

# A Novel Hybrid Modified Binary Particle Swarm Optimization Algorithm for the Uncertain p-Median Location Problem

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*Here, we investigate the classical p-median location problem on a network in which the vertex weights and the distances between vertices are uncertain. We propose a programming model for the uncertain p-median location problem with tail value at risk objective. Then, we show that it is NP-hard. Therefore, a novel hybrid modified binary particle swarm optimization algorithm is presented to obtain the approximate optimal solution of the proposed model. The algorithm contains the tail value at risk simulation and the expected value simulation. Finally, by computational experiments, the algorithm is illustrated to be efficient.*

**Keywords:** Location problem, p-median, Uncertainty theory, Tail value at risk, Uncertain programming, Binary particle swarm optimization.

Manuscript was received on 01/08/2019, revised on 05/11/2019 and accepted for publication on 09/12/2019.

## 1. Introduction

Location problems are important optimization problems recently been studied by many researchers. A famous location problems is the p-median location problem (pMLP) introduced as follows: Assume that  $N = (V, E)$  is an undirected connected network with vertex set  $V$ ,  $|V| = n$ , and edge set  $E$ . The distance between two points on  $N$  is equal to the length of the shortest path connecting the two points. Each vertex is associated with a nonnegative weight the demand of the client at the vertex. In a pMLP, the aim is to find p locations for establishing facilities on edges or vertices of  $N$  such that the sum of the weighted distances from the clients to the closest facility becomes minimum. The pMLP model is probably the most widely used model and most extensively researched in location problems [1, 17, 18, 32].

Concerning the median location problems on networks, Hakimi [23, 24] proved that optimal locations of the facilities exist at the vertices of the network. Kariv and Hakimi [26] showed that pMLP on a general network is NP-hard. Also, they designed an  $O(p^2n^2)$  time algorithm for the problem on tree networks. Goldman in 1971 presented simple one-pass solution algorithms for 1MLP on a tree network and a network which contained exactly one cycle [22]. The first algorithm is based on a reduction procedure for a useful simplification of problems involving general networks. Goldman showed that the 1-median on a tree is independent of edge lengths and based on this

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condition he presented a linear time algorithm for the problem. For 2MLP on trees, by using the properties relating to the 1-median and the 2-median, Gavish and Sridhar developed an improved algorithm based on edge-deletion method in  $O(n \log n)$  time [21]. Tamir [45] improved the time complexity of pMLP on trees to  $O(pn^2)$ . In 2005, Benkoczi and Bhattacharya [6] designed an  $O(n \log^{p+2} n)$  time algorithm to solve pMLP on trees. Drezner [15] proposed optimal solution procedures when  $p = 2$ . Also, problems, with  $p = 2, 3$ , were solved by Schobel and Scholz [42]. For 1MLP on a cactus network, a linear time algorithm was developed in [9].

In the context of the median location problems in the plane, the interested reader is referred to [7, 8, 10, 11, 12, 16, 33, 34, 41, 43].

In real life, we are usually faced with various types of uncertainty. For example, some parameters of the location problems, such as the vertex weights, the travel times between vertices, the establishment costs of facilities and the network modification costs may not be known certainly. The uncertainty theory introduced by Liu [29] is a suitable tool to deal with such parameters. Gao [19] modeled the single facility location problems with uncertain demands. Wen et al. [49] investigated the capacitated facility location-allocation problem with uncertain demands. Nguyen and Chi [37] studied I1-MLP on a tree with uncertain costs and showed that the inverse distribution function of the minimum cost can be obtained at  $O(n^2 \log n)$  time. For a survey on uncertain location problems, see e.g., [20, 25, 31, 40, 46, 52].

Note that uncertainty leads to risk. Liu [30] introduced the risk concept in the uncertain environment. Measuring the risk is an important step in the decision-making process. The risk metrics contain techniques and data sets used to calculate the risk value of the problem under investigation. Tail value at risk (*TVaR*) metric [39] is a measure of risk that is widely acceptable among industry segments and market participants. For a survey on risk management in location problems with random and fuzzy variables, the reader is referred to [4, 5, 47, 48, 50].

Here, we concentrate on pMLP with uncertain vertex weights and uncertain distances. We propose a model for the uncertain p-median location problem (UpMLP) with tail value at risk objective. Then, we show that the problem is NP-hard. Considering the uncertain and NP-hardness of UpMLP, evolutionary and meta-heuristic algorithms can be efficiently used for successful generation of approximate optimal solutions. Hence, we present a hybrid modified binary particle swarm optimization (HMBPSO) algorithm to obtain the approximate optimal solution of the proposed model, containing the tail value at risk simulation and the expected value simulation.

To our knowledge, two papers considered the implementation of metaheuristic algorithms for location problems. Alizadeh and Bakhteh [2] studied the general inverse pMLPs on networks and presented a modified firefly algorithm for the problem under investigation. Mirzapolis Adeg et al. [36] investigated the general inverse ordered pMLP on crisp networks and designed a modified particle swarm optimization algorithm for it. There is no work on implementation of hybrid metaheuristic algorithms on inverse pMLPs in uncertain networks. However, many papers can be found in the literature for other classical location problems on uncertain networks; see, e.g., [3, 25, 44, 49, 51].

The rest of our work is organized as follows. In the next section, we first introduce uncertainty theory and the *TVaR* metric in an uncertain environment. Then, we introduce the uncertain optimization model and present a new model with *TVaR* objective. In Section 3, we propose a programming model for UpMLP and solve it with *TVaR* objective. Then, we show that the problem under investigation is NP-hard. In Section 4, we present a novel HMBPSO algorithm to obtain an

approximate optimal solution of the proposed model. The algorithm contains the tail value at risk simulation and the expected value simulation. To show the effectiveness of the proposed HMBPSO algorithm, we give a numerical example in Section 5. We conclude in Section 6.

## 2. Preliminaries

In this section, we first present some definitions and theorems of uncertainty theory and  $TVaR$  metric. Then, we introduce an uncertain optimization model and present a new model with  $TVaR$  objective [29].

### 2.1. Uncertainty Theory

Let  $\Gamma$  be a nonempty set and  $\Theta$  be a  $\sigma$ -algebra over  $\Gamma$ . An uncertain measure is a set function  $M: \Theta \rightarrow [0,1]$  that satisfies the normality, duality and subadditivity axioms. The triple  $(\Gamma, \Theta, M)$  is called an uncertainty space.

**Definition 2.1.** Let  $(\Gamma, \Theta, M)$  be an uncertainty space. A measurable function  $\theta$  from  $(\Gamma, \Theta, M)$  to the set of real numbers is called an uncertain variable.

**Definition 2.2.** Let  $\theta$  be an uncertain variable. For any real number  $x$ , the function  $Y(x) = M\{\theta \leq x\}$  is called an uncertainty distribution of  $\theta$ .

**Definition 2.3.** Let  $\theta_i, i = 1, \dots, n$ , be the uncertain variables. We call  $\theta_i, i = 1, \dots, n$ , independent if for any Borel sets  $B_1, B_2, \dots, B_n$  of real numbers, we have

$$M\left\{\bigcap_{i=1}^n \{\theta_i \in B_i\}\right\} = \min_{i=1}^n M\{\theta_i \in B_i\}.$$

**Theorem 2.4.** Let  $\theta_i, i = 1, \dots, n$ , be the independent uncertain variables and  $Y_i^{-1}, i = 1, \dots, n$ , be the inverse uncertainty distributions of  $\theta_i, i = 1, \dots, n$ , respectively. Also, let  $f(x_1, x_2, \dots, x_n)$  be a strictly increasing function with respect to  $x_i, i = 1, 2, \dots, m$ , and a strictly decreasing function with respect to  $x_i, i = m + 1, \dots, n$ . Then, the uncertain variable  $\vartheta = f(\theta_1, \theta_2, \dots, \theta_n)$  has the following inverse uncertainty distribution:

$$Y^{-1}(\alpha) = f\left(Y_1^{-1}(\alpha), \dots, Y_m^{-1}(\alpha), Y_{m+1}^{-1}(1-\alpha), Y_{m+2}^{-1}(1-\alpha), \dots, Y_n^{-1}(1-\alpha)\right),$$

having the following expected value:

$$E(\vartheta) = \int_0^1 f\left(Y_1^{-1}(\alpha), \dots, Y_m^{-1}(\alpha), Y_{m+1}^{-1}(1-\alpha), Y_{m+2}^{-1}(1-\alpha), \dots, Y_n^{-1}(1-\alpha)\right) dY.$$

## 2.2. TVaR Metric in an Uncertain Environment

Risk demonstrates a situation, in which there is a chance of loss or danger. The quantification of risk is a key step towards the management and mitigation of risk. In this section, we introduce the definition of the *TVaR* metric to account for the probability of loss and the severity of loss in an uncertain environment [39].

In order to define the *TVaR* metric, we first introduce the definition of loss function [30].

**Definition 2.5.** Consider  $\theta_i$ ,  $i = 1, \dots, n$ , as the uncertain factors of a system. A function  $f$  is said to be a loss function if some specified loss occurs whenever

$$f(\theta_1, \theta_2, \dots, \theta_n) > 0.$$

In the uncertain environment, *TVaR* of loss function is defined as follows [39].

**Definition 2.6.** Let  $\theta_i$ ,  $i = 1, \dots, n$ , be the uncertain factors and  $f$  be the loss function of a system. Then, *TVaR* of  $f$  is defined to be

$$TVaR_\alpha(f) = \frac{1}{\alpha} \int_0^\alpha \sup\{\lambda | M\{f(\theta_1, \theta_2, \dots, \theta_n) \geq \lambda\} \geq \beta\} d\beta.$$

for each given risk confidence level  $\alpha \in (0, 1]$ .

**Theorem 2.7.** Let  $\theta_i$ ,  $i = 1, \dots, n$ , be the uncertain factors of a system and  $Y_i^{-1}$ ,  $i = 1, \dots, n$ , be the inverse uncertainty distributions of  $\theta_i$ ,  $i = 1, \dots, n$ . Also assume that the loss function  $f(x_1, x_2, \dots, x_n)$  is a strictly increasing function with respect to  $x_i$ ,  $i = 1, 2, \dots, n$ . Then, for each risk confidence level  $\alpha \in (0, 1]$ , we have

$$TVaR_\alpha(f) = \frac{1}{\alpha} \int_0^\alpha f(Y_1^{-1}(1 - \beta), Y_2^{-1}(1 - \beta), \dots, Y_n^{-1}(1 - \beta)) d\beta.$$

## 2.3. Uncertainty Optimization

Let  $x = (x_1, x_2, \dots, x_n)$  be a decision vector, and  $\theta = (\theta_1, \theta_2, \dots, \theta_n)$  be an uncertain vector. Consider the following optimization model:

$$\begin{aligned} \min & f(x, \theta) \\ \text{s. t. } & g_j(x, \theta) \leq 0 \quad j = 1, \dots, p, \\ & z_l(x) \leq 0 \quad l = 1, \dots, m, \\ & x \geq 0. \end{aligned} \quad (1)$$

where  $f$  and  $g_j$ ,  $j = 1, \dots, p$ , are uncertain functions and  $z_l$ ,  $l = 1, \dots, m$ , are crisp functions. Since the objective function of the model (1) involves uncertainty, it cannot be directly optimized. Therefore, by considering  $f(x, \theta)$  as a loss function, we minimize its *TVaR*. In addition, we use the expected value of uncertain constraints to get a crisp feasible set. Thus, the model (1) can be reformulated as

$$\begin{aligned}
& \min TVaR_\alpha(f(x, \theta)) \\
& \text{s. t. } E(g_j(x, \theta)) \leq 0 \quad j = 1, \dots, p, \\
& \quad \quad z_l(x) \leq 0 \quad l = 1, \dots, m, \\
& \quad \quad x \geq 0.
\end{aligned} \tag{2}$$

According to theorems 2.4 and 2.7, we can rewrite problem (2) as follows:

$$\begin{aligned}
& \min \frac{1}{\alpha} \int_0^\alpha f(x, \gamma_1^{-1}(1 - \beta), \gamma_2^{-1}(1 - \beta), \dots, \gamma_n^{-1}(1 - \beta)) d\beta \\
& \text{s. t. } \int_0^1 g_j(\gamma_1^{-1}(\gamma), \dots, \gamma_m^{-1}(\gamma), \gamma_{m+1}^{-1}(1 - \gamma), \gamma_{m+2}^{-1}(1 - \gamma), \dots, \\
& \quad \quad \gamma_n^{-1}(1 - \gamma)) d\gamma \leq 0 \quad j = 1, \dots, p, \\
& \quad \quad z_l(x) \leq 0 \quad l = 1, \dots, m, \\
& \quad \quad x \geq 0.
\end{aligned} \tag{3}$$

where  $g_j(x, \theta_1, \theta_2, \dots, \theta_n)$  is strictly increasing with respect to  $\theta_1, \theta_2, \dots, \theta_m$  and strictly decreasing with respect to  $\theta_{m+1}, \theta_{m+2}, \dots, \theta_n$ .

### 3. Problem Formulation

Let  $N = (V, E)$  be an undirected connected network with vertex set  $V$ ,  $|V| = n$ , edge set  $E$  and a constant  $p \leq n$ . The length of each edge  $e \in E$  is positive and each vertex  $v_i \in V$  is associated with a nonnegative weight as the demand of the client at the vertex. Let  $d(x, y)$  denote the distance between  $x, y \in N$  which is equal to the distance of the shortest path connecting these two points. In the classical pMLP, the aim is to locate  $p$  pairwise different facilities  $m_1, \dots, m_p$  on the network  $N$  (i.e., on edges or vertices) such that the sum of weighted distances from each vertex to its closest facility is minimized. In fact, the optimal solution of the following problem is a p-median:

$$\min_{X_p \subset N, |X_p|=p} \sum_{v_i \in V} w_i d(v_i, x_p), \tag{4}$$

where

$$d(v_i, X_p) = \min_{j=1,2,\dots,p} d(v_i, m_j) \quad X_p = \{m_1, \dots, m_p\}.$$

Hakimi [26] proved the existence of an optimal solution among the set of vertices.

Let  $d_{ij} = d(v_i, v_j)$  be the distance between vertices  $v_i \in V$  and  $v_j \in V$ . Also, let  $w = \{w_i | v_i \in V\}$  and  $d = \{d_{ij} | v_i, v_j \in V\}$  denote the set of vertex weights and the set of distances between vertices, respectively. Then, the optimal objective value of pMLP is a function of  $w$  and  $d$ , which we denoted by  $f_p(w, d)$  hereafter.

In the following, we are going to present a 0-1 linear programming formulation of the problem under investigation. Let  $y_{ij}$  be a variable that is equal to 1, if the demands of the vertex  $v_i$  are served by a facility at the vertex  $v_j$ , and 0, otherwise. Also, let the variable  $x_j$  be equal to 1, if there is an open facility at the vertex  $v_j$ , and 0, otherwise. Then, the 0-1 linear programming formulation of the classical discrete pMLP can be stated as follows:

$$\min \sum_{i=1}^n \sum_{j=1}^n w_i d_{ij} y_{ij} \tag{5}$$

$$\text{s. t. } \sum_{j=1}^n y_{ij} = 1, \quad \forall i = 1, \dots, n, \tag{5.1}$$

$$y_{ij} \leq x_j, \quad \forall i, j = 1, \dots, n, \tag{5.2}$$

$$\sum_{j=1}^n x_j = p \tag{5.3}$$

$$y_{ij}, x_j \in \{0,1\} \quad \forall i, j = 1, \dots, n. \tag{5.4}$$

This objective is to minimize the total weighted distance between each demand vertex and the nearest facility. Constraints (5.1) require each demand vertex to be assigned to exactly one facility. Constraints (5.2) allow for the demand of vertex  $v_i$  to be assigned to a vertex  $v_j$  only if there is an open facility in this location. Constraints (5.3) state that exactly  $p$  facilities are to be located. Finally, the last constraints are the standard integrality conditions [13,35].

Let  $x_{ij}$  be the demand of customer in vertex  $v_i$  which is provided by facility in  $v_j$ . Then, the model can be rewritten as follows:

$$\min \sum_{i=1}^n \sum_{j=1}^n d_{ij} x_{ij}$$

$$\text{s. t. } \sum_{j=1}^n x_{ij} = w_i \quad \forall i = 1, \dots, n,$$

$$x_{ij} \leq w_i x_j \quad \forall i, j = 1, \dots, n, \tag{6}$$

$$\sum_{j=1}^n x_j = p$$

$$x_{ij} \geq 0, x_j \in \{0,1\} \quad \forall i, j = 1, \dots, n.$$

Now, consider  $N = (V, E)$  as an undirected connected network with uncertain vertex weights and uncertain distances. Some assumptions are listed as follows.

1. The weight of each vertex  $v_i$  is a positive uncertain variable  $\eta_i$ .
2. The distance between two vertices  $v_i$  and  $v_j$  is a positive uncertain variable  $\theta_{ij}$ .
3. All the uncertain variables  $\eta_i$  and  $\theta_{ij}$  are independent.

4. The uncertain variables  $\eta_i$  and  $\theta_{ij}$  have regular uncertainty distributions  $\Phi_i$  and  $Y_{ij}$ , respectively.

Therefore, the classical discrete pMLP with uncertain data can be formulated as follows:

$$\begin{aligned}
 & \min \sum_{i=1}^n \sum_{j=1}^n \theta_{ij} x_{ij} \\
 & \text{s. t. } \sum_{j=1}^n x_{ij} = \eta_i \quad \forall i = 1, \dots, n, \\
 & \quad \quad x_{ij} \leq \eta_i x_j \quad \forall i, j = 1, \dots, n, \\
 & \quad \quad \sum_{j=1}^n x_j = p \\
 & \quad \quad x_{ij} \geq 0, x_j \in \{0,1\} \quad \forall i, j = 1, \dots, n.
 \end{aligned} \tag{7}$$

**Definition 3.1.** Let  $V_p \subseteq V$ ,  $|V_p| = p$  and

$$x_j = \begin{cases} 1, & v_j \in V_p \\ 0, & \text{o. w.} \end{cases} \quad x_{ij} = \begin{cases} E(\eta_i), & \text{if the location of the nearest facility} \\ & \text{to the vertex } v_i \text{ in } V_p \text{ is the vertex } v_j, \\ 0, & \text{o. w.} \end{cases}$$

Using the above notations,  $V_p \subseteq V$  is called an expected p-facility location if and only if

$$\begin{cases} \sum_{j=1}^n x_{ij} - E(\eta_i) = 0, \quad \forall i = 1, \dots, n, \\ x_{ij} - E(\eta_i)x_j \leq 0, \quad \forall i, j = 1, \dots, n, \\ \sum_{j=1}^n x_j = p, \\ x_{ij} \geq 0, x_j \in \{0,1\}, \quad \forall i, j = 1, \dots, n. \end{cases}$$

Now, let  $V_p$  be an expected p-facility location. Define

$$S(V_p) = \sum_{i=1}^n \sum_{j=1}^n \theta_{ij} x_{ij}.$$

It is clear that  $S(V_p)$  is also an uncertain variable.

**Definition 3.2.** For a risk confidence level  $\alpha \in (0,1]$ , an expected p-facility location  $V_p^*$  is called  $TVaR_\alpha$ -p-median if

$$TVaR_\alpha(S(V_p^*)) \leq TVaR_\alpha(S(V_p))$$

holds for any expected p-facility location  $V_p$ .

Therefore, for a risk confidence level  $\alpha \in (0,1]$ , the  $TVaR_\alpha$ -p-median of uncertain network  $N$  is the optimal expected p-facility location of the following model:

$$\begin{aligned}
 & \min \sum_{i=1}^n \sum_{j=1}^n \left( \frac{1}{\alpha} \int_0^\alpha \gamma_{ij}^{-1} (1 - \beta) d\beta \right) x_{ij} \\
 & \text{s. t. } \sum_{j=1}^n x_{ij} = \int_0^1 \Phi_i^{-1} (1 - \gamma) d\gamma \quad \forall i = 1, \dots, n, \\
 & \quad \quad x_{ij} \leq \left( \int_0^1 \Phi_i^{-1} (1 - \gamma) d\gamma \right) x_j \quad \forall i, j = 1, \dots, n, \\
 & \quad \quad \sum_{j=1}^n x_j = p \\
 & \quad \quad x_{ij} \geq 0, x_j \in \{0,1\} \quad \forall i, j = 1, \dots, n.
 \end{aligned} \tag{8}$$

Thus, we can obtain the p-median with minimal risk measure and expected value of constraints by setting  $d_{ij}^\alpha = \left( \frac{1}{\alpha} \int_0^\alpha \gamma_{ij}^{-1} (1 - \beta) d\beta \right)$ ,  $v_i, v_j \in V$ , for each risk confidence level  $\alpha \in (0,1]$ , and  $w_i = \int_0^1 \Phi_i^{-1} (1 - \gamma) d\gamma$ ,  $v_i \in V$ . Kariv and Hakimi [26] showed that pMLP on general networks is NP-hard. Therefore, we conclude the following proposition immediately.

**Proposition 3.3.** UpMLP with  $TVaR$  criterion on general networks is NP-hard.

The above proposition implies that it is not possible to present exact polynomial time methods to solve UpMLP on general networks. Therefore, we propose an efficient HMBPSO algorithm for approximating the optimal solution of (8).

### 4. HMBPSO Algorithm

Kennedy and Eberhart [27] in 1995 developed the particle swarm optimization (PSO) algorithm as a nature-inspired evolutionary computation algorithm. Consider the following model:

$$\begin{aligned}
 & \min f(z) \\
 & \text{s. t. } z \in Z,
 \end{aligned}$$

where  $Z$  is the continuous and restricted space. In a PSO algorithm, a potential solution is presented as a particle  $z_l \in Z$  and a direction  $v_l \in R$  in which the particle will move. A swarm of particle is defined to be a set  $\{z_1, z_2, \dots, z_N\}$ , in which  $N$  is the number of particles. Each particle  $z_l$  retains a record of the position of its previous best performance in a vector called  $P_{best,l}$ . The particle with best performance so far in the population is maintained in a vector  $G_{best}$ . An iteration involves evaluating each  $z_l$ , and then randomly setting  $v_l$  in the direction of particle  $z_l$ 's best previous position  $P_{best,l}$  and the best previous position  $G_{best}$  of any particle in the population.

Group particle optimization algorithm is inherently a continuous optimization algorithm. However, many optimization problems have a discrete space. Kennedy and Eberhart in [28] presented the PSO algorithm with discrete binary variables. They used the following transfer function, to calculate the particle position:

$$M(v_{lk}) = \frac{1}{1 + \exp(-v_{lk})}, \quad (9)$$

where  $v_{lk}$  is the probability of changing the  $k$ -th component of  $z_l$  towards one. Therefore, after calculating  $v_l$  for each particle  $z_l$ , the new position of the particle is obtained as follows:

$$z_{lk} = \begin{cases} 1, & \text{rand} \leq M(v_{lk}) \\ 0, & \text{rand} > M(v_{lk}), \end{cases} \quad (10)$$

in which rand is a random value between zero and one.

Now, consider the model (8) of UpMLP. First, for each risk confidence level  $\alpha \in (0,1]$ , we obtain the shortest distance between vertices by applying Algorithm 1. Then, we consider a particle of UpMLP as  $z_l = (x_1, x_2, \dots, x_n)$  such that  $\sum_1^n x_j = p$ ,  $x_j \in \{0,1\}$ . Using the distances of vertices obtained by Algorithm 1, we set

$$x_{ij} = \begin{cases} w_i, & \text{if } x_j = 1 \text{ and it is the location of} \\ & \text{the nearest facility to the client } i, \\ 0, & \text{o. w.} \end{cases} \quad (11)$$

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**Algorithm 1:** Find shortest distance between two vertices  $v_i$  and  $v_j$  for risk confidence level  $\alpha \in (0,1]$ .

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- 1: Set  $d_{ij}^\alpha = 0$ .
  - 2: **for**  $r = 1, \dots, M$  **do**
  - 3:   compute  $d_{ij}^r = Y_{ij}^{-1}(1 - \frac{r}{M}\alpha)$ ,
  - 4:   set  $d_{ij}^\alpha = d_{ij}^\alpha + \frac{r}{M}\alpha d_{ij}^r$ .
  - 5: **end for**
  - 6: Compute  $d_{ij}^\alpha = \frac{1}{\alpha} d_{ij}^\alpha$ .
  - 7: Report  $d_{ij}^\alpha$  as the shortest distance between two vertices  $v_i$  and  $v_j$ .
- 

Note that we set  $x_{ij} = w_i$ , ( $x_j = 1$ ), only for one nearest facility to the client  $i$ . We calculate the weight of vertex  $v_i$ , i.e.,  $w_i$ , by using Algorithm 2 [38].

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**Algorithm 2:** Find the weight of vertex  $v_i$ .

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- 1: Set  $w_i = 0$ .
  - 2: **for**  $k = 1, \dots, 99$  **do**
  - 3:   compute  $w_k = 0.01(\Phi_i^{-1}(1 - 0.0k))$ ,
  - 4:   set  $w_i = w_i + w_k$ .
  - 5: **end for**
  - 6: Report  $w_i$  as the weight of vertex  $v_i$ .
-

Therefore, we first randomly generate the particle  $z_l$  of UpMLP. Repeat this process  $N$  times and get  $N$  initial feasible particles  $z_1, z_2, \dots, z_N$ . Then, we assume that the fitness of each  $z_l$  is  $TVaR$ , i.e.,  $Fit(z_l) = TVaR_\alpha(z_l)$ . Thus, the particle with smallest objective value has the best fitness. The fitness of each particle  $z_l$  is obtained as follows:

$$TVaR_\alpha(z_l) = \sum_{i=1}^n \sum_{j=1}^n d_{ij}^\alpha x_{ij}. \quad (12)$$

In the process of updating the  $(i + 1)$ th iteration, we first denote  $P_{best,l}(i)$  for each particle  $z_l$  and  $G_{best}(i)$ , and then obtain the new directions of the particles using the following equation:

$$v_l(i + 1) = v_l(i) + C_1 r_1 [P_{best,l}(i) - z_l(i)] + C_2 r_2 [G_{best}(i) - z_l(i)], \quad (13)$$

where,

$$P_{best,l}(i) = \begin{cases} z_l(i), & Fit(z_l(i)) = Fit(z_l(i - 1)) \\ P_{best,l}(i - 1), & \text{o. w.}, \end{cases}$$

and  $G_{best}(i) = P_{best,k}(i)$ , with  $k = \operatorname{argmin}\{P_{best,l}(i) : l = 1, \dots, N\}$ .

In addition,  $r_1$  and  $r_2$  are uniformly distributed random numbers in the interval  $[0,1]$  and  $C_1$  and  $C_2$  are learning rates, to well adjust the convergence of the particles. In the basic PSO algorithm, the values of  $C_1$  and  $C_2$  are fixed. But, if these coefficients depend on the repetition of the algorithm, we will get better results. Therefore, here, we propose modifications of  $C_1$  and  $C_2$  of the basic PSO. In other words, we consider

$$C_1 = C_2 = 2 + \left( \frac{t}{t + 1} \right),$$

in which  $t = 1, 2, \dots, MaxIt$ , and  $MaxIt$  indicates the number of generations of the HMBPSO algorithm. It can easily be seen that  $C_1, C_2 \in [2,3]$ .

After obtaining  $v_l$  by (13), we use (10) to get a new particle of the next generation. We obtain a new generation of particles by repeating the above process  $N$  times.

Now, based on all the above explanations, we summarize the HMBPSO algorithm for solving the model (8) as follows.

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**Algorithm 3:** Obtain an approximate optimal solution of the model (8)

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- 1: Initialize the feasible particles  $z_1, z_2, \dots, z_N$ .
  - 2: Compute  $x_{ij}$ ,  $i, j = 1, \dots, n$  by (11).
  - 3: Compute the particles fitness by using (12) and evaluate each particle of UpMLP.
  - 4: Update all the particles by using equations (13) and (10).
  - 5: Repeat Steps 3 and 4 for  $MaxIt$  times.
  - 6: Return  $G_{best}$  as the optimal solution of the model (8), and  $TV aR_\alpha(G_{best}) = Fit(G_{best})$  as the corresponding optimal value.
-

## 5. Computational Experiments

In this section, we give a numerical example to show the efficiency of the HMBPSO algorithm. The results of the numerical experiments are obtained on a PC with processor Intel(R) Core(TM) i3 CPU 2.27GHZ and 4GB of RAM under Windows 7.

Consider  $N$  as a network with 18 vertices and 19 edges (see Figure 1). Let  $N$  be a medicines distribution system. In this system, assume that

- (1) the vertices indicate urban areas,
- (2) at each area, there is a drugstore,
- (3) warehouses of the distribution company and drugstores are considered as facilities and clients, respectively,
- (4) the weight  $\eta_i$  of the vertex  $v_i$  is equal to the average monthly purchase of residents of this area from the drugstore located at vertex  $v_i$ ,
- (5) the vertex weights and the distances between vertices (the travel times between vertices) of  $N$  are uncertain variables,
- (6) Our aim is to find three vertices on the network  $N$  to locate warehouses of the distribution company to minimize the sum of  $TVaR$  of weighted distances from each urban area to its closest warehouse.

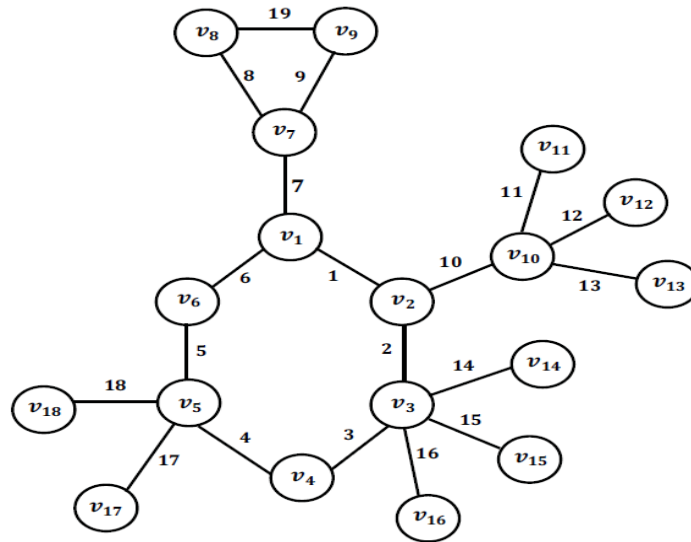
We applied the HBPSO and HMBPSO algorithms for different instances of UpMLPs and observed that the HMBPSO algorithm in comparison with the HBPSO algorithm can solve these problems more efficiently. In the following, among all the tested instances, we present the instance which the HMBPSO algorithm uses for solving U3MLP with  $TVaR$  objective at a risk confidence level of  $\alpha = 0.8$  on given network  $N$ .

Let the edge lengths  $\theta_e, e \in E$ , be linear uncertain variables (see Table 1). For  $\alpha \in (0,1)$ , the inverse uncertainty distribution of a linear uncertain variable  $\mathcal{L}(a, b)$  is

$$\phi^{-1}(\alpha) = (1 - \alpha)a + \alpha b.$$

Therefore, if  $\theta_e = \mathcal{L}(a, b)$  is the uncertain variable of the edge  $e \in E$ , then

$$d_e^\alpha = \frac{1}{\alpha} \left[ \int_0^\alpha (\beta a + (1 - \beta)b) d\beta \right] = \frac{\alpha}{2} (a - b) + b.$$



**Figure 1.** The network  $N$

**Table 1.** The input data of the network  $N$

|            |  |
|------------|--|
| $\theta_e$ | $(\mathcal{L}(8, 10), \mathcal{L}(18, 21), \mathcal{L}(19, 21), \mathcal{L}(4, 6), \mathcal{L}(3, 4), \mathcal{L}(14, 16), \mathcal{L}(28, 30), \mathcal{L}(10, 12), \mathcal{L}(17, 18), \mathcal{L}(6, 8), \mathcal{L}(22, 24), \mathcal{L}(7, 9), \mathcal{L}(15, 17), \mathcal{L}(18, 21), \mathcal{L}(26, 28), \mathcal{L}(28, 30), \mathcal{L}(16, 18), \mathcal{L}(4, 6), \mathcal{L}(4, 6))$ |
| $\eta_i$   | $(\mathcal{L}(24, 26), \mathcal{L}(27, 28), \mathcal{L}(3, 5), \mathcal{L}(27, 28), \mathcal{L}(19, 20), \mathcal{L}(1, 4), \mathcal{L}(8, 10), \mathcal{L}(16, 18), \mathcal{L}(29, 30), \mathcal{L}(29, 30), \mathcal{L}(4, 6), \mathcal{L}(30, 31), \mathcal{L}(29, 30), \mathcal{L}(14, 16), \mathcal{L}(24, 26), \mathcal{L}(4, 6), \mathcal{L}(12, 13), \mathcal{L}(27, 28))$                  |

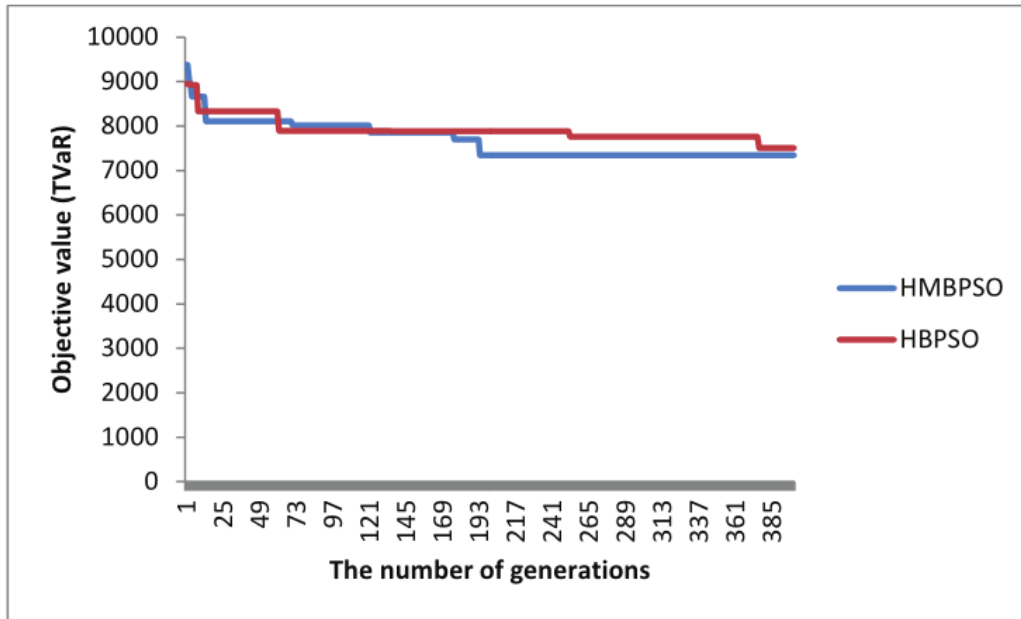
Thus, with the obtained crisp length edges and using the Dijkstra’s algorithm, we find the shortest distance between vertices (Dijkstra’s algorithm is a successful algorithm for finding shortest path between two vertices [14]). Also, let the vertex weights  $\eta_i$  be linear uncertain variables (see Table 1). Therefore, we find the expected value of the vertex weight  $\eta_i$  as follows:

$$w_i = \int_0^1 (\gamma a + (1 - \gamma)b)d\gamma = \frac{(a + b)}{2}.$$

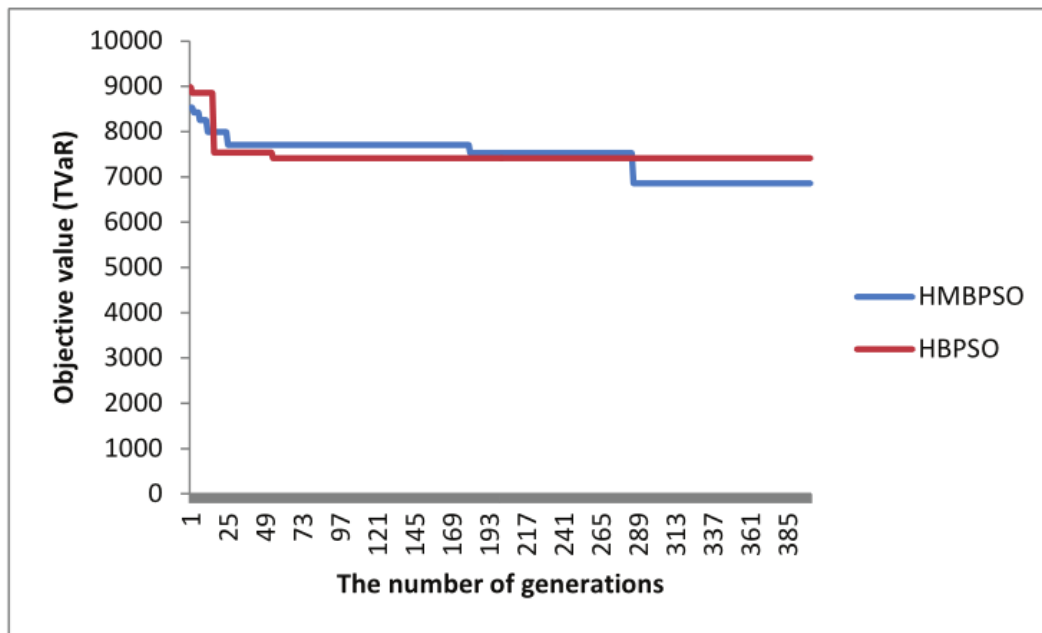
We use the HMBPSO and HBPSO algorithms for approximating the optimal solution and compare the obtained results.

**Table 2.** Performance results of HBPSO and HMBPSO on U3MLP

| Approach | $N$ | Optimal solution   | Obj.v. | R.t.    |
|----------|-----|--|--------|---------|
| HMBPSO   | 50  | (1, 2, 2, 18, 18, 2, 2, 2, 1, 2, 2, 2, 2, 18, 2, 1, 18, 18)  | 7341   | 33.679s |
| HBPSO    | 50  | (2, 2, 2, 18, 18, 18, 2, 7, 7, 2, 7, 7, 2, 2, 2, 18, 7, 18)  | 7509   | 40.95s  |
| HMBPSO   | 100 | (1, 3, 3, 3, 1, 1, 1, 8, 8, 1, 1, 1, 3, 3, 3, 3, 1, 1)       | 6860.3 | 78.93s  |
| HBPSO    | 100 | (1, 1, 4, 4, 10, 4, 1, 10, 1, 10, 10, 10, 10, 1, 1, 4, 4, 4) | 7409.8 | 81.3s   |



**Figure 2.** Convergence comparison with population of 50



**Figure 3.** Convergence comparison with population of 100

Table 2 compares the best objective values in each population size calculated by the HBPSO and HMBPSO algorithms with  $N = 50, 100$  and  $MaxIt = 400$ . Note that in Table 2, 'Obj.v.' is the objective value of the problem and 'R.t.' is the running time of the algorithm. From Table 2, we can conclude that the HMBPSO algorithm finds the solution of the problem under investigation more efficiently. The convergence comparisons are shown in figures 2 and 3, respectively. Based on the obtained results, it is obvious that the HMBPSO algorithm outperforms. Note that, there is no guarantee

that the HMBPSO and HBPSO algorithms with the smaller initial population will find better optimal solution.

## 6. Conclusion

We investigated the pMLP model with uncertain vertex weights and uncertain distances. We showed that UpMLP with  $TVaR$  objective is NP-hard. Thus, we proposed a novel HMBPSO algorithm for approximating the optimal solution of the problem, containing the tail value at risk simulation and the expected value simulation. Finally, by computational experiments, the efficiency of the algorithm was illustrated.

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