S}\right) - Q_{F,S}\right) \\ +\sum_{i=1}^{I} H_i\left[Q_{R,S} - \gamma_i(\mu)\left(\frac{1}{2}\left(Q_{F,S}\right) - Q_{F,S}\right)\right] \end{array}\right\}\end{array}\right)\frac{1}{b_S - a_S}$$

$$-\left(\begin{array}{c}\left\{P - P(\mu)\left(\frac{1}{2}\right) + \sum_{i=1}^{I} D_i\left[\gamma_i(\mu)\left(\frac{1}{2}\right)\right]\right\} - \left\{T + \gamma_i \sum_{i=1}^{I} C_i\right\} \\ -\left\{T(\mu)\left(\frac{1}{2}\right) + \pi(1-\mu)\left(\frac{1}{2}\right) + \sum_{i=1}^{I} H_i\left[\gamma_i(\mu)\left(\frac{1}{2}\right)\right]\right\}\end{array}\right) Q_{F,S}\frac{1}{b_S - a_S}$$

$$\frac{\partial \emptyset_2}{\partial Q_{R,S}} = \left(\begin{array}{c}\left\{\begin{array}{c} P\left(Q_{F,S}\right) + P(\mu)\left(\frac{1}{2}\left(Q_{F,S} + \frac{Q^{+}_{F,S}}{\mu}\right) - Q_{F,S}\right) + \\ \sum_{i=1}^{I} D_i\left[Q_{R,S} - \gamma_i(\mu)\left(\frac{1}{2}\left(Q_{F,S} + \frac{Q^{+}_{F,S}}{\mu}\right) - Q_{F,S}\right)\right]\end{array}\right\} - \\ \left\{\begin{array}{c} \sum_{i=1}^{I} C_i\left(Q_{R,S}\right) + \left(T + \sum_{i=1}^{I} C_i\right)Q_{F,S} + T(\mu)\left(\frac{1}{2}\left(Q_{F,S} + \frac{Q^{+}_{F,S}}{\mu}\right) - Q_{F,S}\right) \\ +\pi(1-\mu)\left(\frac{1}{2}\left(Q_{F,S} + \frac{Q^{+}_{F,S}}{\mu}\right) - Q_{F,S}\right) \\ +\sum_{i=1}^{I} H_i\left[Q_{R,S} - \gamma_i(\mu)\left(\frac{1}{2}\left(Q_{F,S} + \frac{Q^{+}_{F,S}}{\mu}\right) - Q_{F,S}\right)\right]\end{array}\right\}\end{array}\right)\left(\frac{1}{\mu}\right)\frac{1}{b_S - a_S}$$

$$+\left(\begin{array}{c}\left\{P(\mu)\left(\frac{1}{2\mu}\right) + \sum_{i=1}^{I} D_i\left[1 - \gamma_i(\mu)\left(\frac{1}{2\mu}\right)\right]\right\} \\ -\left\{\begin{array}{c}\sum_{i=1}^{I} C_i + T(\mu)\left(\frac{1}{2\mu}\right) + \\ \pi(1-\mu)\left(\frac{1}{2\mu}\right) + \sum_{i=1}^{I} H_i\left[1 - \gamma_i(\mu)\left(\frac{1}{2\mu}\right)\right]\end{array}\right\}\end{array}\right)\left(Q_{F,S} + \frac{Q^{+}_{F,S}}{\mu}\right)\frac{1}{b_S - a_S}$$

$$-\left(\left\{\sum_{i=1}^{I} D_i\right\} - \left\{\sum_{i=1}^{I} C_i + \sum_{i=1}^{I} H_i\right\}\right)\left(Q_{F,S}\right)\frac{1}{b_S - a_S}$$

$$\frac{\partial \emptyset_3}{\partial Q_{F,S}} = \left(P - T - \sum_{i=1}^{I} C_i + \pi\right)\frac{b_S}{b_S - a_S} - \left(P - T - \sum_{i=1}^{I} C_i + \frac{1}{2}\pi\right)\left(Q_{F,S} + \frac{Q_{F,S}^{+}}{\mu}\right)\frac{1}{b_S - a_S}$$

$$- \left(\begin{array}{c}\left\{P\left(Q_{F,S}\right) + P\left(Q_{F,S}^{+}\right)\right\} - \\ \sum_{i=1}^{I} C_i\, Q_{R,S} + \left(T + \sum_{i=1}^{I} C_i\right) Q_{F,S} + T\left(Q_{F,S}^{+}\right) + \pi\left(\frac{1}{2}\left(Q_{F,S} + \frac{Q_{F,S}^{+}}{\mu}\right) - Q_{F,S} - Q_{F,S}^{+}\right)\end{array}\right)\frac{1}{b_S - a_S}$$

$$\frac{\partial \emptyset_3}{\partial Q_{R,S}} = \left(P - \left\{\sum_{i=1}^{I} C_i + T - \pi\right\}\right)\frac{b_S}{b_S - a_S} - \left(P - \left\{T + \sum_{i=1}^{I} C_i + \frac{1}{2\mu}\pi\right\}\right)\left(Q_{F,S} + \frac{Q_{F,S}^{+}}{\mu}\right)\frac{1}{b_S - a_S}$$

$$- \left(\begin{array}{c}\left\{P\left(Q_{F,S}\right) + P\left(Q_{F,S}^{+}\right)\right\} \\ - \left\{\sum_{i=1}^{I} C_i\, Q_{R,S} + \left(T + \sum_{i=1}^{I} C_i\right) Q_{F,S} + T\left(Q_{F,S}^{+}\right) + \pi\left(\frac{1}{2}\left(Q_{F,S} + \frac{Q_{F,S}^{+}}{\mu}\right) - Q_{F,S} - Q_{F,S}^{+}\right)\right\}\end{array}\right)\left(\frac{1}{\mu}\right)\frac{1}{b_S - a_S}$$

Appendix B. Derivatives for the Barrier function

$$\frac{\partial B}{\partial Q_{F,S}} = r\left[\frac{T}{\left(B - \sum_{i=1}^{I} C_i\left(Q_{R,S}\right) - T\left(Q_{F,S}\right)\right)^2} + \frac{1-\mu}{\left(\frac{Q_{R,S}}{\gamma_i} - Q_{F,S} - \mu\left(x_S - Q_{F,S}\right)\right)^2} + \frac{-1}{\left(Q_{F,S}\right)^2}\right]$$

$$\frac{\partial B}{\partial Q_{R,S}} = r\left[\frac{\sum_{i=1}^{I} C_i}{\left(B - \sum_{i=1}^{I} C_i\left(Q_{R,S}\right) - T\left(Q_{F,S}\right)\right)^2} + \frac{-\frac{1}{\gamma_i}}{\left(\frac{Q_{R,S}}{\gamma_i} - Q_{F,S} - \mu\left(x_S - Q_{F,S}\right)\right)^2} + \frac{-1}{\left(Q_{R,S}\right)^2}\right]$$

$$\frac{\partial^2 B}{\partial^2 Q_{F,S}} = r\left[\frac{-2T^2\left(Q_{F,S} - B - \sum_{i=1}^{I} C_i\left(Q_{R,S}\right)\right)}{\left(B - \sum_{i=1}^{I} C_i\left(Q_{R,S}\right) - T\left(Q_{F,S}\right)\right)^4} + \frac{-(1-\mu)\left(2Q_{F,S} + 2\mu\left(Q_{F,S}\right) - 2\mu^2(x_s) - 2\frac{1}{\gamma_i}Q_{R,S} + 2\frac{1}{\gamma_i}Q_{R,S}(\mu) - 2\mu(x_s) + 4\mu\left(Q_{F,S}\right)\right)}{\left(\frac{Q_{R,S}}{\gamma_i} - Q_{F,S} - \mu\left(x_S - Q_{F,S}\right)\right)^4} + \frac{2}{\left(Q_{F,S}\right)^3}\right]$$

$$\frac{\partial^2 B}{\partial^2 Q_{R,S}} = r\left[\frac{-\sum_{i=1}^{I} C_i\left(2\left(\sum_{i=1}^{I} C_i\right)^2 Q_{R,S} - 2B\sum_{i=1}^{I} C_i - 2T\left(Q_{F,S}\right)\sum_{i=1}^{I} C_i\right)}{\left(B - \sum_{i=1}^{I} C_i\left(Q_{R,S}\right) - T\left(Q_{F,S}\right)\right)^4} + \frac{\frac{1}{\gamma_i}\left(2\left(\frac{1}{\gamma_i}\right)^2 Q_{R,S} - 2\frac{1}{\gamma_i}Q_{F,S} - 2\frac{1}{\gamma_i}\mu\left(x_S - Q_{F,S}\right)\right)}{\left(\frac{Q_{R,S}}{\gamma_i} - Q_{F,S} - \mu\left(x_S - Q_{F,S}\right)\right)^4} + \frac{2}{\left(Q_{R,S}\right)^3}\right]$$

$$\frac{\partial^2 B}{\partial Q_{F,S}\partial Q_{R,S}} = r\left[\begin{array}{c}\dfrac{-T\left(2\left(\sum_{i=1}^{I} C_i\right)^2 Q_{R,S} - 2B\sum_{i=1}^{I} C_i - 2T\left(Q_{F,S}\right)\sum_{i=1}^{I} C_i\right)}{\left(B - \sum_{i=1}^{I} C_i\left(Q_{R,S}\right) - T\left(Q_{F,S}\right)\right)^4} + \\ \dfrac{-(1-\mu)\left(2\left(\frac{1}{\gamma_i}\right)^2 Q_{R,S} - 2\frac{1}{\gamma_i}Q_{F,S} - 2\frac{1}{\gamma_i}\mu\left(x_S - Q_{F,S}\right)\right)}{\left(\frac{Q_{R,S}}{\gamma_i} - Q_{F,S} - \mu\left(x_S - Q_{F,S}\right)\right)^4}\end{array}\right]$$